ANNA UNIVERSITY
Regulations - 2021

Common syllabus for B.E. / B.Tech. Degree Course to all the Engineering Colleges in Tamilnadu
(Non - Autonomous)
Common to all Branches

ENGINEERING PHYSICS
Subject Code: PH 3151

OBJECTIVES:

- To make the students effectively to achieve an understanding of mechanics.
- To enable the students to gain knowledge of electromagnetic waves and its applications.
- To introduce the basics of oscillations, optics and lasers.
- Equipping the students to be successfully understand the importance of quantum physics.
- To motivate the students towards the applications of quantum mechanics.

UNIT I: MECHANICS 9

Multiparticle dynamics: Center of mass (CM) – CM of continuous bodies – motion of the CM – kinetic energy of system of particles. Rotation of rigid bodies: Rotational kinematics – rotational kinetic energy and moment of inertia - theorems of M .I –moment of inertia of continuous bodies – M.I of a diatomic molecule - torque – rotational dynamics of rigid bodies – conservation of angular momentum – rotational energy state of a rigid diatomic molecule - gyroscope - torsional pendulum – double pendulum –Introduction to nonlinear oscillations.

UNIT II: ELECTROMAGNETIC WAVES 9

The Maxwell's equations - wave equation; Plane electromagnetic waves in vacuum, Conditions on the wave field - properties of electromagnetic waves: speed, amplitude, phase, orientation and waves in matter - polarization - Producing electromagnetic waves - Energy and momentum in EM waves: Intensity, waves from localized sources, momentum and radiation pressure - Cell-phone reception. Reflection and transmission of electromagnetic waves from a non-conducting medium-vacuum interface for normal incidence.

UNIT III: OSCILLATIONS, OPTICS AND LASERS 9

Simple harmonic motion - resonance –analogy between electrical and mechanical oscillating systems - waves on a string - standing waves - traveling waves - Energy transfer of a wave - sound waves - Doppler effect. Reflection and refraction of light waves - total internal reflection - interference – Michelson interferometer –Theory of air wedge and experiment. Theory of laser - characteristics - Spontaneous and stimulated emission - Einstein's coefficients - population inversion - Nd-YAG laser, CO_2 laser, semiconductor laser –Basic applications of lasers in industry.

UNIT IV: BASIC QUANTUM MECHANICS 9

Photons and light waves - Electrons and matter waves – Compton effect - The Schrodinger equation (Time dependent and time independent forms) - meaning of wave function - Normalization – Free particle - particle in a infinite potential well: 1D, 2D and 3D Boxes - Normalization, probabilities and the correspondence principle.

UNIT V: APPLIED QUANTUM MECHANICS 9

The harmonic oscillator (qualitative)- Barrier penetration and quantum tunneling (qualitative) - Tunneling microscope - Resonant diode - Finite potential wells (qualitative) - Bloch's theorem for particles in a periodic potential – Basics of Kronig-Penney model and origin of energy bands.

OUTCOMES:

After completion of this course, the students should be able to

- Understand the importance of mechanics.
- Express their knowledge in electromagnetic waves.
- Demonstrate a strong foundational knowledge in oscillations, optics and lasers.
- Understand the importance of quantum physics.
- Comprehend and apply quantum mechanical principles towards the formation of energy bands.

Total: 45 Periods

1. Mechanics

Multiparticle dynamics: Center of mass (CM) – CM of continuous bodies – motion of the CM – kinetic energy of system of particles. Rotation of rigid bodies: Rotational kinematics – rotational kinetic energy and moment of inertia – theorems of M.I – moment of inertia of continuous bodies – M.I of a diatomic molecule – torque – rotational dynamics of rigid bodies – conservation of angular momentum – rotational energy state of a rigid diatomic molecule – gyroscope – torsional pendulum – double pendulum – Introduction to non-linear oscillations.

Introduction

- Inertia is the reluctance of the system to change its state of rest or of uniform motion. In case of linear motion mass is the inertia of the system.
- But in the case of rotational motion inertia of rigid body plays a very important role in the industrial applications where both rotational and translational operations are to be coupled.
- Its principle is also used widely in the design, fabrication and working of several machines of common use namely cranes, motors, lathes, lifting devices, soil testing machines, hydraulic systems, in the design of and fixing the position of huge reactors, etc.
- Thus, being the student of technology one should understand the role of moment of inertia in rotational motion as well as the relation between translational and rotational motion of a rigid body.
- Many equations of rotational motion could be written by simply replacing the parameters in the equations of linear motion with the appropriate analogous parameters in the rotational motion.

Some of the pairs of analogous parameters are given below, followed by their definitions.

Rigid Body Motion

Translational (Linear) motion	Rotational motion
Displacement	Angular displacement
Velocity	Angular velocity
Acceleration	Angular acceleration
Momentum	Angular momentum
Force	Torque
Mass	Moment of inertia

Definition of Terminology in Linear Motion

Displacement: The change of position of a particle in a particular direction is called displacement. If the direction is along x-axis, then the displacement is x. It is measured in **metre (m)**.

Velocity (v): The rate of change of displacement is called velocity. It is given by,

$$\boxed{\vec{v} = \frac{d\vec{x}}{dt}}$$

S.I unit of velocity is **m / s or ms^{-1}**

Acceleration ($\vec{a}$): The rate of change of velocity is called acceleration.

$$\boxed{\vec{a} = \frac{d\vec{v}}{dt}}$$

S.I unit of acceleration is **m / s^2 (or) ms^{-2}**

Momentum ($\vec{p}$): The product of mass and velocity of the particle or of a body is called momentum. It is given by

$$\vec{p} = m\vec{v}$$

$$\boxed{\vec{p} = m\frac{d\vec{x}}{dt}}$$

S.I unit is **kgms^{-1}**

Force (F) : The product of mass and acceleration of a particle or a body is called force. It is given by,

$$\boxed{\vec{F} = m\,\vec{a}}$$

$$\vec{F} = m\frac{d\vec{v}}{dt} \qquad \left(\because \vec{a} = \frac{d\vec{v}}{dt}\right)$$

$$= m\frac{d}{dt}\left(\frac{dx}{dt}\right)$$

$$\boxed{\vec{F} = m\frac{d^2\vec{x}}{dt^2}}$$

S.I unit is $\mathbf{kg\,ms^{-2}}$ **(or) newton (or) N**

Definition of terminology in rotational motion

Angular displacement (θ) **:** The angle described by a rigid body from its rest position in given time is called angular displacement. Its unit is radian.

Angular velocity (ω) **:** The rate of change of angular displacement is called angular velocity. It is given by

$$\boxed{\omega = \frac{d\theta}{dt}}$$

S.I unit of angular velocity is **radian / s**

Angular acceleration (α) **:** The rate of change of angular velocity is called angular acceleration. It is given by

$$\boxed{\alpha = \frac{d\omega}{dt}}$$

S.I unit of angular acceleration is $\mathbf{radian / s^2}$

Angular momentum (L) : The moment of momentum is called angular momentum. It is given by

$$\vec{L} = \vec{r} \times \vec{p}$$

$$\vec{L} = \vec{r} \times m\,\vec{v} \qquad [\because \vec{p} = m\,\vec{v}]$$

$$\boxed{\vec{L} = m\,(\vec{r} \times \vec{v})}$$

It is unit is $\mathbf{kg\,m^2\,s^{-1}}$

Torque (τ) **:** The moment of the applied force about the axis of rotation is called torque. It is given by

$$\boxed{\tau = \vec{r} \times \vec{F}}$$

$\vec{F}$ – Force

$\vec{r}$ – ⊥r distance

S.I. unit is **Nm**

(i) Relation between linear velocity and angular velocity

Linear velocity, $\vec{v} = \dfrac{dx}{dt}$

But, $dx = r\,d\,\theta$ $\left(\because d\,\theta = \dfrac{dx}{r}\right)$

$$\therefore\ \vec{v} = r\,\frac{d\,\theta}{dt}$$

$$\boxed{\vec{\boldsymbol{v}} = \boldsymbol{r}\,\omega}$$

ω – angular velocity $\left(\because \dfrac{d\,\theta}{dt} = \omega\right)$

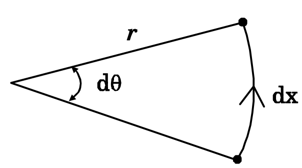


Fig. 1.1

(ii) Relation between linear acceleration and angular acceleration ($\vec{a}$ and α)

The linear acceleration is given by

$$\vec{a} = \frac{d\vec{v}}{dt}$$

$$= \frac{d}{dt}(r\,\omega) = r\frac{d\omega}{dt}$$

$$\boxed{\vec{a} = r\,\alpha} \qquad \left(\because \vec{\alpha} = \frac{d\omega}{dt} \right)$$

1.1 MULTIPARTICLE DYNAMICS

A mechanical system consists of two or more particles is called multiparticle system.

Let us consider two particles of mass m_1 and m_2 moving in one-dimension with co-ordinates x_1 and x_2 as shown in fig. 1.2.

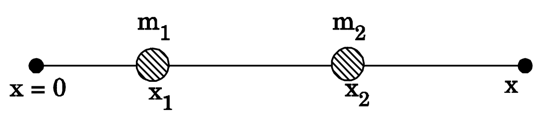


Fig. 1.2

$\vec{F}_1$ is the force acting on body m_1, then according to Newton's Law,

$$m_1\vec{a}_1 = \vec{F}_1$$

$$m_1\frac{d^2x_1}{dt^2} = F_1 \quad \text{(or)} \quad m_1\ddot{x}_1 = F_1 \qquad \text{... (1)}$$

where $\frac{d^2x_1}{dt^2}$ is denoted as $\ddot{x}_1$ $\left(\because \vec{a} = \frac{d^2\vec{x}_1}{dt^2} \right)$

The force $\vec{F}_1$ can be divided into two parts like

$$\boxed{\vec{F}_1 = \vec{F}_{12} + \vec{F}_{1e}} \qquad \text{... (2)}$$

where $\vec{F}_{12}$ is the internal force on particle 1 by particle 2. $\vec{F}_{1e}$ is the sum of all external force (like gravity, etc).

Eqn (1) becomes

$$m_2 \ddot{x}_2 = \vec{F}_{12} + \vec{F}_{1e} \qquad \dots (3)$$

Similarly, for particle 2,

$$m_2 \vec{a}_2 = \vec{F}_2$$

$$m_2 \frac{d^2 x_2}{dt^2} = m_2 \ddot{x}_2 = \vec{F}_2 = \vec{F}_{21} + \vec{F}_{2e} \qquad \dots (4)$$

Total force on system of the two particle is given by adding Eqns (3) and (4), we get

$$m_1 \ddot{x}_1 + m_2 \ddot{x}_2 = \vec{F}_{12} + \vec{F}_{21} + \vec{F}_{1e} + \vec{F}_{2e} \qquad \dots (5)$$

By Newton's third law of motion $\vec{F}_{12} = -\vec{F}_{21}$

$$\Rightarrow \qquad m_1 \ddot{x}_1 + m_2 \ddot{x}_2 = \vec{F}_{12} + (-\vec{F}_{12}) + \vec{F}_{1e} + \vec{F}_{2e} \qquad \dots (6)$$

$$\boxed{\boldsymbol{m_1 \ddot{x}_1 + m_2 \ddot{x}_2 = \vec{F}_{1e} + \vec{F}_{2e} = \vec{F}_e}} \qquad \dots (7)$$

where $\vec{F}_e$ is net external force on system of two particles.

Let the total mass of the system (body) is equal to

$$M = m_1 + m_2 \qquad \dots (8)$$

Multiplying and dividing eqn (7) by M in L.H.S,

$$M \frac{(m_1 \ddot{x}_1 + m_2 \ddot{x}_2)}{M} = \vec{F}_e \qquad \dots (9)$$

or

$$M \ddot{X} = \vec{F}_e \qquad \dots (10)$$

$$M \left(\frac{m_1 \frac{d^2 x_1}{dt^2} + m_2 \frac{d^2 x_2}{dt^2}}{M} \right) = \vec{F}_e$$

$$M \frac{d^2}{dt^2}\left(\frac{m_1 x_1 + m_2 x_2}{M}\right) = \vec{F}_e$$

$$M \frac{d^2 X}{dt^2} = \vec{F}_e$$

or $$M\ddot{x} = \vec{F}_e$$

where, $$X = \frac{m_1 x_1 + m_2 x_2}{M}$$

$$= \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} \quad \text{... (11)}$$

is called the center of mass (*CM*).

Thus, *CM* has a location X which is a weighted average of x_1 and x_2:

$$X = \frac{m_1 x_1}{m_1 + m_2} + \frac{m_2 x_2}{m_1 + m_2} \quad \text{... (12)}$$

If $m_1 = m_2 = m$, then $M = 2m$ and

$$X = \frac{x_1 + x_2}{2} \quad \text{... (13)}$$

lies midway between the two particles.

If $m_1 > m_2$, then X is closer to x_1 and vice-versa.

According to eqn.(10), *CM* is a fictitious body whose acceleration is controlled by only the external force and not by the internal forces between the particles.

In general, the *CM* for a system (body) of N number of particles is obtained by extending eqn (1) as

$$\boxed{X = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3 + \ldots\ldots + m_N x_N}{m_1 + m_2 + m_3 + \ldots\ldots m_N}}$$

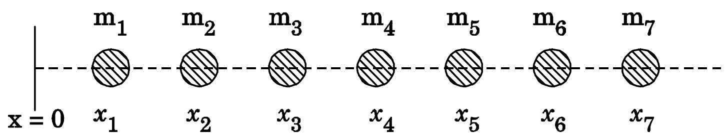


Fig. 1.3

$$X = \frac{\sum_{i=1}^{N} m_i x_i}{\Sigma m_i}$$

$$\text{or} \qquad X = \frac{\sum_{i=1}^{N} m_i x_i}{M} \qquad (\because \Sigma m_i = M)$$

For a continuous distribution of masses, for example in an one-dimension, the *CM* is represented as

$$X = \lim_{\Delta m \to 0} \frac{\sum_{i=1}^{N} \Delta m_i x_i}{M} \qquad \dots (14)$$

Here, we consider an infinite sum. Further for continuous distribution of mass, the summation is changed into integration.

$$\therefore \qquad \boxed{X = \frac{\int x\, dm}{\int dm} = \frac{\int x\, dm}{M}} \qquad (\because \int dm = M)$$

Example

So in the case of a uniform rod

Mass per unit length of the rod $= \dfrac{M}{L}$

Mass of the elemental length dx of the rod $dx = \left(\dfrac{M}{L}\right) dx$

Mass of the whole length of the rod is obtained by integrating between limits $x = 0$ and $x = L$)

$$X = \frac{\int_0^L \frac{x\,M}{L}\,dx}{M} = \left| \frac{1}{L}\left(\frac{x^2}{2}\right)\right|_0^L = \frac{L}{2} \qquad \text{... (15)}$$

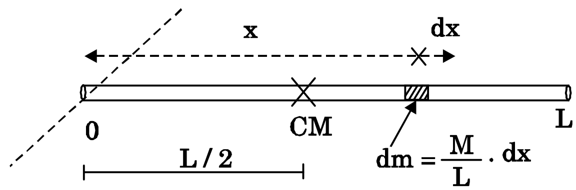


Fig. 1.4

For N particles in three dimensions (3D)

For x-direction

$$X_{CM} = \frac{\sum_i m_i x_i}{M}$$

For y-direction,

$$Y_{CM} = \frac{\sum_i m_i y_i}{M}$$

For z-direction,

$$Z_{CM} = \frac{\sum_i m_i z_i}{M} \qquad \text{... (16)}$$

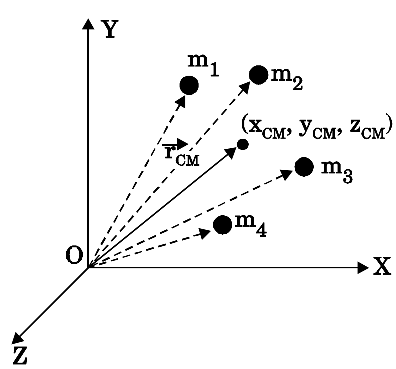


Fig. 1.5 Centre of mass for distributed point masses

Thus in vector rotation $\vec{r_i} = x_i \hat{i} + y_i \hat{j} + z_i \hat{k}$ is the position vector of the particle. So the position of the CM is

$$\vec{r}_{CM} = \frac{\sum_i m_i \vec{r_i}}{M}$$

and for continuous object as shown in fig. 1.6.

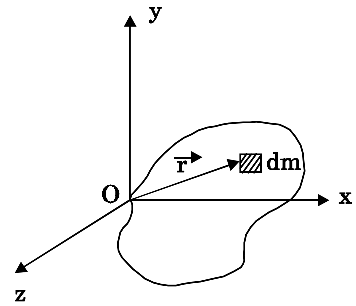


Fig. 1.6

$$\vec{r}_{CM} = \frac{1}{M}\int \vec{r}\, dm \quad \text{... (17)}$$

Centre of Mass : Definition

Consider the motion of a system consisting of a large number of particles. There is one point in it which behaves as though the entire mass of the system were concentrated there and all the external forces were acting at this point. This point is called the centre of mass of the system.

Note

The external force acting on a system (body) at an arbitrary point (other than CM), produces both rotational and transnational motions. On the other hand the force acting at CM of a system produces only transnational motion.

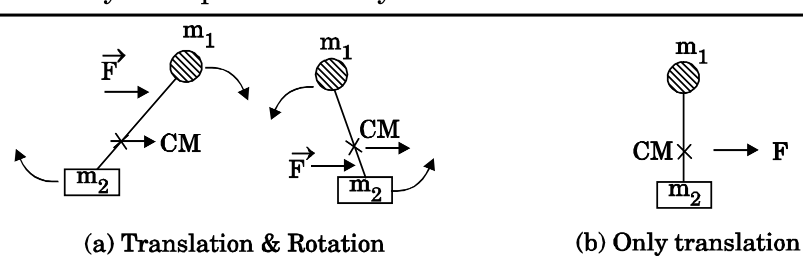


Fig. 1.7

1.2 CENTRE OF MASS (CM)

Every system or body is a collection of large number of tiny particles. In translatory motion of a body, every particle experiences equal displacement with time. Therefore, the motion of the whole body may be represented by a particle.

But when the body rotates or vibrates during translatory motion, then its motion can be represented by a point that moves in the same way as that of a single particle subjected to the same external forces would move. **This point is called centre of mass of a system.**

Definition

***A point in the system at which whole mass of the body is supposed to be concentrated is called centre of mass of the body*.**

Therefore, if a system contains two or more particles, its translatory motion can be described by the motion of the centre of mass of the system.

Examples for motion of centre of mass

(i) Motion of planets and its satellite

- Let us consider the motion of the centre of mass of the Earth and moon system (Fig 1.8). The moon moves round the Earth in a circular orbit and the Earth moves round the sun in an elliptical orbit.

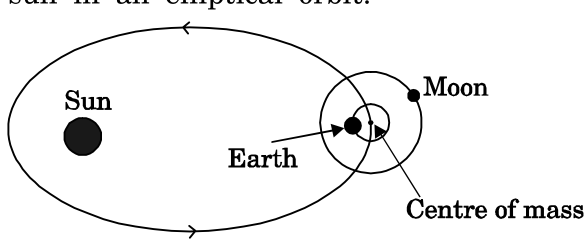


Fig. 1.8 Centre of mass of Earth - Moon system

It is more correct to say that the Earth and the moon both move in circular orbits about their common centre of mass in an elliptical orbit round the Sun.

For the system consisting of the Earth and the moon, their mutual gravitational attractions are the internal forces. The gravitational attraction by the Sun attraction on both the Earth and moon are the external forces acting on the centre of mass of the system.

(ii) Projectile Trajectory

- When a cracker is fired at an angle with the horizontal and when it explodes in the air, then different pieces of the cracker follows different parabolic paths.

The centre of mass of all the pieces of the cracker continue to move in the initial parabolic path as shown by a dotted line in fig. 1.9. This is because the different pieces of the cracker move under the effect of internal forces only.

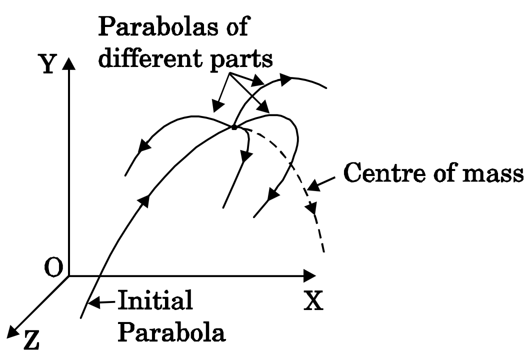


Fig. 1.9

(iii) Decay of a Nucleus

- Let us consider a spontaneous decay of a radioactive nucleus into two fragments as shown in fig. 1.10. The two fragments move apart in opposite directions obeying the laws of conservations of energy and momentum.

Since the parent nucleus is at rest, therefore, the centre of mass of the system remains at rest.

The nucleus decays under the effect of internal forces. The heavier fragment moves with less speed while the lighter fragment moves with high speed.

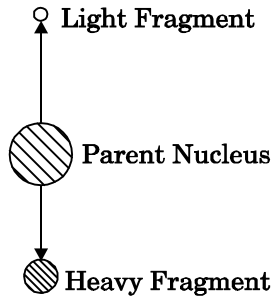


Fig. 1.10

Centre of mass of two point masses

With the equations for centre of mass, let us find the centre of mass of two point masses m_1 and m_2, which are at positions x_1 and x_2 respectively on the X-axis.

For this case, we can express the position of centre of mass in the following three ways based on the choice of the co-ordinate system.

(i) When the masses are on positive *X*-axis

The origin is taken arbitrarily so that the masses m_1 and m_2 are at positions x_1 and x_2 on the positive X-axis as shown in figure 1.11 (a).

The centre of mass is on the positive X-axis at X_{CM} and it is given by the equation,

$$X_{CM} = \frac{m_1 x_2 + m_2 x_2}{m_1 + m_2}$$

(ii) When the origin coincides with any one of the masses

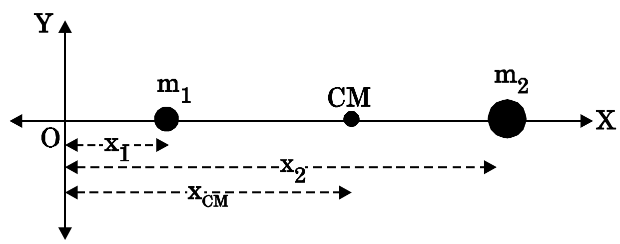
The calculation is minimised if the origin of the coordinate system is made to coincide with any one of the masses as shown in figure 1.11(b).

When the origin coincides with the point mass m_1, its position x_1 is zero, (i.e. $x_1 = 0$). Then,

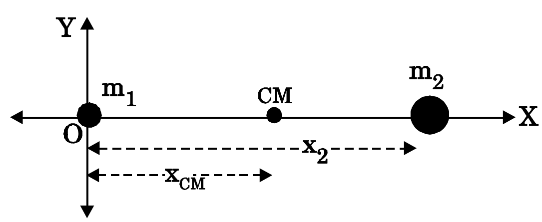
$$X_{CM} = \frac{m_1 (0) + m_2 x_2}{m_1 + m_2}$$

The equation further is simplified as,

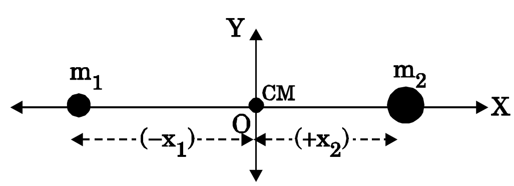
$$\boxed{X_{CM} = \frac{m_2 x_2}{m_1 + m_2}}$$



(a) When the masses are on positive X axis



(b) When the origin coincides with any one of the masses



(c) When the origin coincides with the center of mass itself

Fig. 1.11 Centre of mass of two point masses determined by shifting the origin

(iii) When the origin coincides with the centre of mass itself

If the origin of the coordinate system is made to coincide with the centre of mass then, $X_{CM} = 0$ and the mass m_1 is found to be on the negative X-axis as shown in fig. 1.11(c). Hence, its position x_1 is negative, (i.e., $-x_1$).

$$0 = \frac{m_2(-x_1) + m_2 x_2}{m_1 + m_2}$$

$$0 = m_1(-x_1) + m_2 x_2$$

$$\boxed{m_1 x_1 = m_2 x_2}$$

The equation given above is known as ***principle of moments.***

1.3 MOTION OF CENTRE OF MASS

When a rigid body moves, it centre of mass will also move along the body.

For kinematic quantities like velocity $(\vec{v}_{CM})$ and acceleration $(\vec{a}_{CM})$ of the centre of mass, we can differentiate the expression for position of centre of mass with respect to time once and twice respectively.

For simplicity, let us take the motion along X direction only.

$$\vec{v}_{CM} = \frac{d\vec{x}_{CM}}{dt}\frac{d}{dt}\left(\frac{\Sigma m_i x_i}{\Sigma m_i}\right) = \frac{\Sigma m_i\left(\frac{d\vec{x}_i}{dt}\right)}{\Sigma m_i}$$

$$\vec{v}_{CM} = \frac{\Sigma m_i \vec{v}_i}{\Sigma m_i} \qquad \text{... (1)}$$

$$\vec{a}_{CM} = \frac{d}{dt}\left(\frac{d\vec{x}_{CM}}{dt}\right) = \left(\frac{d\vec{v}_{CM}}{dt}\right) = \frac{\Sigma m_i \left(\frac{d\vec{v}_i}{dt}\right)}{\Sigma m_i}$$

$$\boxed{\vec{a}_{CM} = \frac{\Sigma m_i \vec{a}_i}{\Sigma m_i}} \quad \text{... (2)}$$

In the absence of external force, $\vec{F}_{ext} = 0$, the individual rigid bodies of a system can move or shift only due to the internal forces.

This does not affect the position of the centre of mass. This mean that the centre of mass is in a state of rest or uniform motion.

Hence, $\vec{a}_{CM}$ is zero when centre of mass is at rest or has uniform motion ($\vec{v}_{CM} = 0$ or $\vec{v}_{CM} =$ constant). There is no acceleration of centre of mass, ($\vec{a}_{CM} = 0$).

From equations (1) and (2),

$$\vec{v}_{CM} = \frac{\Sigma m_i \vec{v}_i}{\Sigma m_i} = 0 \text{ (or) } \vec{v}_{CM} = \text{constant}$$

It shows that

$$\boxed{\vec{a}_{CM} = \frac{\Sigma m_i \vec{a}_i}{\Sigma m_i} = 0} \quad \text{... (3)}$$

Here, the individual particles may still move with their respective velocities and accelerations due to internal forces.

In the presence of external force, (i.e. $\vec{F}_{ext} \neq 0$), the centre of mass of the system will accelerate as given by the following equation.

$$\vec{F}_{ext} = (\Sigma m_i) \vec{a}_{CM} ; \quad \vec{F}_{ext} = M \vec{a}_{CM} ;$$

$$\boxed{\vec{a}_{CM} = \frac{\vec{F}_{ext}}{M}} \quad \ldots (4)$$

1.4 CENTRE OF MASS (CM) OF CONTINUOUS BODIES (RIGID BODIES)

Experimental location of the centre of mass

- The centre of masses of homogeneous and regular shaped bodies coincides with their geometrical centre. Hence it can be easily located; but if the body is of irregular shape the location of its centre of mass is difficult.

However, since the centre of mass is found to coincide with centre of gravity of the bodies, it can be easily obtained either by pivoting the body to a balanced position or suspending it from some fixed point in it. Basically, both are similar methods.

As an illustration, we describe below the method of suspension to locate the centre of mass of any regular or irregular shaped body.

The body of first hung from some (any) point P and a vertical line PQ is drawn when the body is in equilibrium. The body is then hung from some other point R and a vertical line RS is drawn (Fig 1.12).

The point of intersection C of these two lines PQ and RS gives the position of centre of mass.

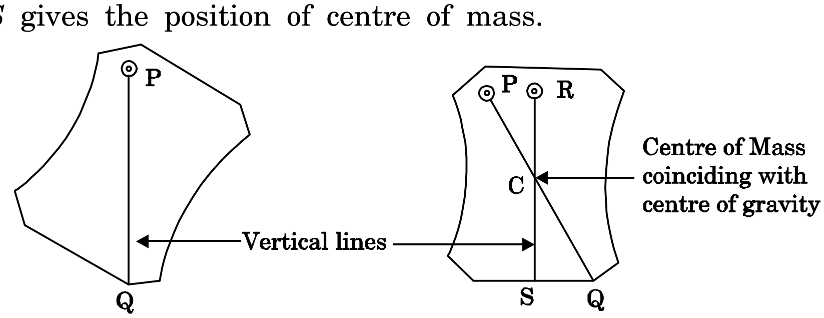


Fig. 1.12 Body hung from different points *P* and *R*

Centre of mass of a solid cone

Fig. 1.13 shows a right circular solid cone of base radius a and height h. Let ρ be the density of the material of the cone.

If the solid cone is homogeneous, then its mass $m = \frac{1}{3} \pi a^2 h \cdot \rho$.

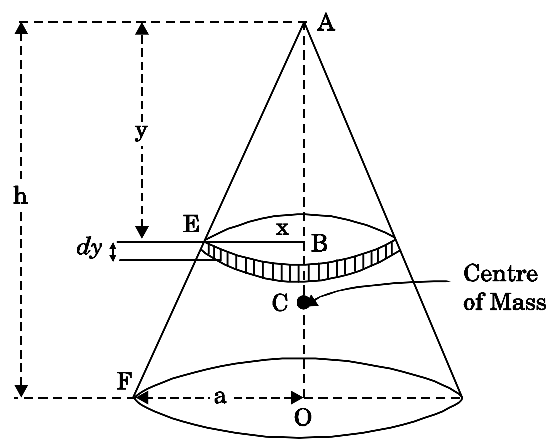


Fig. 1.13 Right circular solid cone

The centre of mass lies on the axis of symmetry AO. The cone is considered to be made up of large number of circular discs, each of thickness dy.

Let us consider one such elementary disc of radius x at a distance y from the vertex A of the solid cone. The mass of this elementary disc is

$$dm = \rho (\pi x^2) dy \quad \text{... (1)}$$

From the figure 1.13,

$$\frac{x}{a} = \frac{y}{h} \quad \text{[In similar triangles } AEB \text{ and } AFO \text{]}$$

$$x = \frac{a}{h} y$$

$$\therefore\ dm = \rho \cdot \pi \left(\frac{a}{h} y \right)^2 \cdot dy \qquad \text{... (2)}$$

Now from equation (1), we have for the distance of centre of mass on the axis of symmetry AO as measured from the vertex A as

$$Y_{CM} = \frac{1}{M} \int y\, dm \qquad \text{... (3)}$$

On substituting the value of dm, we get

$$Y_{CM} = \frac{1}{M} \int_0^h y\, \rho\, \pi \left(\frac{a}{h} y \right)^2 dy$$

or
$$Y_{CM} = \frac{\rho\, \pi\, a^2}{M\, h^2} \int_0^h y^3\, dy \qquad \text{... (4)}$$

where the limits of y is taken from $y = 0$ to $y = h$ to cover the entire solid cone filled with such elementary discs. It gives

$$Y_{CM} = \frac{\rho\, \pi\, a^2}{Mh^2} \left[\frac{y^4}{4} \right]_0^h$$

$$= \frac{\rho\, \pi\, a^2}{Mh^2} \frac{h^4}{4} = \frac{\rho\, \pi\, a^2\, h^2}{4M} \qquad \text{... (5)}$$

But M = Total mass of the solid cone $= \frac{1}{3} \pi\, a^2\, h\rho$

hence
$$Y_{CM} = \frac{\rho\, \pi\, a^2\, h^2 \cdot 3}{4 \cdot \pi\, a^2\, h\rho}$$

The CM of cone from its vertex Y_{CM} is written as R_{CM}

$$\boldsymbol{R_{CM} = \frac{3}{4}h}$$

Thus, CM of a solid cone is at a distance of $\frac{3}{4}h$ from vertex of the cone along its axis.

Centre of mass of a triangular lamina

The medians of the triangle are axes of symmetry in the base of triangular sheets. We simply draw any two medians of the triangle which intersect at a point. This point is the centre of mass of the triangular body (Fig. 1.14).

We know that the medians bisect each other in the ratio of 2 : 1 the position of centre of mass on any medians is obtained by dividing that median in the ratio 2 : 1 the larger portion being towards the vertex. That point is the centre of mass.

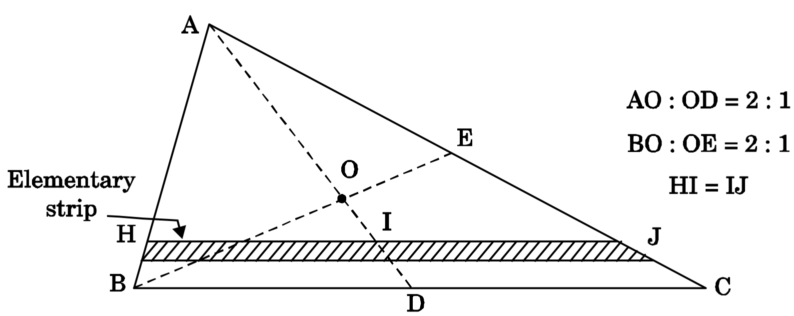


Fig. 1.14 The point of intersection of the medians of a triangle gives the position of centre of mass

Howevever, the position of centre of mass can also be calculated by assuming the triangle to be made up of large number of strips parallel to one side of the triangle and placed one above the other as shown in the fig. 1.14.

Centre of mass of some regular objects

Fig 1.15. shows the centre of masses of some regular shaped homogeneous rigid bodies.

- For a rigid body, the centre of mass is a point at a fixed position with respect to the body as a whole. Depending on the shape of the body and the way the mass is distributed in it, the centre of mass is a point may or may not be within the body.
- If the shape is symmetrical and the mass distribution is uniform, we can usually find the location of the centre of mass quite easily.
- For a long thin rod of uniform cross section and density, the centre of mass is at the geometrical centre.
- For a thin circular plane ring, It is again at the geometrical centre of the circle.
- For a flat circular disc or rectangle, again the centre of mass is at the geometrical centre.

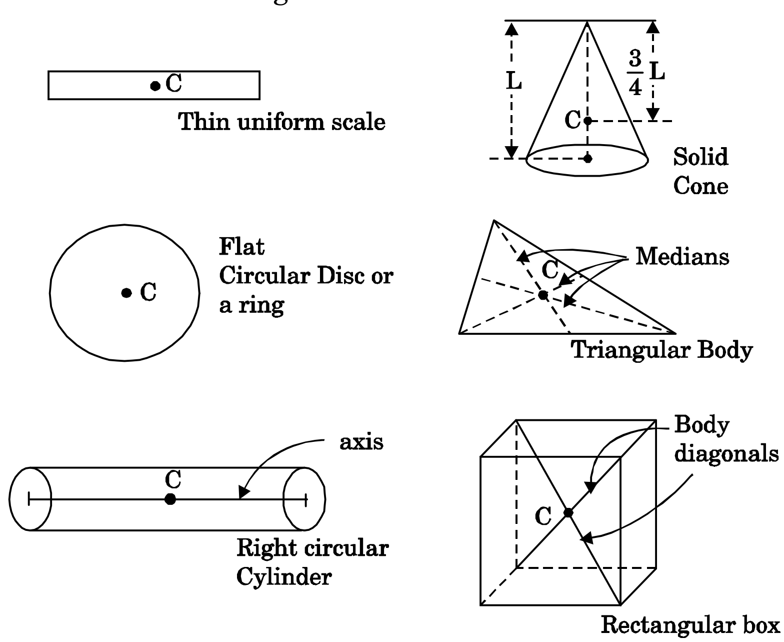


Fig. 1.15 The dot shows of the centre of mass of the rigid body

S.No.	Shape of the body	Position of centre of mass
1.	uniform rod	Middle point of rod
2.	Circular disc	Centre of the disc
3.	Circular ring	Centre of the ring
4.	Sphere	Centre of sphere
5.	Hollow sphere	Centre of sphere
6.	Cylinder	Middle point of the axis
7.	Cubical Block	Point of intersection of diagonals joining opposites corners
8.	Plane lamina	Point of intersection of two diagonals
9.	Cone of pyramid	On line joining the apex to the centre of the base of the cone at a distance $1/4^{th}$ of the length of this line.
10.	Triangular plane lamina	Point of intersection of medians of triangle.

Generally, the centre of mass coincides with the centre of gravity of regularly shaped bodies. In case of rigid body, the internal forces are taken as zero and only the external forces acting on the body are considered.

Note

Difference between centre of gravity and centre of mass

- The centre of gravity of a body is a point, where the whole weight of the body supposed to be concentrated.
- The centre of mass of a body is that point, where the whole mass of the body is supposed to be concentrated.

For uniform geometrically shaped, bodies the centre of gravity coincides with centre of mass. However, they do not coincide in bodies whose density is not uniform throughout.

1.5 KINETIC ENERGY OF THE SYSTEM OF PARTICLES

Let there are n number of particles in a system of particles and these particles possess some motion. The motion of the i^{th} particle of this system depends on the external force $\vec{F}_i$ acting on it.

Let at any time if the velocity of i^{th} particle be $\vec{v}_i$ then its kinetic energy would be

$$E_{Ki} = \frac{1}{2} m v_i^2$$

$$E_{Ki} = \frac{1}{2} m (v_i \cdot v_i) \qquad \text{... (1)}$$

Let $\vec{r}_i$ be the position vector of the i^{th} particle w.r.t. O and $\vec{r'}_i$ be the position vector of the centre of mass w.r.t. $\vec{r}_i$, as shown in the figure 1.16, then

$$\boxed{\vec{r}_i = \vec{r'}_i + \vec{R}_{CM}} \qquad \text{... (2)}$$

where $\vec{R}_{CM}$ is the position vector of centre of mass of the system w.r.t. O.

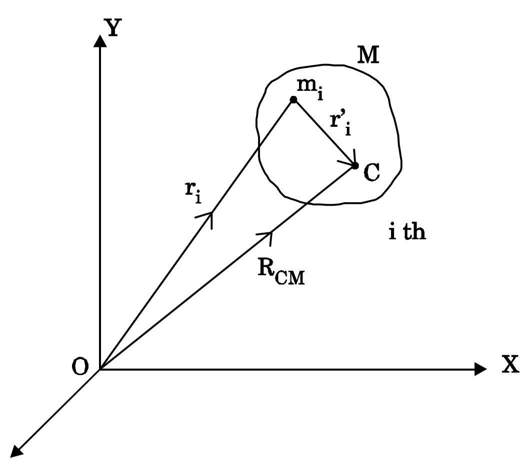


Fig. 1.16

Differentiating the equation 2 we get

$$\frac{d\vec{r}_i}{dt} = \frac{d\vec{r'}_i}{dt} + \frac{d\vec{R}_{CM}}{dt}$$

or,

$$\boxed{v_i = v'_i + v_{CM}} \qquad \text{... (3)}$$

where v_i is the velocity of i^{th} particle and v_{CM} is the velocity of centre of mass of system of particle.

Putting equation 3 in 1 we get,

$$E_{Ki} = \frac{1}{2} m_i [(v'_i + v_{CM}) \cdot (v'_i + v_{CM})] = \frac{1}{2} m_i [(v_i'^2 + 2v'_i \cdot v_{CM} + v_{CM}^2)]$$

$$E_{Ki} = \frac{1}{2} m_i v_i'^2 + m_i v'_i \cdot v_{CM} + \frac{1}{2} m_i v_{CM}^2 \qquad \text{... (4)}$$

The sum of kinetic energy of all the particles can be obtained from equation 4

$$E_K = \sum_{i=1}^{n} E_{Ki} = \sum_{i=1}^{n} \left[\frac{1}{2} m_i v_i'^2 + m_i v'_i \cdot v_{CM} + \frac{1}{2} m_i v_{CM}^2 \right]$$

$$E_K = \sum_{i=1}^{n} \frac{1}{2} m_i v_i'^2 + \sum_{i=1}^{n} m_i v'_i \cdot v_{CM} + \sum_{i=1}^{n} \frac{1}{2} m_i v_{CM}^2$$

$$E_K = \frac{1}{2} v_{CM}^2 \sum_{i=1}^{n} m_i + \sum_{i=1}^{n} m_i v_i'^2 + v_{CM} \sum_{i=1}^{n} m_i v'_i$$

$$E_K = \frac{1}{2} v_{CM}^2 M + \sum_{i=1}^{n} \frac{1}{2} m_i v_i'^2 + v_{CM} \frac{d}{dt} \sum_{i=1}^{n} m_i \vec{r'}_i \qquad \text{... (5)}$$

$$\left(\because v'_i = \frac{dr'_i}{dt} \right)$$

Now last term in equation (5) is equal to zero

$$\text{i.e.,} \quad \sum_{i=1}^{n} m_i \vec{r'}_i = 0$$

$$\because \quad \sum_{i=1}^{n} m_i \vec{r'}_i = \sum_{i=1}^{n} m_i (\vec{r}_i - \vec{R}_{CM}) \qquad (\because r'_i = r_i - R_{CM})$$

$$= \sum_{i=1}^{n} m_i \vec{r}_1 - \sum_{i=1}^{n} m_i \vec{R}_{CM}$$

$$= MR_{CM} - MR_{CM} = 0 \qquad (\because \Sigma\, m_i \vec{r}_i = M \vec{R}_{CM})$$

Therefore, kinetic energy of the system of particles is,

$$E_K = \frac{1}{2} M\vec{v}^2_{CM} + \frac{1}{2} \sum_{i=1}^{n} m_i \vec{v'}^2_i = E_{K_{CM}} + E'_K \qquad \text{... (6)}$$

where,

$$\boxed{E_{K_{CM}} = \frac{1}{2} \vec{v}^2_{CM} M}$$

is the kinetic energy obtained as if all the mass were concentrated at the centre of mass

$$\boxed{E'_k = \sum_{i=1}^{n} \frac{1}{2} m_i \vec{v'}^2_i} \qquad \text{... (7)}$$

is the kinetic energy of the system of particle w.r.t. the centre of mass.

Hence it is clear from equation (6) that kinetic energy of the system of particles consists of two parts: the kinetic energy obtained as if all the mass were concentrated at the centre of mass plus the kinetic energy of motion of all particles about the centre of mass.

If there were no external force acting on the particle system then the velocity of the centre of mass of the system will remain constant and kinetic energy of the system would also remain constant.

1.6 ROTATION OF RIGID BODIES

Rigid body

A rigid body is defined as that body which does not undergo any change in shape or volume when external forces are applied on it.

- When forces are applied on a rigid body, the distance between any two particles of the body will remain unchanged however, large the forces may be.
- Actually, no body is perfectly rigid. Every body can be deformed more or less by the application of the external force. The solids, in which the changes produced by external forces are negligibly small, are usually considered as rigid body.

Rotational motion

When a body rotates about a fixed axis, its motion is known as rotatory motion.

A rigid body is said to have pure rotational motion, if every particle of the body moves in a circle, the centre of which lies on a straight line called the axis of rotation (Fig 1.17).

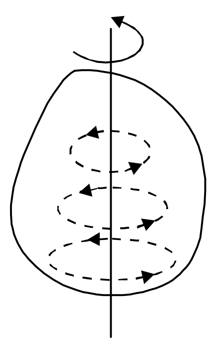


Fig. 1.17 Rotational motion

- The axis of rotation may lie inside the body or even outside the body. The particles lying on the axis of rotation remains stationary.

The position of particles moving in a circular path is conveniently described in terms of a radius vector r and its angular displacement θ.

Let this consider a rigid body that rotates about a fixed axis XOX' passing through O and perpendicular to the plane of the paper as shown in fig. 1.18.

Let the body rotate from the position A to the position B. The different particles at $P_1, P_2, P_3, \ldots$ in the rigid body covers unequal distances $P_1 P_1', P_2 P_2', P_3 P_3' \ldots$ in the same interval of time.

Thus their linear velocities are different. But in the same time interval, they all rotate through the same angle θ and hence the angular velocity is the same for the all the particles of the rigid body.

Thus, *in the case of rotational motion, different constituent particles have different linear velocities but all of them have the same angular velocity*.

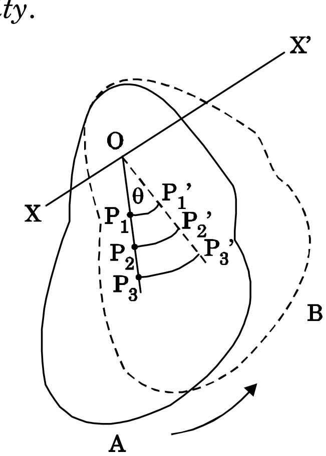


Fig. 1.18 Rotational motion of a rigid body

Equation of rotational motion

As in linear motion, for a body having uniform angular acceleration, we shall derive the equations of motion.

Let us consider a particle start rotating with angular velocity ω_0 and angular acceleration α. At any instant t, let ω be the angular velocity of the particle and θ be the angular displacement produced by the particle.

Therefore change in angular velocity in time t, $= \omega - \omega_0$

But, angular acceleration $= \dfrac{\text{change in angular velocity}}{\text{time taken}}$

(i.e) $$\alpha = \frac{\omega - \omega_0}{t} \quad \text{... (1)}$$

$$\alpha t = \omega - \omega_0$$

or $$\boxed{\omega = \omega_0 + \alpha t} \quad \text{... (2)}$$

Average angular velocity $= \left(\dfrac{\omega + \omega_0}{2}\right)$

Total angular displacement

$= \text{average angular velocity} \times \text{time taken}$

(i.e) $$\theta = \left(\frac{\omega + \omega_0}{2}\right)t \quad \text{... (3)}$$

Substituting ω from the equation (2),

$$\theta = \left(\frac{\omega_0 + \alpha t + \omega_0}{2}\right)t$$

$$= \left(\frac{2\omega_0 + \alpha t}{2}\right)t$$

$$= \left(\frac{2\omega_o}{2} + \frac{\alpha t}{2} \right) t$$

$$\boxed{\theta = \omega_0 t + \frac{1}{2} \alpha t^2} \qquad \text{... (4)}$$

From equation (1), $t = \left(\frac{\omega - \omega_0}{\alpha} \right)$ ... (5)

Using equation (5) in (3),

$$\theta = \left(\frac{\omega + \omega_0}{2} \right) \left(\frac{w - \omega_0}{\alpha} \right) = \frac{(\omega^2 - \omega_0^2)}{2\alpha}$$

$$2 \alpha \theta = \omega^2 - \omega_0^2$$

$$\boxed{\omega^2 = \omega_0^2 + 2\alpha \theta} \qquad \text{... (6)}$$

The equations (2), (4) and (6) are the equations of rotational motion.

1.7 ROTATIONAL KINEMATICS

Rotational Kinetic Energy and Moment of inertia

Consider a rigid body rotating about a fixed axis *XOX′*. The rigid body consists of a large number of particles. Let $m_1, m_2, m_3, \ldots$ etc. be the masses of the particles situated at distances. $r_1, r_2, r_3, \ldots$ etc. from the fixed axis. (Fig. 1.19)

All the particles rotate with the same angular velocity ω. But the linear velocities of the particles are different.

Kinetic Energy of first particle $= \frac{1}{2} m_1 v_1^2$

$$= \frac{1}{2} m (r_1 \omega)^2 \qquad \{ \because v_1 = r_1 \omega \}$$

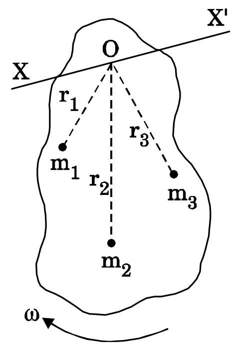


Fig. 1.19

$\therefore$ Kinetic energy of the first particle $= \frac{1}{2} m_1 r_1^2 \omega^2$

Similarly

Kinetic energy of the second particle $= \frac{1}{2} m_2 r_2^2 \omega^2$

Kinetic energy of the third particle $= \frac{1}{2} m_3 r_3^2 \omega^2$

and so on.

Kinetic energy of the whole body is equal to the sum of the kinetic energy of all particles of the body

$\therefore$ Kinetic energy of the rigid body

$$E_K = \frac{1}{2} m_1 r_1^2 \omega^2 + \frac{1}{2} m_2 r_2^2 \omega^2 + \frac{1}{2} m_3 r_3^2 \omega^2 + \ldots\ldots$$

$$= \frac{1}{2} \omega^2 (m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \ldots)$$

$$E_K = \frac{1}{2} \omega^2 \Sigma mr^2$$

The term Σmr^2 is called moment of inertia of a body about the given axis of rotation and denoted by I.

i.e., $$I = \Sigma m r^2$$

$\therefore$ The kinetic energy of the rigid body $E_k = \frac{1}{2}\omega^2 I$

i.e., $$\boxed{E_K = \frac{1}{2} I \omega^2}$$

Moment of Inertia or Rotational inertia

Inertia of a body is its inability to change by itself its state of rest or of uniform motion in a straight line.

Similarly **Moment of inertia of a body is its inability to change by itself its state of rest or of rotatory motion about an axis**.

An external force is necessary to change its state.

- **There is a tendancy to resist changes in uniform rotational motion**. For example if a fan is switched off it continues to rotate for some more before it comes to rest.

- **The property of a body by which it resists change uniform rotational motion is called rotational inertia or moment of inertia**.

The moment of inertia of a body depends on the mass of the body and also the distribution of the mass with respective the axis of rotation.

Moment of inertia of a particle

The moment of inertia of a particle about an axis is defined as the product of the mass of the particle and square of the distance of the particle from the axis of rotation.

If '*m*' is the mass of the particle and '*r*' is the distance of the particle from the axis of rotation, (Fig 1.20) then

The moment of inertia of the particle

$$\boxed{I = mr^2}$$

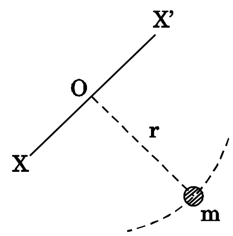


Fig. 1.20

The unit of moment of inertia is **kg m²**

Moment of inertia of a rigid body

Consider a rigid body of mass M, rotating about an axis XX'. The body is supposed to be made of a large number of particles.

If m_1, m_2 are the masses of the particle and $r_1, r_2, r_3 \ldots$ their respective distances from the axis of rotation, then by definition, the moments of inertia of the particle about the axis of rotation are $m_1 r_1^2, m_2 r_2^2, m_3 r_3^2 \ldots$ respectively.

Then moment of inertia of the whole body is the sum of the moment of inertia of all the particles.

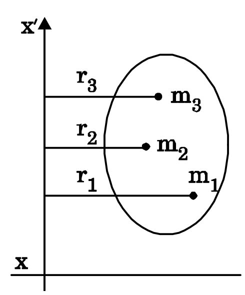


Fig. 1.21

i.e. $$I = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \ldots$$

$$I = \Sigma m r^2$$

The moment of inertia of a rigid body about a given axis is the sum of products of masses of its particles and the square of their respective distances from the axis of rotation.

The unit of moment of inertia is $\mathbf{kgm^2}$.

When angular velocity $\omega = \mathbf{1}$ radian / sec

Rotational kinetic energy $= E_R = \frac{1}{2} I \times \omega^2 = \frac{1}{2} I \times 1^2 = \frac{1}{2} I$ ... (2)

Therefore, $\boxed{I = 2E_R}$

It shows that moment of inertia of a body is equal to twice the kinetic energy of a rotating body when angular velocity is one radian per second.

Note

1. Moment of inertia depends on mass, distribution of mass and on the position of axis of rotation.
2. Moment of inertia does not depend on angular velocity, angular acceleration, torque, angular momentum and rotational kinetic energy.
3. Moment of inertia of a continuous distribution of mass, treating the element of mass dm at position r as particle (Fig 1.22)

$$dI = dmr^2 \text{ i.e., } I = \int r^2\, dm$$

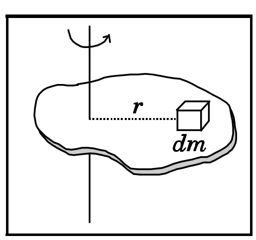


Fig. 1.22

4. It is not a vector as direction (clockwise or anti-clockwise) is not to be specified and also not a scalar as it has different values in different directions.
5. In case of a hollow and solid body of same mass, radius and shape for a given axis, moment of inertia of hollow body is greater than that for the solid body because it depends upon the mass distribution.

Physical significance of Moment of Inertia

The property which opposes the change in rotational motion of the body is called the moment of inertia. Greater is moment of inertia of the body about the axis of rotation, greater is the torque required to rotate the body.

Thus it is clear that *the moment of inertia of a body has the same role in rotational motion as that of mass (or inertia) is linear motion*.

It is clear that the moment of inertia of a body depends on

(i) ***mass of body*** and

(ii) ***distribution of mass about the axis of rotation.***

The two bodies of same mass may have different moment of inertia

Practical utility of Moment of Inertia

1. Fly-wheel

- A fly-wheel is such a heavy wheel whose most of the mass is concentrated at the rim, so that its moment of inertia is quite large. This wheel is attached to the shaft of the engine in which the torque rotating the shaft changes periodically.

2. Wheels of vehicles

In cycle, riksha, car, motor car, scooter etc. the moment of inertia of wheels is increased by concentrating most of the mass at the rim of the wheel and connecting the rim to the axle of the wheel through the spokes.

Due to large moment of inertia of wheels, they cause greater opposition to the change in their state of rotational motion.

Radius of Gyration

If the whole mass of the rigid body 'M' is assumed to be concentrated at a distance of 'K' from the axis of rotation then

$$\boxed{I = M \cdot K^2}$$

Here $M = \Sigma m$

K is known as radius of gyration.

Definition

The radius of gyration is defined as the distance from the axis of rotation to the point where the entire mass of the body is assumed to be concentrated.

If the body consists of n particles of equal mass m, then the moment of inertia.

$$I = \Sigma mr^2$$

$$= mr_1^2 + m_2^2 + mr_3^2 + \ldots + mr_3^2 + \ldots + mr_n^2$$

put $m_1 = m_2 = m_3 = \ldots = m_n = m$ and multiplying and dividing by n, we get

$$= nm\left[\frac{r_1^2 + r_2^2 + r_3^2 + \ldots + r_n^2}{n}\right] = MK^2 \qquad \ldots (4)$$

where $M = nm$ is the mass of the body and

$$K = \sqrt{\frac{r_1^2 + r_2^2 + r_3^2 + \ldots}{n}} \qquad \ldots (5)$$

***K* is called the Radius of Gyration of the body about the axis of rotation. It is equal to the root mean square distance of all particles from the axis of rotation of the body.**

Hence the radius of gyration of a body about a given axis is equal to root mean square distance of the constituent particles of the body from the given axis.

1. Radius of gyration (K) depends on shape and size of the body, position and configuration of the axis of rotation, distribution of mass of the body w.r.t. the axis of rotation.
2. S.I. unit : Metre.
3. **Significance of radius of gyration:** Through this concept a real body (particularly irregular) is replaced by a point mass for dealing its rotational motion.

Example: In case of disc rotating about an axis through its centre of mass and perpendicular to its plane

$$MK^2 = \frac{MR^2}{2}$$

$$\therefore \; K^2 = \frac{R^2}{2}$$

$$K = \frac{R}{\sqrt{2}}$$

So instead of disc we can assume a point mass M at a distance $\frac{R}{\sqrt{2}}$ from the axis of rotation for dealing the rotational motion of the disc.

1.8 THEOREMS ON MOMENT OF INERTIA (MI)

There are two important theorems which help to find the moment of inertia of a body about some other axis if moment of inertia about any symmetrical axis of the body is given. These are called theorems of parallel and perpendicular axes.

They are

1. **Parallel axes theorem** and
2. **Perpendicular axes theorem**

1. Theorem (Principle) of Parallel axes

Statement

The moment of inertia of a body about any axis is equal to the sum of its moment of inertia about a parallel axis passing through its centre of gravity of the body and the product of its mass of the body with the square of the distance between the two axes.

Explanation

Let G be the centre of gravity of a rigid body of mass M. Let AB be an axis parallel to axis CD. (Fig 1.23)

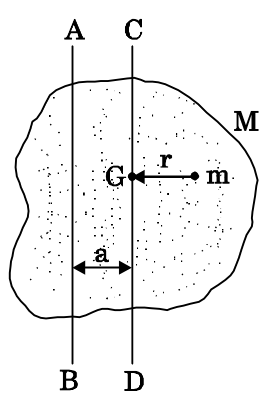


Fig. 1.23

Let a be the distance between the axes.

If I and I_G are the moments of inertia of the body about the axes AB and CD respectively, then by the theorem of parallel axes,

$$\boxed{I = I_G + Ma^2.}$$

Proof

Consider a particle of mass m at a distance r from CD.

M.I. of this particle about the axis $CD = mr^2$

$\therefore$ M.I. of the whole body about CD, $I_G = \Sigma\, mr^2$

M.I. of the particle about the axis $AB = m\,(r + a)^2$

$\therefore$ M.I. of the whole body about $AB, I = \Sigma\, m\,(r + a)^2$

$$I = \Sigma\, m\,(r^2 + a^2 + 2ar)$$

$$= \Sigma\, mr^2 + a^2\,\Sigma\, m + 2a\,\Sigma\, mr$$

ie, $$I = I_G + Ma^2 + 2a\,\Sigma\, mr$$

$\Sigma\, mr$ represents the algebraic sum of the moments of all the mass particles of the body about an axis through the centre of gravity of the body. Since the body always balances about an axis through its centre of gravity. $\Sigma\, mr$ should be zero.

Therefore, $\boxed{I = I_G + Ma^2}$

2. Theorem (Principle) of Perpendicular axes

Statement

It states that the moment of inertia of a plane lamina about an axis perpendicular to its plane is equal to the sum of the moments of inertia of the plane lamina about any two mutually perpendicular axes in its own plane and intersecting each other at the point where the perpendicular axis passes through it.

Explanation

Let OX and OY be two mutually perpendicular axes in the plane of the lamina, intersecting each other at the point O. OZ is the axes perpendicular to both OX and OY. (Fig 1.24)

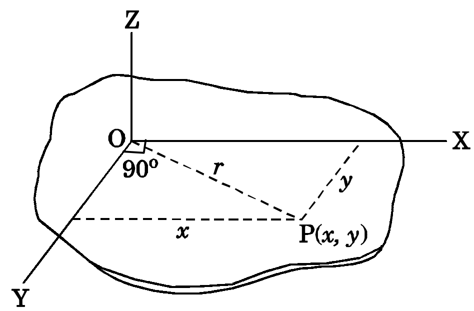


Fig. 1.24

Let I_x and I_y be the moments of inertia of the lamina about the axis OX and OY respectively.

If I_z is the moment of inertia about the axis OZ, passing through O and perpendicular to the plane of the lamina, then by the theorem of perpendicular axes,

$$\boxed{I_z = I_x + I_y}$$

Proof

Consider a particle P of the lamina of mass m at a distance r from O. Let x and y be the distances of the particle from OY and OX respectively.

M.I. of this particle about $OX = my^2$

M.I. of the entire lamina about $OX, I_x = \Sigma\, my^2$

Similarly M.I. of the lamina about $OY, I_y = \Sigma\, mx^2$

M.I. of the lamina about OZ axis through O and perpendicular to the plane of the lamina, $I_z = \Sigma\, mr^2$ ... (1)

But

$$r^2 = x^2 + y^2 \quad \text{... (2)}$$

Substituting eqn (2) in eqn (1), we have

$$I_z = \Sigma m (x^2 + y^2)$$

$$I_z = \Sigma mx^2 + \Sigma my^2$$

$$I_z = I_y + I_x$$

ie, $\boxed{I_z = I_x + I_y}$

Calculation of Moment of Inertia of a body

The moment of inertia of a continuous homogeneous body (Rigid body) with definite geometrical shape can be calculated as follows.

- Find the moment of inertia of an infinitesimal element of the body about the given axis.

i.e., Multiply the mass dm of the element by x^2, the square of the distance from the given axis.

- Then, integrate the expression between the limits to get moment of inertia of whole of the body.

$$I = \int dm \cdot x^2$$

where the integral is taken over the whole body.

In fact some times the theorems parallel and perpendicular axes are also used to calculate the moment of inertia.

1.9 MOMENT OF INERTIA OF CONTINUOUS BODIES (RIGID BODIES)

1. Moment of inertia of a thin uniform rod

(a) About an axis through its centre and perpendicular to its length

Let *AB* be a thin uniform rod of length l and mass M. The rod is free to rotate about an axis *PQ* perpendicular to its length and passing through its centre *O*. (Fig. 1.25)

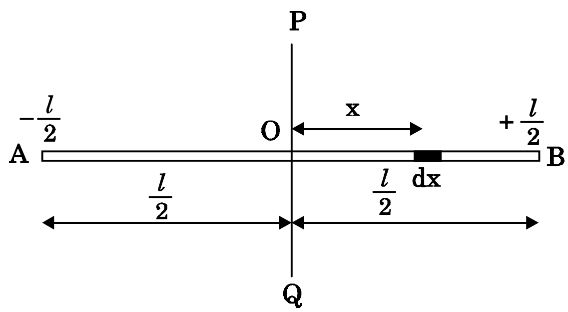


Fig. 1.25 Moment of Inertia of a thin uniform rod - about an axis through its centre and perpendicular to its length

Mass per unit length (linear density) of the rod,

$$m = \frac{M}{l} \quad \text{... (1)}$$

Consider a small element of length dx of the rod at a distance x, from O.

Mass of the element $= m \cdot dx$

M.I. of this element about the axis *PQ*

$$= \text{mass} \times (\text{distance})^2.$$

$$= mdx \cdot x^2$$

$$= mx^2\, dx \quad \text{... (2)}$$

The rod *AB* is considered to be made up of a number of such small elements of length dx.

Hence the moment of inertia I of the rod about the axis PQ is obtained by integrating the moments of inertia of all such elements lying between the limits

$$x = -\frac{l}{2} \text{ and } x = \frac{l}{2}$$

ie,
$$I = \int_{-l/2}^{l/2} mx^2\, dx \qquad \ldots (3)$$

$$= m\left[\frac{x^3}{3}\right]_{-l/2}^{l/2}$$

$$m\left[\frac{(l/2)^3 - (-l/2)^3}{3}\right] = m\frac{\left[\frac{l^3}{8} + \frac{l^3}{8}\right]}{3}$$

$$= \frac{m}{3}\left[\frac{2\,l^3}{8}\right] = \frac{ml^3}{12}$$

$$I = ml \cdot \frac{l^2}{12} \qquad (\because\ ml = M)$$

$$\boxed{\boldsymbol{I = \frac{M\,l^2}{12}}} \qquad \ldots (4)$$

(b) About an axis passing through one end of the rod and perpendicular to its length

Let AB be a thin uniform rod of length l and mass M. O is its centre. As the rod is uniform, its centre and centre of gravity coincide. PQ is an axis passing through O and perpendicular to the length of the rod. (Fig. 1.26)

$$\text{M.I. of the rod about } PQ = \frac{M\,l^2}{12} \qquad \ldots (1)$$

Let RS be an axis passing through one end A and perpendicular to the length of the rod.

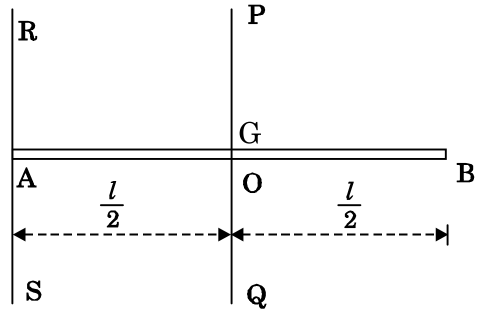


Fig. 1.26 Moment of Intertia of a thin uniform rod - about an axis passing through one end of the rod and perpendicular to its length

Let I be the moment of inertia of the rod about this axis RS.

By the theorem of parallel axes,

M.I. of the rod about the axis RS = M.I. of the rod about the axis PQ passing through the centre of gravity + Mass × square of the distance of the axis from the C.G.

Therefore,

$$I = \frac{M l^2}{12} + M \cdot \left(\frac{l}{2}\right)^2$$

$$I = \frac{M l^2}{12} + \frac{M l^2}{4}$$

$$I = \frac{M l^2 + 3 l^2}{12} = \frac{4 M l^2}{12}$$

ie, $$\boxed{I = \frac{M l^2}{3}}$$

2. Moment of inertia of thin ring (or hoop)

(a) About an axis through its centre and perpendicular to its plane

Let M be the mass and R, the radius of a thin circular ring with centre O. Let I be the moment of inertia of the ring about an axis passing through O and perpendicular to the plane of the ring.

Consider a particle of mass m of the ring. Its moment of inertia about the axis passing through O and perpendicular to the plane of the ring is

$$I = mR^2 \quad \text{... (1)}$$

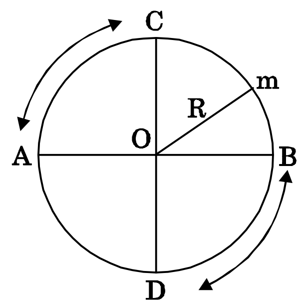


Fig. 1.27 Moment of Inertia of thin ring - about an axis through its centre and perpendicular to its plane

$\therefore$ M.I. of the ring about the axis, $I = \Sigma\, mR^2$. ... (2)

ie, $\boxed{I = MR^2}$ $(\because\ \Sigma m = M)$

(b) About a diameter

AB and CD represent two mutually perpendicular diameters of the ring. The moment of inertia of the ring about any diameter is the same. (Fig, 1.27)

Let I be the moment of inertia about AB. Then about CD also, the moment of inertia is I.

By the theorem of perpendicular axes, the moment of inertia of the ring about an axis passing through O and perpendicular to its plane is equal to the sum of the moments of inertia about two mutually perpendicular axes in the plane of the ring.

i.e., $I_z = I_x + I_y$ ... (3)

But, $I_z = MR^2$ and $I_x = I_y = I$

$$\therefore \; MR^2 = I + I$$

or, $$2I = MR^2$$

ie, $$\boxed{I = \frac{MR^2}{2}} \quad \text{... (4)}$$

(c) About a tangent in the plane of the ring

Let EF be a tangent to the ring at A, in the plane of the ring and also it is parallel to the diameter CD.

Let I be the moment of inertia of the ring about the tangent EF.

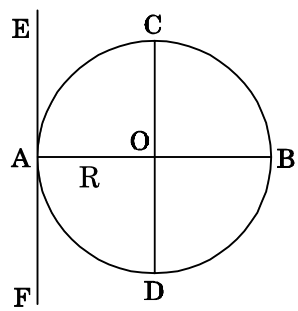


Fig. 1.28 Moment of Inertia of thin ring - about a tangent in the plane of the ring

By parallel axis theorem, M.I. about EF

= M.I. about CD. + Mass × square of the distance OA

$$\text{ie, } I = \frac{MR^2}{2} + MR^2$$

$$= \frac{MR^2 + 2\,MR^2}{2}$$

$$\boxed{I = \frac{3}{2} MR^2} \quad \text{... (5)}$$

3. Moment of inertia of a thin circular disc

(a) About an axis through its centre and perpendicular to its plane

Let M be the mass and R, the radius of a thin circular disc. The disc is free to rotate about an axis AB passing through its centre O, and perpendicular to its plane.

$$\text{Mass per unit area of the disc} = \frac{M}{\text{Area of the disc}} = \frac{M}{\pi R^2} \quad \text{... (1)}$$

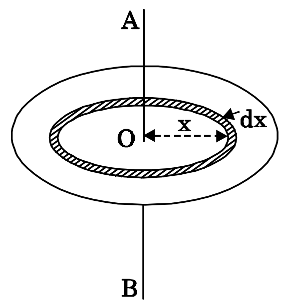


Fig. 1.29 Moment of Inertia of a thin circular disc - about an axis through its centre

Consider a narrow circular strip of the disc with centre O and radius x. Let dx be its radial width.

Area of the strip = $2\pi x\, dx$.

$$\text{Mass of the strip} = \frac{M}{\pi R^2}\, 2\pi x\, dx$$

$$= \frac{2M}{R^2} x\, dx \quad \text{... (2)}$$

Moment of inertia of this circular strip about the axis AB

$$= \frac{2M}{R^2} x dx\, x^2$$

$$= \frac{2M}{R^2} x^3\, dx \quad \text{... (3)}$$

The circular disc is considered to be made up of a large number of such narrow strips, concentric with O and radii ranging from 0 to R.

Hence, the moment of inertia I of the disc about the axis AB is obtained by integrating the above expression between limits $x = 0$ and $x = R$.

$$\text{ie. } I = \int_0^R \frac{2M}{R^2} x^3 \, dx \qquad \text{... (4)}$$

$$= \frac{2M}{R^2} \int_0^R x^3 \, dx$$

$$= \frac{2M}{R^2} \left[\frac{x^4}{4} \right]_0^R$$

$$= \frac{2M}{R^2} \frac{R^4}{4}$$

ie,

$$\boxed{I = \frac{MR^2}{2}} \qquad \text{... (5)}$$

(b) About a diameter

Let AB and CD be two perpendicular diameters of the disc. The moment of inertia of the disc about any diameter is the same as that about any other diameter. Hence the moments of inertia about AB and CD are equal.

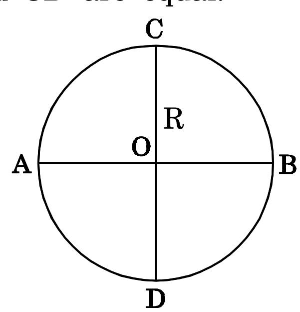


Fig. 1.30 Moment of Inertia of a thin circular disc - about a diameter

Let each of them be I.

The moment of inertia of the disc about an axis, through the centre and perpendicular to the plane of the disc is

$$= \frac{MR^2}{2} \qquad \ldots (6)$$

Therefore, by the theorem of perpendicular axes,

$$I_z = I_x + I_y \qquad \ldots (7)$$

$$\text{But, } I_z = \frac{MR^2}{2} \text{ and}$$

$$I_x = I_y = I$$

$$\frac{MR^2}{2} = I + I$$

$$\frac{MR^2}{2} = 2I$$

$$I = \frac{MR^2}{2 \times 2}$$

or,

$$\boxed{I = \frac{MR^2}{4}} \qquad \ldots (7)$$

4. Moment of Inertia of Solid sphere

(a) About a diameter

Let M be the mass of a uniform solid sphere of radius R and centre O. The sphere is free to rotate about its diameter AB.

Mass per unit volume (density) of the sphere

$$\frac{\text{Mass}}{\text{Volume}} = \frac{M}{\frac{4}{3}\pi R^3}$$

$$= \frac{3M}{4\pi R^3} \qquad \ldots (1)$$

Consider a thin circular slice (elemental volume) of the sphere at a distance x from O. Let dx be the thickness of the slice. This slice is a thin uniform disc. The radius of this disc is given by $y^2 = (R^2 - x^2)$ $\quad (\because R^2 = x^2 + y^2)$

Area of this disc $= \pi y^2 = \pi (R^2 - x^2)$ $\quad (\therefore y^2 = R^2 - x^2)$

Volume of the disc = Area × thickness $= \pi (R^2 - x^2)\, dx$ ... (2)

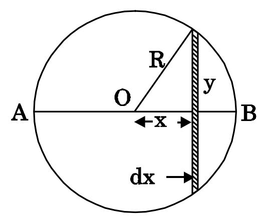


Fig. 1.31 Moment of Inertia of a solid sphere - about a diameter

Mass of the elemental disc $= \dfrac{3M}{4\pi R} \times \pi (R^2 - x^2)\, dx,$

$$= \frac{3M}{4R^3} (R^2 - x^2)\, dx$$

M.I. of this disc about the axis AB

$$= \frac{\text{Mass} \times (\text{Radius})^2}{2},$$

$$= \frac{3M}{4R^3} (R^2 - x^2)\, dx \times \frac{y^2}{2}$$

$$= \frac{3M}{4R^3} (R^2 - x^2)\, dx \times \frac{R^2 - x^2}{2}$$

$$= \frac{3M}{8R^3} (R^2 - x^2)^2\, dx \qquad \text{... (4)}$$

$(\because y^2 = R^2 - x^2)$

The sphere is considered to be made up of large number of such discs, with x varying from $-R$ to $+R$.

Hence, the moment of inertia I of the sphere is obtained by integrating the above expression between limits $x = -R$ and $x = +R$.

ie, Moment of inertia of the solid sphere about a diameter.

$$I = \int_{-R}^{+R} \frac{3M}{8R^3} (R^2 - x^2)^2 \, dx$$

$$= 2 \int_{0}^{R} \frac{3M}{8R^3} (R^2 - x^2)^2 \, dx$$

ie $$I = \frac{3M}{4R^3} \int_{0}^{R} (R^4 - 2R^2 x^2 + x^4) \, dx \quad \text{... (5)}$$

$$= \frac{3M}{4R^3} \left[R^4 x - 2R^2 \frac{x^3}{3} + \frac{x^5}{5} \right]_0^R$$

$$= \frac{3M}{4R^2} \left[R^5 - \frac{2R^5}{3} + \frac{R^5}{5} \right]$$

$$= \frac{3M}{4R^2} \left[\frac{15R^5 - 10R^5 + 3R^5}{15} \right]$$

$$= \frac{3M}{4R^3} \times \frac{8R^5}{15}$$

ie, $$\boxed{I = \frac{2}{5} MR^2} \quad \text{... (6)}$$

(b) About a tangent

A tangent drawn to the sphere at any point is parallel to one of its diameter. The distance between the tangent and the diameter is R, the radius of the sphere. (Fig. 1.32)

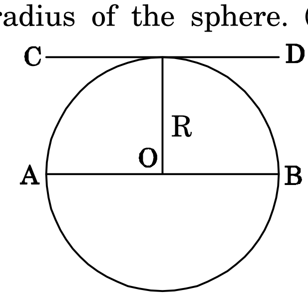


Fig. 1.32 Moment of Inertia of solid sphere about a tangent

By the theorem of parallel axes, the M.I. of the sphere about the tangent $CD, I =$ M.I. of the sphere about a parallel axis AB passing through the centre + Mass of the sphere × Square of the distance between the axes.

$$\therefore \qquad I = \frac{2}{5} MR^2 + MR^2$$

$$= \frac{2\,MR^2 + 5\,MR^2}{5}$$

ie,
$$\boxed{I = \frac{7}{5} MR^2}$$

5. Moment of Inertia of a Solid cylinder

(a) About an axis passing through the centre and perpendicular to its length

Let M be the mass of a solid cylinder of length l and radius R. The cylinder is free to rotate about an axis. AB passing through the centre and perpendicular to its length.

Mass per unit length (linear density) of the cylinder,

$$m = \frac{M}{l} \qquad \dots (1)$$

The cylinder is imaged to be divided into a large number of thin circular discs with their plane at right angles to the length of the cylinder.

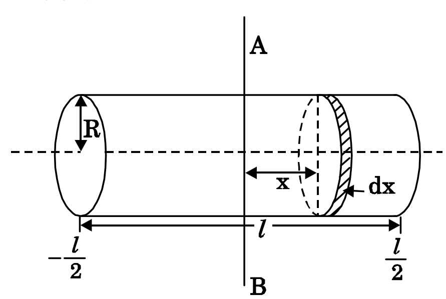


Fig. 1.33 Moment of Inertia of a solid cylinder - about an axis passing through the centre and perpendicular to its length

Consider one such circular disc of thickness dx at a distance x from the axis AB.

Mass of the disc = $m\,dx$

Moment of inertia of this disc about its own diameter.

$$= \frac{\text{Mass} \times (\text{Radius})^2}{4},$$

$$= \frac{m\,dx\,R^2}{4}$$

$$= \frac{m\,R^2\,dx}{4} \qquad \text{... (2)}$$

$\therefore$ Moment of inertia of this disc about the axis AB, parallel to the diameter = Moment of inertia about the diameter + Mass × square of the distance between the axes. (using parallel axes theorem).

$$= \frac{m\,R^2\,dx}{4} + m\,dx\,x^2 \qquad \text{... (3)}$$

Therefore, the moment of inertia of the solid cylinder about axis AB, is obtained by integrating the above expression between limits $x = -l/2$ and $x = +l/2$.

$\therefore$ M.I. of the cylinder about AB,

$$I = \int_{-l/2}^{l/2} \left(\frac{m\,R^2\,dx}{4} + m\,x^2\,dx \right) \qquad \text{... (4)}$$

$$= \int_{-l/2}^{l/2} \frac{m\,R^2\,dx}{4} + \int_{-l/2}^{l/2} m\,x^2\,dx$$

$$= 2\int_{0}^{l/2} \frac{m\,R^2\,dx}{4} + 2\int_{0}^{l/2} m\,x^2\,dx \qquad \text{... (5)}$$

$$I = \frac{m\,R^2}{2}\,[\,x\,]_0^{l/2} + 2m\left[\frac{x^2}{3}\right]_0^{l/2} \qquad \ldots (6)$$

$$= \frac{m\,R^2}{2} \times \frac{l}{2} + 2m \times \frac{l^3}{3 \times 8}$$

$$= \frac{M}{l} \times \frac{R^2}{2} \times \frac{l}{2} + \frac{2M}{l} \times \frac{l^3}{3 \times 8} \qquad \left(\because m = \frac{M}{l}\right)$$

$$= \frac{MR^2}{4} + \frac{M\,l^2}{12}$$

ie,
$$\boxed{\boldsymbol{I = M\left(\frac{R^2}{4} + \frac{l^2}{12}\right)}} \qquad \ldots (7)$$

(b) About the axis of the cylinder

A solid cylinder is considered to be a thick circular disc. It is composed of a large number of thin circular discs placed one above the other.

Let each such elementary disc be of mass m and radius R, equal to the radius of the cylinder.

The moment of inertia of each such disc about an axis passing through its centre and perpendicular to its plane

$$= \frac{mR^2}{2} \qquad \ldots (8)$$

$\therefore$ M.I. of the solid cylinder about its axis $= \Sigma\,\dfrac{mR^2}{2}$

$$\boxed{I = \frac{MR^2}{2}} \qquad \ldots (9)$$

where M is the mass of the cylinder.

Moment of Inertia of Certain Rigid Bodies of mass M

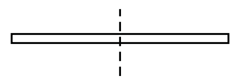
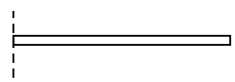
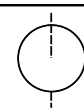
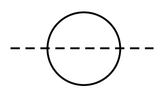
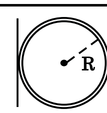
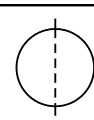
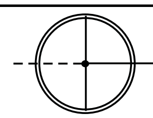
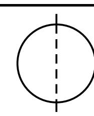
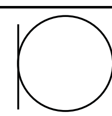
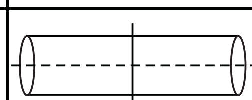
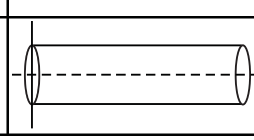
Body	Location Axis	Figure	MI
A thin uniform rod of length l	Passing through the C.G and $\perp r$ to the length		$\frac{M l^2}{12}$
A thin uniform rod of length l	Passing through one end and $\perp r$ to the length		$\frac{M l^2}{3}$
Circular thin ring of radius R	Passing through the centre and $\perp r$ to the plane		MR^2
Circular thin ring of radius R	About any diameter		$\frac{MR^2}{2}$
Circular thin ring of radius R	About a tangent in the plane of ring		$\frac{3}{2} MR^2$
Circular thin circular disc of radius R	About an axis through it centre		$\frac{MR^2}{4}$
Circular thin circular disc of radius R	About any diameter		$\frac{MR^2}{4}$
Solid sphere of radius R	About any diameter		$\frac{2}{5} MR^2$
Solid sphere of radius R	About any tangent		$\frac{7}{5} MR^2$
Solid cylinder of radius R and length l	Passing through CM and $\perp r$ to the length		$M\left[\frac{l^2}{12} + \frac{R^2}{4}\right]$
Solid cylinder of radius R and length l	About symmetry axis		$\frac{MR^2}{2}$

Fig. 1.34

1.10 MOMENT OF INERTIA OF A DIATOMIC MOLECULE

A diatomic molecule, in its stable equilibrium position consists two atoms that are at a distance 'R' apart. The distance 'R' is called the bond length between the two atoms.

Presently we can consider that it consists of two tiny spheres at either end of a thin weightless rigid rod, as shown in fig.1.35. This kind of arrangement can be called as **rigid rotor**.

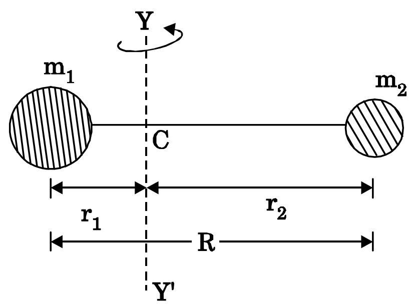


Fig. 1.35 Moment of Inertia of a diatomic molecule

Let 'C' be the center of mass of the molecule and r_1 and r_2, the respective distances of the two atoms from it.

Then

$$r_1 + r_2 = R \quad \text{... (1)}$$

and

$$m_1 r_1 = m_2 r_2 \quad \text{... (2)}$$

where m_1 and m_2 are the masses of two atoms respectively.

From eqn.(1),

$$r_1 = R - r_2 \quad \text{... (3)}$$

and from eqn.(2),

$$r_2 = \frac{m_1 r_1}{m_2} \qquad \text{... (4)}$$

So,

$$r_1 = R - \frac{m_1 r_1}{m_2}$$

$$\therefore \qquad R = r_1 + \frac{m_1 r_1}{m_2} = r_1\left[1 + \frac{m_1}{m_2}\right] \qquad \text{... (5)}$$

(or)

$$r_1 = \frac{R}{\left[1 + \frac{m_1}{m_2}\right]} \qquad \text{... (6)}$$

Now, the moment of inertia of the molecule (i.e., of the two atoms) about an axis passing through the center of mass '*C*' and perpendicular to the bond is given as

$$I = m_1 r_1^2 + m_2 r_2^2 \qquad \text{... (7)}$$

So,

$$I = m_1 r_1 \cdot r_1 + m_1 r_1 \cdot r_2, \qquad [\because \text{ From eqn.(2)}]$$

$$I = m_1 r_1 (r_1 + r_2),$$

(or) by using eqn.(1),

$$\boxed{\boldsymbol{I = m_1 r_1 R}} \qquad \text{... (8)}$$

Substituting eqn.(6) in eqn.(8) gives

$$I = m_1 R\left[\frac{R}{\left(1 + \frac{m_1}{m_2}\right)}\right],$$

So

$$I = \frac{m_1 R^2}{\left(1 + \frac{m_1}{m_2}\right)} = \frac{m_1 R^2}{\frac{m_2 + m_1}{m_2}} = \frac{m_1 m_2 R^2}{m_2 + m_1}$$

(or)

$$I = \left(\frac{m_1 m_2}{m_1 + m_2}\right) R^2,$$

$$\boxed{\boldsymbol{I = \mu R^2}} \qquad ...(9)$$

where $\mu = \frac{m_1 m_2}{m_1 + m_2}$ is called as **reduced mass** of the molecule. Thus the figure 1.35 can also be redrawn as

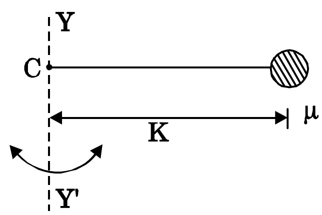


Fig. 1.36

In fig.1.36, $K = R$, which is called radius of gyration, so moment of inertia

$$\boxed{\boldsymbol{I = \mu K^2}} \qquad ...(10)$$

1.11 ROTATIONAL DYNAMICS OF RIGID BODIES

Angular momentum

The moment of linear momentum is known as angular momentum.

Consider a particle of mass m is at a distance r from the axis of rotation. Let v be the linear velocity of the particle (Fig. 1.37). Then,

Angular momentum

$$= \text{linear momentum} \times \text{distance}$$

$$\vec{L} = mv \times r$$

$$= mr\,\omega \times r \qquad \{\because v = r\,\omega\}$$

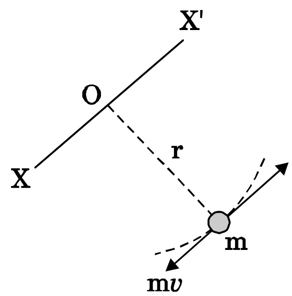


Fig. 1.37

ie., $\boxed{\textbf{Angular momentum} = mr^2\,\omega}$

where ω is the angular velocity of the particle.

Also $\vec{L} = \vec{r} \times \vec{p}$

S.I unit for angular momentum is $\mathbf{kg\,m^2\,s^{-1}}$

Definition

Angular momentum of a particle is defined as its moment of linear momentum it is given by the product of linear momentum and perpendicular distance of its line of action from the axis of rotation. It is denoted by $\vec{L}$.

It is a vector quantity. The direction of angular momentum is given by right hand rule. According to this rule, if the fingers of right hand are curled in the direction of rotation about the axis, then the thumb points on the direction of angular momentum.

In vector notation, angular momentum is given as the vector product of $\vec{r}$ and $\vec{P}$, the linear momentum *i.e.*,

$$\boxed{\vec{L} = \vec{r} \times \vec{p}} \quad \ldots (1)$$

$\vec{r} \rightarrow$ position vector

$\vec{p} \rightarrow$ linear momentum

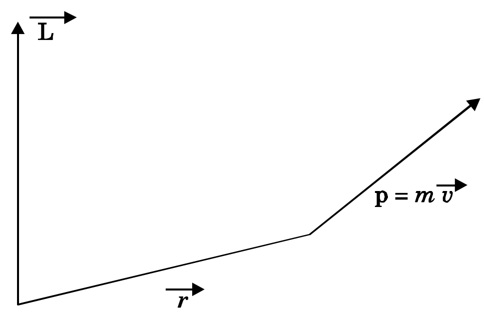


Fig. 1.38

The direction of angular momentum is perpendicular to plane containing $\vec{r}$ and $\vec{p}$.

Expression for Angular momentum of a rigid body

Consider a rigid body rotating about a fixed axis XOX'. The rigid body consists of a large number of particles.

Let $m_1, m_2, m_3, \ldots$ etc. be the masses of the particles situated at distances $r_1, r_2, r_3, \ldots$ etc. from the fixed axis.

All the particles rotate with the same angular velocity ω. (Fig. 1.39)

Angular momentum

$$= \text{linear momentum} \times \text{distance}$$

$$= mv \times r$$

$$= mr\omega \times r = mr\omega \times r$$

$$= mr^2\omega \qquad \{\because v = r\omega\}$$

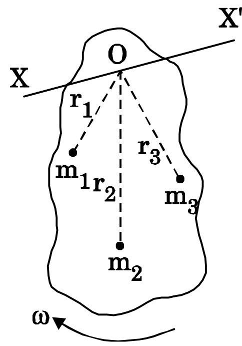


Fig. 1.39

∴ Angular momentum of the first particle = $m_1 r_1^2 \omega$

Angular momentum of the second particle = $m_2 r_2^2 \omega$

Angular momentum of the third particle = $m_3 r_3^2 \omega$

and so on.

The angular momentum of the whole body is equal to the sum of the angular momentum of all the particles of the body.

$$\left.\begin{array}{l}\text{The angular momentum}\\ \text{of the rigid body}\end{array}\right\} = m_1 r_1^2 \omega + m_2 r_2^2 \omega + m_3 r_3^2 \omega$$

$$= \omega (m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \ldots)$$

$$= \omega \Sigma mr^2$$

Let I be the moment of inertia of the rigid body about the fixed axis, $I = \Sigma mr^2$

∴ The angular momentum of the rigid body, = ωI

ie., Angular momentum $\boldsymbol{L = I \omega}$

Torque (τ)

Torque is the turning effect of a force on a body, on which the force acts.

The turning effect of a force depends on

(i) the magnitude of the force and

(ii) the perpendicular distance from the axis of rotation to the line of action of the force

Definition

The moment of the applied force is called torque. It is represented by the symbol 'τ'.

If F is the force acting on a body at a distance r (Fig. 1.40) then,

Torque = Force × distance

i.e., $\tau = F \times r$

The rotational motion comes into picture only when the torque acts on the body.

Torque in vector notation

When a force is applied on a rigid body capable of rotation about some axis, the body rotates about the axis.

The ability of a force to rotate a body about an axis is measured in terms of a quantity called *torque*.

Consider a body capable of rotation about an axis passing through O. Let a force F act at A distant 'r' from O such that the line of action of the force is perpendicular to OA.

The moment of this force F about the axis through O is a measure of the torque. (Fig 1.40)

Torque $\tau = \vec{r} \times \vec{F}$

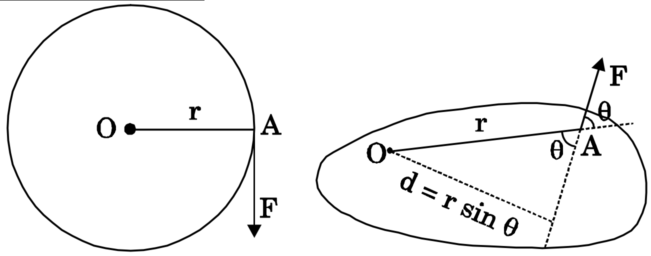


Fig. 1.40

If the direction of the force is inclined at an angle θ with the direction of r, it is measured as the product of the force F and the perpendicular distance from the axis of rotation to the line of action of the force.

$$\therefore \text{ Torque } = F \times ON = F \times d = F \cdot r \sin\theta$$

This can be expressed in vector form as

$$\boxed{\vec{\tau} = \vec{r} \times \vec{F}}$$

Thus torque is the cross product of force $\vec{F}$ and the vector $\vec{r}$ between the axis of rotation and the point of application of the force.

It is acting in a direction perpendicular to the plane containing $\vec{r}$ and $\vec{F}$ and its direction is given by right hand screw rule.

If $\vec{r}$ and $\vec{F}$ perpendicular to each other $\tau = rF$

$(\because \theta = 90° \text{ and } \sin\theta = 1)$

Unit of Torque

Newton - metre (Nm)

Laws of motion applied to rotatory motion

1. A rotating body tends to rotate continuously and uniformly about a fixed axis unless it is acted upon by an external torque.
2. The rate of change of angular momentum is equal to the external torque applied and acts in the direction of torque
3. When a body exerts a torque on another body, the second body exerts an equal and opposite torque on the first body about the same axis of rotation.

Relation between torque and angular momentum

Consider a rigid body capable of rotation about the axis. Let I be the moment of inertia of the body about that axis. Let a constant torque τ acting on it changes its angular velocity from ω_o to ω in time 't'.

Torque = rate of change of angular momentum

$$\tau = \frac{(I\omega - I\omega_o)}{t} = I\frac{(\omega - \omega_o)}{t} = I\alpha,$$

where α is the angular acceleration.

$$\therefore \quad \boxed{\tau = I\alpha}$$

$$\tau = I\frac{d\omega}{dt} \qquad \left(\because \alpha = \frac{d\omega}{dt}\right)$$

$$\tau = \frac{d}{dt}(I\omega) = \frac{dL}{dt}$$

$$\boxed{\tau = \frac{dL}{dt}}$$

1.12 CONSERVATION OF ANGULAR MOMENTUM

The law of conservation of angular momentum states that in the absence of an external torque, the angular momentum of a body or a system of bodies remains conserved.

The equation of motion of angular momentum of a particle is given by

$$\vec{\tau} = \frac{d\vec{L}}{dt} \qquad \text{... (1)}$$

where $\vec{\tau}$ is the torque acting on the particle and $\frac{d\vec{L}}{dt}$ is the rate of change of angular momentum.

If there is no torque i.e. $\vec{\tau} = 0$

$$\frac{d\vec{L}}{dt} = 0$$

or $$\vec{L} = \text{constant} \quad \dots (2)$$

i.e., angular momentum is conserved.

Thus, when the resultant external torque acting on a particle is zero, the total vector angular momentum of the particle remains conserved. **This is called law or principle of conservation of angular momentum.**

Hence, in the absence of any external torque, if the moment of inertia (I) of the system increases, the angular velocity (ω) decreases. Similarly, if moment of inertia (I) decreases, the angular velocity increases.

($\because$ angular momentum, $L = I\omega$)

Final angular momentum = Initial angular momentum

i.e Angular momentum remains unchanged.

If the moment of inertia I of a body changes due to redistribution of mass about the axis, angular velocity ω changes in such a manner that the product $I\omega$ remains a constant.

It is clear that if the M.I. increases, the angular velocity ω decreases and vice versa.

This principle of conservation of angular momentum is utilised by a diver, a skater, or a circus acrobat, while doing various acts of spinning.

Illustrations of explain the law of conservation of angular momentum

(i) Consider a man holding a pair of dumb bells or weights in his outstretched arms extending and standing on a turn table capable rotating freely about a vertical axis passing through its centre. (Fig. 1.41).

The table is set rotating with certain angular velocity. The dumb bells are brought nearer the axis of rotation.

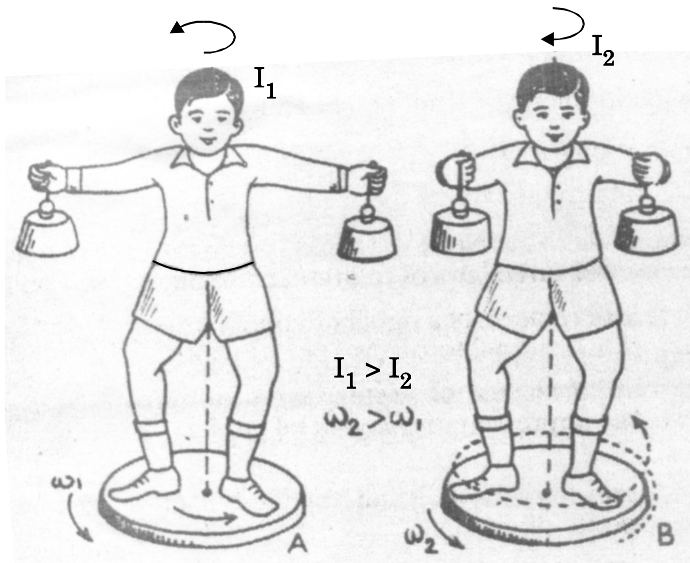


Fig. 1.41 Conservation of angular momentum

As a consequence, the moment of inertia of the system about the axis of rotation decreases and the angular velocity of the turn table increases.

As the man extends out his arms the M.I. increases and the angular velocity of the turn table decreases to maintain the angular momentum a constant.

(ii) In case of an acrobat in a circus, the acrobat leaves the swing with his arms and legs stretched. As soon as he leaves the swing, he possesses some angular momentum.

When the acrobat pulls his hands and legs together inwards, his M.I decreases. Hence the angular velocity of the acrobat increases considerably and the acrobat rolls.

Differences Analogy between linear and rotational motion

The following table summarize the analogy showing the quantities relating to linear motion and the corresponding quantities relating to rotational motion.

Table 1.1

Linear motion	Rotational motion
Displacement = s (or) x	Angular displacement = θ
Velocity, $\vec{v} = \frac{d\vec{s}}{dt} = \frac{d\vec{x}}{dt}$	Angular velocity $\omega = \frac{d\theta}{dt}$
Acceleration, $\vec{a} = \frac{d\vec{v}}{dt}$ also, $\vec{a} = \frac{d^2\vec{s}}{dt^2}$	Angular acceleration $\alpha = \frac{d\omega}{dt}$ also, $\alpha = \frac{d^2\theta}{dt^2}$
Mass = M	Moment of Inertia $I = \Sigma mr^2$
Force $F = ma$	Torque $\tau = I\alpha$
Momentum $\vec{p} = m\vec{v}$	Angular momentum $\vec{L} = I\omega$
Kinetic Energy = $\frac{1}{2}mv^2$	Kinetic energy = $\frac{1}{2}I\omega^2$
Work done = $\vec{F}\cdot\vec{s}$	Work done = $\tau\theta$

1.13 ROTATIONAL ENERGY STATES OF A RIGID DIATOMIC MOLECULE

If we consider two atoms of masses m_1 and m_2 which are situated at a distance r_1 and r_2 with respect to the axis of rotation YY' respectively, then this arrangement is called as rigid rotor. Here 'R' is the bond length between the two atoms ($R = r_1 + r_2$).

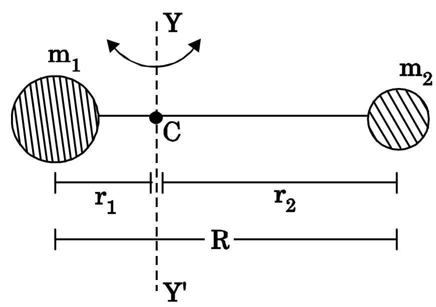


Fig. 1.42 Rotational energy states of a rigid diatomic molecule

If the distance diatomic molecule rotates with respect to the center of mass '*C*', then its kinetic energy is given as

$$E = \frac{1}{2} m_1 r_1^2 \omega^2 + \frac{1}{2} m_2 r_2^2 \omega^2, \qquad \text{... (1)}$$

(or) $$E = \frac{1}{2} (m_1 r_1^2 + m_2 r_2^2) \omega^2,$$

(or) $$E = \frac{1}{2} I \omega^2$$

(because, the moment of inertia $I = m_1 r_1^2 + m_2 r_2^2$)

$$E = \frac{1}{2} I \omega^2, \qquad \text{... (2)}$$

The eqn.(2) can be rewritten as,

$$E = \frac{1}{2I} \cdot I^2 \omega^2 \qquad \text{... (3)}$$

As, $I \omega = L$, (the angular momentum of the rigid rotor), then Eq.(3) becomes

$$\boxed{E = \frac{L^2}{2I}} \qquad \text{... (4)}$$

At atomic level, the rotation leads to quantization of the angular momentum with values given by

$$L^2 = l(l + 1) \hbar^2, \qquad l = 0, 1, 2, \ldots \qquad \text{... (5)}$$

where 'l' is the rotational quantum number. As 'l' varies interms of integer values, so corresponding the energy levels of a rotating molecule are therefore given by

$$\boxed{E_l = \frac{l(l + 1) \hbar^2}{2I}} \qquad \text{... (6)}$$

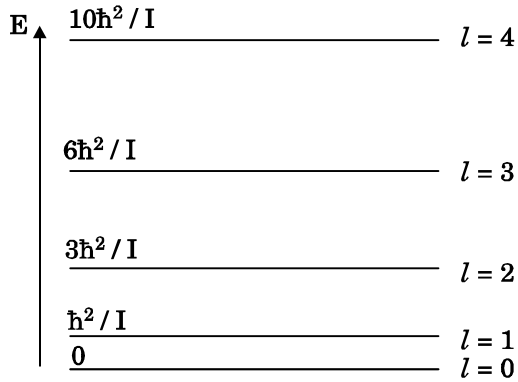


Fig. 1.43

Here $\hbar = \dfrac{h}{2\pi}$ and 'h' is the planck's constant.

The ground level and first four excited rotational energy levels for a diatomic molecule is shown in the fig.1.44.

Note that the levels are not equally spaced.

1.14 GYROSCOPE

Definition

A gyroscope is a device used for measuring or maintaining orientation and angular velocity. It is a spinning wheel or disc in which the axis of rotation (spin axis) is free to assume any orientation by itself.

When rotating, the orientation of this axis is unaffected by tilting or rotation of the mounting, according to the **conservation of angular momentum**

Gyroscopic principle

All spinning objects have gyroscopic properties. The main properties that an object can experience in any gyroscopic motion are rigidity in space and precession.

Description and working

A gyroscope is essentially a heavy wheel rotating at a high speed about an axle passing through its centre of mass and so mounted as to be free to turn about any of three mutually perpendicular axes 1, 2, 3 (Fig.1.44 (a) & (b)).

If the wheel rotates with high angular speed about the axis 1, the base may be turned in any manner without exerting any torque on the wheel.

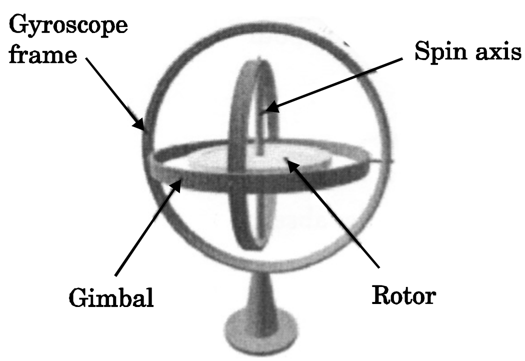


Fig. 1.44 (a)

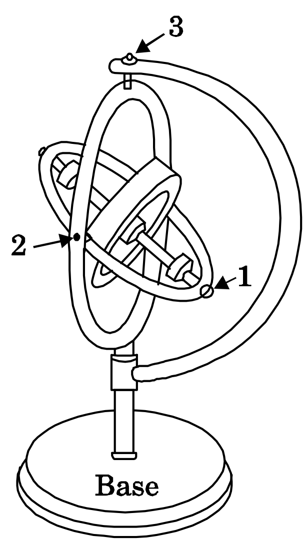


Fig. 1.44 (b)

In other words, so long the wheel rotates rapidly, it maintain its axis of rotation unchanged in space as the support is tilted in any manner.

If, any torque is applied perpendicular to the axis of rotation, there will be a precession inversely proportional to the angular momentum ($I \omega$) of the wheel. However, a heavy wheel rotating at high speed, having a large moment of inertia, would suffer very small precession.

Thus gyroscope is a device characterised by the greater stability of its axis of rotation.

Applications

1. In view of the property of stability, the gyroscope are used as stabilizers in ships, boats and aeroplanes.
2. Due to the inherent stability of the gyroscope, it used as a compass, and a gyro-compass is preferable to the magnetic compass in many respects.
3. Another important application of the directional stability of a rapidly spinning (rotating) body is the rifling of the barrels of the rifles.

   This spin motion prevents the deflection of the bullet from its path due to air and gravity effects, and causes only very little precession. Thus the uniformity of flight of the bullet is increased.
4. The rolling of hoops and the riding of bicycles (which are statically unstable since both of them cannot remain in equilibrium when at rest) are possible because of the gyroscopic effect. This effect produces a movement of the plane of rotation, tending to counterbalance the disturbing action of gravity.

5. Many modern aircraft instruments such as automatic pilot, bomb sights, artificial horizon, turn and back indicators, etc. have been developed on gyroscope-controlled principles.

1.15 TORSION PENDULUM

A circular metallic disc suspended using a thin wire that executes torsional oscillation is called torsional pendulum.

- It executes torsional oscillations, whereas a simple pendulum executes linear oscillations.

Description

A torsional pendulum consists of a metal wire suspended vertically with the upper end fixed. The lower end of the wire is connected to the centre of a heavy circular disc (Fig.1.45).

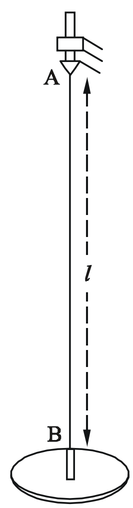


Fig. 1.45 Torsional Pendulum

Expression for the period of Oscillation of a Torsion Pendulum

When the disc is rotated by applying a twist, the wire is twisted through an angle θ. Then, the restoring couple set up in the wire

$$= C\theta \qquad \text{... (1)}$$

where C – couple per unit twist.

If the disc is released, it oscillates with angular velocity $\frac{d\theta}{dt}$ in the horizontal plane about the axis of the wire. These oscillations are known as ***torsional oscillations***.

If $\frac{d^2\theta}{dt^2}$ is the angular acceleration produced in the disc and I its moment of inertia of the disc about the axis of the wire then,

$$\text{Applied couple} = I\frac{d^2\theta}{dt^2} \qquad \text{... (2)}$$

At equilibrium, applied couple = restoring couple

$$I\frac{d^2\theta}{dt^2} = -C\,\theta$$

Here – ve sign signifies that the restoring couple is opposite to applied couple (torque)

$$\frac{d^2\theta}{dt^2} = -\frac{C}{I}\theta \qquad \text{... (3)}$$

This equation represents simple harmonic motion which shows that angular acceleration $\left(\frac{d^2\theta}{dt^2}\right)$ is proportional to angular displacement θ and is always directed towards the mean position.

Hence, the motion of the disc being ***simple harmonic motion,*** the time period of the oscillation is given by

$$T = 2\pi\sqrt{\frac{\text{Displacement}}{\text{Acceleration}}}$$

$$= 2\pi\sqrt{\frac{\theta}{\frac{C}{I}\times\theta}}$$

$$\boldsymbol{T = 2\pi\sqrt{\frac{I}{C}}} \quad \ldots (4)$$

Uses of Torsional Pendulum

Torsional pendulum is used to determine

1. Rigidity modulus of the wire
2. Moment of inertia of the disc
3. Moment of inertia of an irregular body.

Determination of Rigidity Modulus of the Wire

The rigidity modulus of the wire is determined by the following equation

$$T = 2\pi\sqrt{\frac{I}{C}} \quad \ldots (1)$$

Experiment

A circular disc is suspended by a thin wire, whose rigidity modulus is to be determined. The top end of the wire is fixed firmly in a vertical support.

The disc is then rotated about its centre through a small angle and set free. It executes torsional oscillations.

The time taken for 20 complete oscillations is noted. The experiment is repeated and the mean time period (T) of oscillation is determined.

The length l of the wire is measured. This length is then changed by about 10 cm and the experiment is repeated. The readings for five or six different lengths of wire are measured.

The disc is removed and its mass and diameter are measured.

The time period of oscillation is

$$T = 2\pi\sqrt{\frac{I}{C}} \qquad \text{... (2)}$$

Squaring on both sides, we have

$$T^2 = 2^2\pi^2\left(\sqrt{\frac{I}{C}}\right)^2 \qquad \text{... (3)}$$

$$T^2 = \frac{4\pi^2 I}{C} \qquad \text{... (4)}$$

Substituting couple per unit twist $C = \frac{\pi n r^4}{2l}$ in eqn (4),

We have $$T^2 = \frac{4\pi^2 I}{\frac{\pi n r^4}{2l}} = \frac{2l \times 4\pi^2 I}{\pi n r^4} \qquad \text{... (5)}$$

rearranging the equation (5),

The rigidity modulus of the material of the wire

$$\boxed{\boldsymbol{n = \frac{8\pi I}{r^4}\left(\frac{l}{T^2}\right)}} \qquad \text{... (6)}$$

I – moment of inertia of circular disc $= \frac{MR^2}{2}$

where M – Mass of the circular disc

R – Radius of the disc

1.16 DOUBLE PENDULUM

A double pendulum is a pendulum with another pendulum attached to its end.

It is a simple physical system that exhibits rich dynamic behaviour with strong sensitive by to initial conditions.

The pendulum behaves like a linear system for small angles. When the angles are small in the double pendulum, the system behaves like the linear double spring. In this case, the motion is determined by simple sine and cosine functions.

On the otherhand for large angles, the pendulum is non-linear and the phase graph becomes much more complex.

The pendulum rods are treated as massless and rigid.

Kinematics of the double pendulum

The derivation of the equations of motion is shown below, using the direct Newtonian method.

Kinematics deals with the motion of object without regard to forces. In kinematics expressions for the position, velocity, and acceleration are given in terms of the variables that specify the state of the device. (Fig. 1.46)

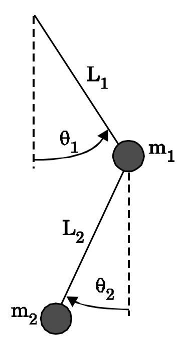


Fig. 1.46 Double Pendulum

- x = horizontal position of pendulum mass
- y = vertical position of pendulum mass
- θ = angle of pendulum (O = vertical downwards, counter-clockwise is positive)
- L = length of rod (constant)

The origin is taken at the pivot point of the upper pendulum. The positions of pendulum 1 and pendulum 2 are found by using simple trigonometry relation.

Let position of pendulum 1 be (x_1, y_1) and pendulum 2 (x_2, y_2). x_1, y_1 and x_2, y_2 are given interms of θ_1 and θ_2 respectively.

$$x_1 = L_1 \sin \theta_1$$

$$y_1 = -L_1 \cos \theta_1$$

$$x_2 = x_1 + L_2 \sin \theta_2$$

$$y_2 = y_1 - L_2 \cos \theta_2$$

The velocity is the derivative with respect time of the position.

$$\frac{dx_1}{dt} = \frac{d\theta_1}{dt} L_1 \cos \theta_1$$

$$\dot{x}_1 = \dot{\theta}_1 L_1 \cos \theta_1$$

$$\dot{y}_1 = \dot{\theta}_1 L_1 \sin \theta_1$$

$$\dot{x}_2 = \dot{x}_1 + \dot{\theta}_2 L_2 \cos \theta_2$$

$$\dot{y}_2 = \dot{y}_1 + \dot{\theta}_2 L_2 \sin \theta_2$$

The acceleration is the second derivative.

$$\ddot{x}_1 = -\dot{\theta}_1^2 L_1 \sin \theta_1 + \ddot{\theta}_1 L_1 \cos \theta_1$$

$$\ddot{y}_1 = \dot{\theta}_1^2 L_1 \cos \theta_1 + \dot{\theta}_1 L_1 \sin \theta_1$$

$$\ddot{x}_2 = \ddot{x}_1 - \dot{\theta}_2^2 L_2 \sin \theta_2 + \ddot{\theta}_2 L_2 \cos \theta_2$$

$$\ddot{y}_2 = \ddot{y}_1 \dot{\theta}_2^2 L_2 \cos \theta_2 + \dot{\theta}_2 L_2 \sin \theta_2$$

Forces in double pendulum

The two pendulum masses are treated as point particles. The free body diagram is drawn for the upper mass and writing an expression for the net force acting on it (Fig. 1.47). These variables are defined as follows.

- T = tension in the rod
- m = mass of pendulum
- g = acceleration due to gravity

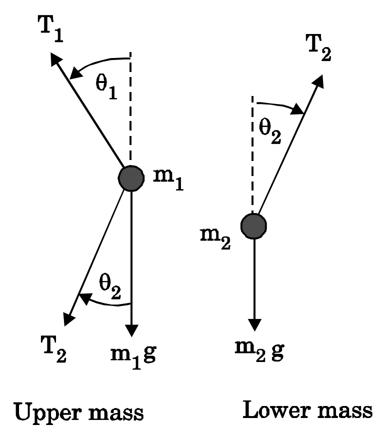


Fig. 1.47 Double pendulum

The forces on the upper pendulum mass are the tension in the upper rod T_1, the tension in the lower rod T_2, and gravity $-m_1 g$.

We write separate equations for the horizontal and vertical forces, since they can be treated independently. The net force on the mass is the sum of these. Here we show the net force and use Netwon's law $F = ma$.

$$m_1 \ddot{x}_1 = -T_1 \sin\theta_2 + T_2 \sin\theta_2 \qquad \text{... (5)}$$

$$m_1 \ddot{y}_1 = T_1 \cos\theta_1 - T_2 \cos\theta_2 - m_1 g \qquad \text{... (6)}$$

For the lower pendulum, the forces are the tension in the lower rod T_2, and gravity $-m_2 g$.

$$m_2 \ddot{x}_2 = -T_2 \sin \theta_2 \quad \text{... (7)}$$

$$m_2 \ddot{y}_2 = T_2 \cos \theta - m_2 g \quad \text{... (8)}$$

In relating these equations to the diagrams, keep in mind that in the example diagram θ_1 is positive and θ_2 is negative, because of the convention that a counter-clockwise angle is positive.

Uses of double pendulum

The double pendulum is widely used in education, research and applications. For example the double pendulum is staple bench top experiment for introducing and studying chaos and state transitions. It has also been used to study chaos both experimentally and numerically.

1.17 INTRODUCTION TO NON - LINEAR OSCILLATIONS

Vibration phenomena is well understood by using *linear vibrations theory* include small amplitude vibrations. Examples: Vibration of long, slender objects like long bridges, aeroplane wings and helicopter blades, small rocking motions of ships in calm waters; the simplest whirling motions of flexible shafts, and so on.

However, interactions between bridges and foundations, between wings / blades and air, between ships and waves, between shafts and bearings, and so on, are all *nonlinear*.

Non - linear systems can show behaviours that linear systems cannot. These include:

(a) multiple steady state solutions, some stable and some unstable, in response to the same inputs.

(b) jump phenomena, involvind discontinuous and significant changes in the response of the system as some forcing parameter is slowly varied.

(c) response at frequencies other than the forcing frequency.

(d) internal resonances, involving the different parts of the system vibrating at different frequencies, all with steady amplitudes (the frequencies are usually in rational ratios, such as 1:2, 1:3, 3:5, etc.),

(e) self sustained oscillations in the absence of explicit external periodic forcing, and

(f) complex, irregular motions that are extremely sensitive to initial conditions (chaos).

Non - linear oscillators

A linear oscillator can oscillate with only one frequency, its motion is *sinusoidal and periodic*.

If the return force in the spring is not linear, the motion will still repeat itself, but it will no longer have only a single frequency in its motion.

The oscillations will repeat over and over, always with the same period, but the position as a function of time will not be given by $y = A \cos(2\pi f_1 t)$, where $f_1 = \frac{1}{P}$. Here P is period.

- In order to describe the motion of the non-linear oscillator, the sine wave different frequencies to be considered.
- Non - linear oscillators describe motion that exactly repeats itself cycle after cycle, we need building blocks that oscillate with exactly the same period P, or at least integer fractions of the period, namely $P, \frac{P}{2}, \frac{P}{3}$, etc.,
- We can write symbolically as $P_n = \frac{P}{n}$ where n is an integer, $n = 1, 2, 3, \ldots$. Since the frequency f_n is the inverse of the period.

$$\boxed{f_n = \frac{n}{P} = n\frac{1}{P} = nf_1} \quad \ldots (1)$$

The first four harmonics. The fourth harmonic is shown below the others for clarity. The dashed vertical line shows half the period, so $t = \frac{P}{2}$.

Note the difference between the behavior of the odd (1 and 3) harmonics and the even harmonics (2 and 4) just after $t = \frac{P}{2}$.

Thus, the complex motion is resultant of sine waves of several frequencies called *harmonics*. The sine waves with frequencies that are integer multiples of the lowest frequency $f_1 = \frac{1}{P}$.

They go through exactly 1, 2, etc. complete oscillations in the period P. The frequency f_1 is called the **fundamental of the harmonic series.** The first four harmonics are shown in fig. 1.48, which are given by

$$\boxed{y_n = A \cos \{ n (2\pi f_1 t) \}, \quad n = 1, 2, 3, 4} \qquad \dots (2)$$

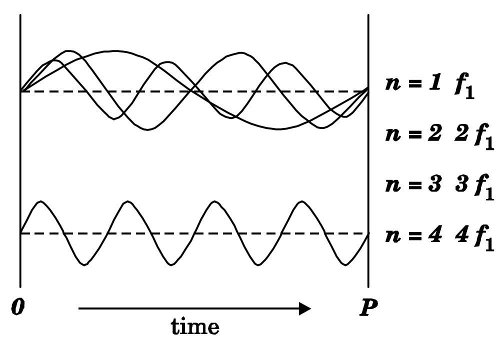


Fig. 1.48 Non - linear oscillator Frequency

It is noted that at the midpoint, all of the harmonics are zero, but the even harmonics have gone through an *integer* number of cycles, and they are going *positive* again while the odd harmonics have gone through $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}$, etc cycles, and are going negative.

If the period of the oscillation is P, then the frequencies present in the motion are

$$\boxed{f_1 = \frac{1}{P},\ f_2 = 2f_1 = \frac{2}{P},\ f_3 = 3f_1,\ f_4 = 4f_1\, 4f,\ \text{etc.}} \quad \ldots (3)$$

To summarize, the motion contains the frequency f_1 which is the inverse of the period, plus harmonics (integer multiples) of this frequency. **This is very different from the simple oscillator**.

In the simple oscillator we had one frequency which only depended on the stiffness and inertia of the system.

Now, the motion of the non-linear oscillator consists of a complex motion made up of harmonics of f_1. The participation of each harmonic in a complex oscillation depends on the details of the nonlinearity.

There are two important characteristics of the nonlinear oscillator.

1. The effects of the nonlinearity becomes much more important as the amplitude is increased.
2. For some types of nonlinearity, the frequency of the oscillator will change with amplitude.

Thus when we drive a nonlinear system, the larger the amplitude the more important the higher harmonics are.

Applications & uses

Non - linear system is a system in which the change of the output is not proportional to change of the input.

Non linear problems are of interact to engineers, biologists, physicists, mathematicians and many other scientists because most systems are inherently non - linear in nature.

SOLVED PROBLEMS

Problem 1.1

Give the location of centre of mass of a (i) hollow sphere (ii) hollow cylinder (iii) hollow cube (iv) ring (v) solid cylinder (vi) solid cube. Does the centre of mass of a body necessarily lie inside the body.

Solution:

(i) CM of hollow sphere is located at its centre.

(ii) CM of hollow cylinder is located at the mid point of its axis of symmetry.

(iii) CM of hollow cube lies at its geometric centre.

(iv) CM of ring is at the centre of ring.

(v) CM of solid cylinder at the mind point of its axis of symmetry.

(vi) CM of solid cylinder at its geometric centre.

It is not necessary that the CM of a body lie inside the body.

Problem 1.2

In an HCl molecule, the separation between the nuclei of two atoms is 1.27 Å. Find the position of CM of the molecule. Given that chlorine atom is 35.5 times heavier than hydrogen atom and that nearly entire mass of the atom is concentrated in it, nucleus.

Solution:

Let the mass of hydrogen atom $m_1 = m$ and it is located at $x_1 = 0$

The mass of chlorine atom $m_2 = 35.5$ m and it is located at

$$x_2 = 1.27 \text{ Å}$$

$$= 1.27 \times 10^{-10} \text{ m}$$

If X is the distance of the CM of HCl molecule, then

$$X = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

$$= \frac{m \times 0 + 35.5\,m \times 1.27 \times 10^{-10}}{m + 35.5\,m}$$

$$= \frac{35.5\text{ m} \times 1.27 \times 10^{-10}}{36.5\text{ m}}$$

$$= \frac{45.08 \times 10^{-10}}{36.5}\text{ m}$$

$$= 1.235 \times 10^{-10}\text{ m}$$

Hence, CM of HCL molecule is situated on the line joining the two nuclei and at a distance of 1.235×10^{-10} m (1.235 Å) from hydrogen nucleus.

Problem 1.3

A machine part is shown in the fig. It consists of a 10 cm long and 10 cm diameter solid cylinder attached at one end of a 8 cm long, 2 cm diameter cylindrical rod. At other end of the rod, a 10 cm diameter and 2 cm wide disc is attached. Find the centre of mass of the machine part assuming it to be made of the homogeneous material.

Solution:

We measure the distance from free end of the cylinder. Let ρ be the density (homogeneous) of the system, then the masses of three components are given by

$$m_1 = \pi r^2 l \rho = \pi \rho \times 25 \times 10 = 250\pi \rho$$

$$m_2 = \pi r^2 l \rho = \pi \rho \times 1 \times 8 = 8\pi \rho$$

$$m_3 = \pi r^2 t \cdot \rho = \pi \rho \times 25 \times 2 = 50\pi \rho$$

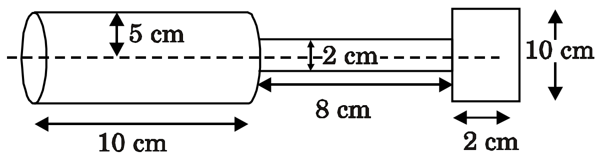


Fig.

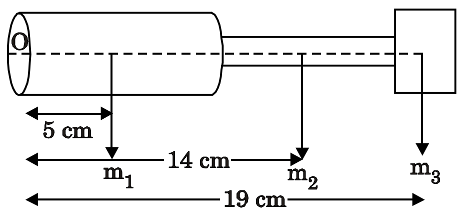


Fig.

The distances from O at which the masses are concentrated are $x_1 = 5$ cm.

$x_2 = 14$ cm, $x_3 = 19$ cm. So distance of centre of mass from O is given by

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{m_1 + m_2 + m_3}$$

$$= \frac{250\pi\rho \times 5 + 8\pi\rho \times 14 + 50\pi\rho \times 19}{250\pi\rho + 8\pi\rho + 50\pi\rho}$$

$$\bar{x} = 7.5\, cm$$

Problem 1.4

Locate the centre of mass of a system of particles of masses $m_1 = 1$ kg, $m_2 = 2$ kg and $m_3 = 3$ kg, situated at the corners of an equilateral triangle of side 1.0 m.

Solution:

We have from fig.

The centre of mass is given by

$$x_{cm} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{m_1 + m_2 + m_3}$$

$$y_{cm} = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3}{m_1 + m_2 + m_3}$$

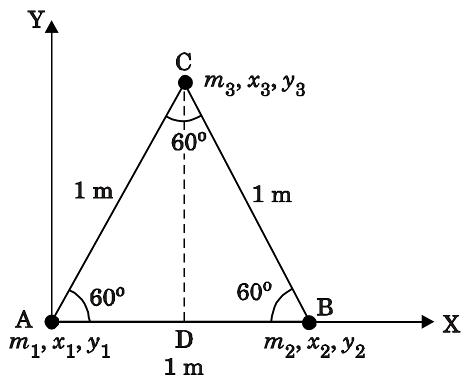


Fig.

In this case $m_1 = 1$ kg, $m_2 = 2$ kg, $m_3 = 3$ kg

$$x_1 = 0\,,\ \ y_1 = 0\,,\ \ x_2 = 1\,,\ \ y_2 = 0\,;$$

So, we have $x_{cm} = \dfrac{1 \times 0 + 2 \times 1 + 3 \times 0.5}{1 + 2 + 3} = \dfrac{3.5}{6}$ m

$$y_{cm} = \frac{1 \times 0 + 2 \times 0 + 3\sqrt{3}/2}{1 + 2 + 3} = \frac{\sqrt{3}}{4}\text{ m}$$

which gives the coordinates of centre of mass as $(3.5/6,\ \sqrt{3}/4)$

Problem 1.5

The radius of gyration of a disc about its centre of mass is 1.41 m. If the mass of the disc 1 kg, what is its moment of inertia?

Solution:

$$I = M K^2$$

Here $M = 1 \text{ kg}, \; K = 1.41 \text{ m}$

therefore, $I = (1)\,(1.41)^2 = \mathbf{2\ kg - m^2}$

Problem 1.6

A body of mass 50 g is revolving about an axis in a circular path. The distance of the centre of mass of the body from the axis of rotation is 50 cm. Find the moment of inertia of the body.

Solution:

Assuming the body as a particle, we have,

$$I = MR^2 = (50 \times 10^{-3})\,(50 \times 10^{-2})^2 = 1.25 \times 10^{-2}\ \mathbf{kg \cdot m^2}$$

Note. If the body has a finite size, then one has to use the theorem of parallel axis.

Problem 1.7

Two masses each of 1 kg mass are attached at the ends of a light rod of length 2 m. Assuming the masses as point masses find moment of inertia of the system about an axis passing through the middle point of the connecting rod and perpendicular to the rod. The mass of the connecting rod is negligible.

Solution:

$$I = \Sigma r^2$$

$$= m_1 r_1^2 + m_2 r_2^2$$

$$m_1 = m_2 = 1 \text{ kg}$$

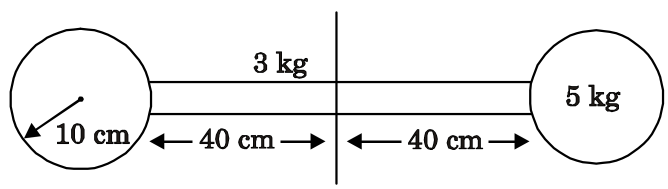
Here $r_1 = r_2 = 1 \text{ m}$

Therefore, $I = 1 \times 1^2 + 1 \times 1^2$

$= 2$ kg–m^2

Problem 1.8

Find the moment of inertia about the geometric centre of the given structure made up of one thin rod connecting two similar solid spheres as shown in figure.



Solution:

The structure is made up of three objects; one thin rod and two solid spheres.

Mass of the rod, $M = 3$ kg and

Total length of the rod, $l = 80$ cm $= 0.8$ m

Moment of inertia of the rod about its centre of mass

$$I_{rod} = \frac{Ml}{12}$$

$$I_{rod} = \frac{1}{12} \times 3 \times (0.8)^2 = \frac{1}{4} \times 0.64$$

$$I_{rod} = 0.16 \text{ kg m}^2$$

Mass of the sphere, $M = 5$ kg

Radius of the sphere, $R = 10$ cm $= 0.1$ m

Moment of inertia of the sphere about its centre of mass

$$I_c = \frac{2}{5} MR^2$$

Moment of inertia of the sphere about geometric centre of the structure is, $I_{sph} = I_c + Ma^2$

where, a = 40 cm + 10 cm = 50 cm = 0.5 m

$$I_{sph} = \frac{2}{5} MR^2 + Ma^2$$

$$I_{sph} = \frac{2}{5} \times 5 \times (0.1)^2 + 5 \times (0.5)^2$$

$$I_{sph} = (2 \times 0.01) + (5 \times 0.25) = 0.02 + 1.25$$

$$I_{sph} = 1.27 \text{ kg m}^2$$

As there are one rod and two similar solid spheres we can write the total moment of inertia (I) of the given geometric structure as,

$$I = I_{\text{rod}} + (2 \times I_{sph})$$

$$I = (0.16) + (2 \times 1.27) = 0.16 + 2.54$$

$$\boxed{I = 2.7 \text{ kg m}^2}$$

Problem 1.9

A flywheel is a uniform disc of mass 72 kg and radius 50 cm. Calculate (a) moment of inertia (b) its kinetic energy when it is rotating at 70 r.p.m.

Solution:

(a) M.I. of the flywheel $= \dfrac{MR^2}{2}$

$$= \frac{72 \times (0.5)^2}{2} = \frac{72 \times 0.5 \times 0.5}{2} = 9 \text{ kg–m}^2$$

(b) K.E. of rotation

$$= \frac{1}{2} I \omega^2 = \frac{1}{2} \times 9 \times \left(2\pi \times \frac{70}{60}\right)^2 = 241.8 \text{ J}$$

Problem 1.10

A hollow sphere of radius r_1 and a solid sphere of radius r_2 have some masses and moments of inertia. Find the ratio of their radii.

Solution:

Assuming hollow sphere to be of negligibly thin wall, it is equivalent to a spherical shell.

Moment of inertia of spherical shell $I_1 = \frac{2}{3} M_1 r_1^2$

Moment of inertia of solid sphere $I_2 = \frac{2}{5} M_2 r_2^2$

Given $M_1 = M_2 = M$ (say) $I_1 = I_2$

$$\therefore \qquad \frac{2}{3} Mr_1^2 = \frac{2}{5} Mr_2^2$$

$$\frac{r_1^2}{r_2^2} = \frac{3}{5} \Rightarrow \frac{r_1}{r_2} = \sqrt{\frac{3}{5}}$$

Problem 1.11

A wheel in the form of uniform disc of diameter one metre and mass 100 kg is rolling on a horizontal plane with a speed of 5 metre per second. Find the kinetic energy of the wheel.

Solution:

The kinetic energy of the wheel is partly due to translatory motion and partly due to rotary motion

Kinetic energy of translation $= \frac{1}{2} mv^2$

$$= \frac{1}{2} \times 100 \times 5^2 = \frac{1}{2} \times 100 \times 5 \times 5 = 1{,}250 \text{ joule}$$

Kinetic energy of rotation $= \frac{1}{2} I \omega^2$

$$I = \frac{MR^2}{2} = \frac{100 \times 0.5^2}{2} = \frac{100 \times 0.5 \times 0.5}{2} = \mathbf{12.5\ kg \cdot m^2}$$

$$\omega^2 = \frac{v^2}{r^2} = \frac{25}{0.5^2} = \frac{25}{0.5 \times 0.5} = 100$$

K.E. of rotation $\frac{1}{2} I\omega^2 = \frac{1}{2} \times 12.5 \times 100 = 625$ joule

Total K.E. = 1250 + 625 = 1175 joule

Problem 1.12

Find the rotational kinetic energy of a ring of mass 9 kg and radius 3 m rotating with 240 rpm about an axis passing through its centre and perpendicular to its plane. (rpm is a unit of speed of rotation which means revolutions per minute).

Solution:

The rotational kinetic energy is, KE $= \frac{1}{2} I \omega^2$

The moment of inertia of the ring is, $I = MR^2$

$$I = 9 \times 3^2 = 9 \times 9 = 81 \text{ kg m}^2$$

The angular speed of the ring is,

$$\omega = 240 \text{ rpm} = \frac{240 \times 2\pi}{60} \text{ rads}^{-1}$$

$$\text{KE} = \frac{1}{2} \times 81 \times \left(\frac{240 \times 2\pi}{60}\right)^2 = \frac{1}{2} \times 81 \times (8\pi)^2$$

$$KE = \frac{1}{2} \times 81 \times 64 \times (\pi)^2 = 2592 \times (\pi)^2$$

$$KE \approx 25920 \text{ J} \quad \because (\pi)^2 \approx 10$$

$$KE = 25.920 \text{ kJ}$$

Problem 1.13

A constant torque of 400 N-m turns a wheel about its centre. The moment of inertia of wheel about this axis is 20 kg–m^2. Calculate (i) angular acceleration (ii) angular velocity and the KE gained in 2s starting from rest.

Solution:

(i) Since $\tau = I\alpha$

$$\alpha = \frac{\tau}{l} = \frac{400}{20}$$

$$\mathbf{m}\alpha = 20 \, rad \, s^{-2}$$

(ii) Using $\omega = \omega_0 + \alpha t$

Assuming $\omega_0 = 0$, $\omega = \alpha t = 20 \times 2 =$ **40 rad s^{-1}**

(iii) Gain in $KE = \frac{1}{2} I \omega^2 - \frac{1}{2} \omega_0^2$

Assuming $\omega_0 = 0$

$$\text{Gain in KE} = \frac{1}{2} I \omega^2 = \frac{1}{2} (20) (40)^2 = 1.6 \times 10^4 \text{ J}$$

Problem 1.14

Calculate the angular momentum of a disc whose rotational kinetic energy is 10 kJ and moment of inertia about its axis of rotation is 4.5×10^{-4} kg–m^2.

Solution:

$$E = \frac{1}{2} I \omega^2, \quad L = I \omega$$

$$2E = I \omega^2$$

Multiply both sides by I

$$2IE = I^2 \omega^2$$

$$I \omega = \sqrt{2IE}$$

therefore, $L = \sqrt{2IE}$

$$= \sqrt{2\,(4.5 \times 10^{-4})\,(10 \times 10^3)}$$

$$\mathbf{L = 3.0\ kg\,m^2\,s^{-1}}$$

Problem 1.15

A small object of mass 20 kg is rotating in a circle of diameter 0.20 m at the rate of 2000 revmin. Find the rotational kinetic energy of the object.

Solution:

Here $M = 20$ kg, radius $r = \dfrac{0.02}{2} = -\,0.10$ km

$\therefore$ Moment of inertia, $I = MR^2$

$$= 20 \times (0.10)^2$$

$$= 0.20 \text{ kg} - \text{m}^2$$

Angular velocity $\omega = 2\pi n = 2\pi \times \left(\frac{2000}{600}\right)$ rad/s

$\therefore$ Rotational kinetic energy $K_{rot} = \frac{1}{2} I \omega^2$

$$K_{rot} = \frac{1}{2} \times (0.20) \times \left(\frac{2\pi \times 2000}{60}\right)^2$$

$$= 0.10 \times \frac{16}{36} \pi^2 \times 10^4 \text{ Joule}$$

$$= 444\,\pi^2 \text{ Joule} = 444 \times 9.8 = 4382 \text{ Joule}$$

Problem 1.16

A body of mass 20 kg is rotating in a circular path of diameter 0.20 m at the rate of 100 revolutions in 3 second. Find
(i) Rotational kinetic energy of the body
(ii) Angular momentum of the body ($\pi^2 = 9.86$)

Solution:

Mass of body $M = 20$ kg

radius of circular path, $r = \frac{0.20}{2} = 0.10$ m

No. of revolutions per second $n = \frac{100}{3}$ rev/sec.

$\therefore$ Angular velocity $\omega = 2\pi n = 2\pi \times \frac{100}{3} = \frac{200}{3} \pi$ rad / s

Moment of inertia of body,

$$I = MR^2 = 20 \times (0.10)^2 = 0.20 \text{ kg–m}^2$$

(i) Rotational kinetic energy, $K = \frac{1}{2} I \omega^2$

$$= \frac{1}{2} \times 0.20 \times \left(\frac{200}{3}\pi\right)^2$$

$$= 0.10 \times \left(\frac{200}{3}\right)^2 \pi^2 = \frac{0.10 \times 200 \times 200}{9} \times 9.86$$

$$= 4.382 \times 10^3 \text{ Joule}$$

(ii) Angular momentum of body, $J = I\omega$

$$= 0.20 \times \frac{200}{3}\pi = \frac{0.20 \times 200 \times 3.14}{3}$$

$$= \mathbf{41.87\ kg-m^2/s}$$

Problem 1.17

A mass of 2 kg is rotating in a circular path of radius 0.8 m with an angular velocity of 44 rad/s. If the radius of path becomes 1.0 m find the new angular velocity.

Solution:

The angular momentum J of mass is conserved

i.e. $J = I\omega = \text{constant}$

i.e. $I_1 \omega_1 = I_2 \omega_2$

or $\omega_2 = \frac{I_1 \omega_1}{I_2}$

Here $I_1 = mr_1^2 = 2 \times (0.8)^2 = 1.28 \text{ kg–m}^2, \omega_1 = 44 \text{ rad/s}$

$$I_2 = mr_2^2 = 2 \times (1.0)^2 = 2 \text{ kg–m}^2$$

$\therefore$ New angular velocity, $\omega_2 = \frac{1.28 \times 44}{2} = 28.16 \text{ rad/s}$

Problem 1.18

A solid cylinder of mass 25 kg rotates about its axis with angular speed 125 s^{-1}. The radius of the cylinder is 0.5 m. What is the kinetic energy associated with the rotation of the cylinder? What is the magnitude of angular momentum of the cylinder about its axis?

Solution:

Mass of cylinder $M = 25$ kg

angular speed $\omega = 125\ s^{-1}$

radius of cylinder $R = 0.5$ m

$\therefore$ Moment of inertia of the cylinder about its axis

$$I = \frac{1}{2}MR^2 = \frac{1}{2} \times 25 \times (0.5)^2$$

$$= 3.125 \text{ kg m}^2$$

$$\therefore \text{ K.E. of rotation, } = \frac{1}{2} I \omega^2$$

$$= \frac{1}{2} \times 3.125 \times (125)^2$$

$$= 2.44 \times 10^4 \text{ J}$$

Angular momentum, $L = I\omega$

$$= 3.125 \times 125$$

$$= 390.6 \text{ Js}$$

Problem 1.19

A circular metal hoop of mass 100 g and radius 10 cm rotates about its centre at the rate of 10 rotations per second. The axis of rotation is normal to the plane of the hoop. Find
(a) Moment of inertia about the axis of rotation.
(b) Torque required to increase in 10 s the rate of rotation from 10 to 20 per second.

Solution:

(a) $I = MR^2$

$= 0.1 \times 0.1^2$

$= 1 \times 10^{-3}$ kg–m^3

(b) Initial angular momentum

$$L_1 = I\,\omega_1$$

$$= (1 \times 10^{-3}) \times (2\pi \times 10)$$

$$= 2\pi \times 10^{-2} \text{ J–s}$$

Final angular momentum

$$L_2 = I\,\omega_2$$

$$= (1 \times 10^{-3}) \times (2\pi \times 20)$$

Change in angular momentum

$$= L_2 - L_1$$

$$= 4\pi \times 10^{-2} - 2\pi \times 10^{-2}$$

$$= 2\pi \times 10^{-2} \text{ J–s}$$

$$\therefore \text{ Torque } = \frac{L_2 - L_1}{t} = \frac{2\pi \times 10^{-2}}{10}$$

$$= 2\pi \times 10^{-3} \text{ N–m}$$

Problem 1.20

A sphere of mass 50 g and radius 8 cm is revolving about an axis in a circular path. The distance of the centre of mass of the body from the axis of rotation is 50 cm. Find the moment of inertia of the body.

Solution:

Moment of inertia of the spherical body about in axis passing through its centre of mass, is given by

$$I_c = \frac{2}{5} MR^2 = \frac{2}{5} (50 \times 10^{-3})(8 \times 10^{-2})^2$$

$$= 1.28 \times 10^{-4} \text{ kg–m}^2$$

$$I = I_c + Ma^2$$

$$= 1.28 \times 10^{-4} + (50 \times 10^{-3})(50 \times 10^{-2})^2$$

$$= 1.28 \times 10^{-4} + 125 \times 10^{-4}$$

$$= 1.26 \times 10^{-2} \text{ kg–m}^2$$

Note: If the size of the body is very small then only the body can be treated as a point mass.

Problem 1.21

The angular velocity of a flywheel decreases uniformly from 600 rpm to 300 rpm in 5 s. Find (a) angular acceleration, (b) the number of revolutions made by the wheel in the 5s interval, (c) How many more seconds are required for the wheel to come to rest, (d) if the moment of inertia of the flywheel is 0.5 kg m², what torque was acting?

Solution:

(a) Using $a = \dfrac{\omega_2 - \omega_1}{t}$ with $\omega_1 = \dfrac{2\pi \times 600}{60} = 20\pi \text{ rad s}^{-1}$

and $\omega_2 = \dfrac{2\pi \times 300}{60} = 10\pi \text{ rad s}^{-1}$ we have

$$= \frac{10\pi - 20\pi}{5} = -2\pi$$

$$= -\ 6.28 \text{ rad s}^{-2}$$

(b) Let the number of revolutions done be n. Since the angle turned in 5s is $2\pi n$ rad on using the relation

$$0 = \omega_1 t + \frac{1}{2}\alpha t^2, \text{ we have}$$

$$2\pi n = 20\pi \times 5 - \frac{1}{2} \times 2\pi \times 5^2$$

$$n = 37.5 \text{ rev}$$

(c) Suppose the wheel takes another t seconds to come to rest from the angular speed of $10\pi \text{ rad s}^{-1}$, then on using $\omega_2 = \omega_1 + \alpha t$, we have

$$0 = 10\pi - 2\pi t$$

$$t = 5s$$

(d) Using $\tau = I\alpha$ with $I = 0.5$ kg m^2 and $\alpha = -6.28$ rad s^{-2}, we have find that the retarding torque is

$$\tau = 2 \times 0.5 \times 6.28$$

$$= 6.28 \text{ N–m}$$

Problem 1.22

The angular momentum of rotation of a body 1s 75.36 Js and its rate of rotation is 24 rev per s. Calculate its moment of inertia.

Solution:

$$L = I\omega$$

Here $L = 75.36\ Js$

$$\omega = 2\pi n = (2\pi)(24) \text{ rad s}^{-1}$$

therefore, $I = \dfrac{L}{\omega} = \dfrac{75.36}{12 \times 3.14 \times 24}$

$$= 0.5 \text{ kg m}^2$$

Problem 1.23

A torque of 75 N-m applied to the rotor of a generator imparts an angular acceleration of 10 rad s^{-2}. What is the moment of inertia of the wheel?

Solution:

$$\tau = I\alpha$$

Here $\tau = 75$ N.m, $\alpha = 10$ rad s^{-2}

therefore $I = \dfrac{\tau}{\alpha} = \dfrac{75}{10}$

$$I = 7.5 \text{ kg–m}^2$$

Problem 1.24

A fly wheel of moment of inertia 0.3 kg–m^3 is rotating at the rate of 200 rpm. What torque is needed to bring it at rest in 10 s?

Solution:

$$\text{Here } \omega_o = \frac{2\pi n}{t}$$

$$= \frac{2 \times 3.14 \times 200}{60}$$

$$= 20.93 \text{ rad/s}$$

$$\omega = 0, \; t = 10 \, s$$

$$\alpha = \frac{\omega - \omega_p}{t}$$

$$= \frac{0 - 20.93}{10}$$

$$= -2.093 \text{ rad s}^{-2}$$

The torque τ required to produce a retardation of 2.093 rad s^{-2} is given by

$$\tau = I\alpha = (0.3)(2.093)$$

$$= 0.63 \text{ N–m}$$

Problem 1.25

Consider the earth as a uniform sphere of mass 5.98 × 10^{24} kg and radius 6.37 × 10^6 m spinning on its axis at the rate of per day. Find the angular momentum associated with this spinning

Solution:

$$\text{M.I. of the earth about diameter} = \frac{2}{5} MR^2$$

$$= \frac{2}{5} \times 5.98 \times 10^{24} \times (6.37 \times 10^6)^2 \text{ kg–m}^2$$

$$\text{Angular velocity } \omega = \frac{2\pi}{24 \times 60 \times 60} \text{ rad s}^{-1}$$

$$\text{Angular momentum} = I\,\omega$$

$$= \frac{2}{5} \times 5.98 \times 10^{24} \times 10^6)^2 \times \frac{2\pi}{24 \times 60 \times 60}$$

$$= 7.055 \times 10^{33} \text{ Nms}^{-1} \text{ (Js}^{-1})$$

Problem 1.26

A grindstone is in the force form of a circular disc of diameter 0.5 m and of mass 10 Kg. What constant torque should be applied so that if attains an angular velocity of 120 rpm in 4 seconds. At what rate is work done by torque at the end of 4 seconds?

Solution:

Angular velocity after 4 secs $\omega = 4\pi$ radians / sec.

We have, $\omega = \omega_0 + \alpha t$

$$\alpha = \pi \text{ radian / sec}^2$$

$$\tau = I \cdot \alpha$$

$$= \frac{MR^2}{2} \times \alpha = \frac{10}{2} \times \frac{(5)^2}{4} \times \pi$$

$$= \frac{5 \times .25 \times \pi}{4}$$

$$= \frac{1.25}{4\pi} = 0.3125\,\pi \text{ Nm}$$

The rate at which the work is done

$$= \tau \times \omega = \frac{1.25}{4}\pi \times 4\pi$$

$$= 1.25\pi^2 \text{ watts}$$

Problem 1.27

A constant torque of 200 Nm acting on a wheel at rest rotates in through an angle of 400 radians in 10 s. Calculate (a) the angular acceleration (b) moment of inertia of the wheel. If the same torque continues to act, what would be angular velocity after 20 seconds?

Solution:

Let α be the angular acceleration

$$\theta = \omega_o t + \frac{1}{2}\alpha t^2$$

$$\theta = 400 \text{ radians, t} = 10 \text{ sec}, \ \omega_o = 0$$

$$400 = \frac{1}{2}\alpha \times 10^2$$

$$\alpha = \frac{400 \times 2}{100}$$

$$\alpha = 8 \text{ radian / sec}^2$$

$$\tau = I\,\alpha$$

$$I = \frac{T}{\alpha} = \frac{200}{8} = 25 \text{ kg m}^2$$

Angular velocity after 20 sec. is

$$\omega = \omega_0 + \alpha t$$

$$= 0 + 8 \times 20$$

$$= 160 \text{ radian / sec}$$

Part - 'A'
'2' Marks Questions with Answers

1. Define multiparticle dynamics.

The study of dynamics of a system which consists of two or more particle is known multiparticle dynamics.

2. Define centre of mass of the system.

Consider the motion of a system consisting of a large number of particles. There is one point in it which behaves as though the entire mass of the system were concentrated there and all the external forces were acting at this point. This point is called the centre of mass of the system.

3. What is centre of mass (CM)?

A point in the system at which whole mass of the body is supposed to be concentrated is called centre of mass of the body.

4. Give the example for motion of centre of mass.

Examples for motion of centre of mass

(i) Motion of planets and its satellite

(ii) Projectile Trajectory

(iii) Decay of a Nucleus

5. How centre of mass is determined for rigid body and regular shape?

Centre of mass of some regular objects.

- For a rigid body, the centre of mass is a point at a fixed position with respect to the body as a whole. Depending on the shape of the body and the way the mass is distributed in it, the centre of mass is a point may or may not be within the body.

- If the shape is symmetrical and the mass distribution is uniform, we can usually find the location of the centre of mass quite easily.
- For a long thin rod of uniform cross section and density, the centre of mass is at the geometrical centre.
- For a thin circular plane ring, It is again at the geometrical centre of the circle.
- For a rectangle, again the centre of mass is at the geometrical centre.

6. What is difference between centre of gravity and centre of mass?

- The centre of gravity of a body is a point, where the whole weight of the body supposed to be concentrated.
- The centre of mass of a body is that point, where the whole mass of the body is supposed to be concentrated.

For uniform geometrically shaped, bodies the centre of gravity coincides with centre of mass. However, they do not coincide in bodies whose density is not uniform throughout.

7. Define rigid body.

A rigid body is defined as that body which does not undergo any change in shape or volume when external forces are applied on it.

8. Define rigid body rotation.

When a body rotates about a fixed axis, its motion is known as rotatory motion.

A rigid body is said to have pure rotational motion, if every particle of the body moves in a circle, the centre of which lies on a straight line called the axis of rotation (Fig).

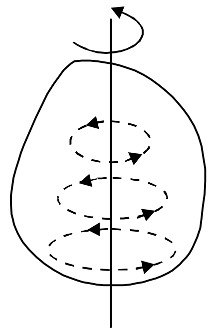


Fig. Rotational motion

9. Write down the equation of motion for rotational motion.

$$\omega = \omega_0 + \alpha t \quad \text{... (1)}$$

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2 \quad \text{... (2)}$$

$$\omega^2 = \omega_0^2 + 2\alpha\theta \quad \text{... (3)}$$

10. Define moment of inertia of a body.

The property of a body by which it resists change uniform rotational motion is called rotational inertia or moment of inertia.

11. Define moment of inertia of a particle

The moment of inertia of a particle about an axis is defined as the product of the mass of the particle and square of the distance of the particle from the axis of rotation.

If 'm' is the mass of the particle and 'r' is the distance of the particle from the axis of rotation, then

The moment of inertia of the particle

$$\boxed{I = mr^2}$$

12. Define moment of inertia of a rigid body.

The moment of inertia of a rigid body about a given axis is the sum of products of masses of its particles and the square of their respective distances from the axis of rotation.

13. What factors the moment of inertia depends?

Moment of inertia depends on mass, distribution of mass and on the position of axis of rotation.

14. What are the physical significance of moment of inertia?

The property which opposes the change in rotational motion of the body is called the moment of inertia. Greater is moment of inertia of the body about the axis of rotation, greater is the torque required to rotate the body.

Thus it is clear that the moment of inertia of a body has the same role in rotational motion as that of mass (or inertia) is linear motion.

15. What is radius of gyration?

The radius of gyration is defined as the distance from the axis of rotation to the point where the entire mass of the body is assumed to be concentrated.

K is called the Radius of Gyration of the body about the axis of rotation. It is equal to the root mean square distance of all particles from the axis of rotation of the body.

16. What are the theorems on moment of inertia?

There are two important theorems which help to find the moment of inertia of a body about some other axis if moment of inertia about any symmetrical axis of the body is given. These are called theorem of parallel and perpendicular axes.

They are

1. Parallel axes theorem and
2. Perpendicular axes theorem

17. State parallel axis theorem.

The moment of inertia of a body about any axis is equal to the sum of its moment of inertia about a parallel axis passing through its centre of gravity of the body and the product of its mass of the body with the square of the distance between the two axes.

18. State perpendicular axis theorem.

It states that the moment of inertia of a plane lamina about an axis perpendicular to its plane is equal to the sum of the moments of inertia of the plane lamina about any two mutually perpendicular axes in its own plane and intersecting each other at the point where the perpendicular axis passes through it.

19. Define angular momentum.

Angular momentum of a particle is defined as its moment of linear momentum it is given by the product of linear momentum and perpendicular distance of its line of action from the axis of rotation. It is denoted by $\vec{L}$.

20. Define torque.

The moment of the applied force is called torque. It is represented by the symbol 'τ'.

If F is the force acting on a body at a distance r then,

Torque = Force × distance

i.e., $\vec{\tau} = \vec{F} \times \vec{r}$

The rotational motion is due to only when the torque acts on the body.

21. State conversation of angular momentum.

The law of conservation of angular momentum states that in the absence of an external torque, the angular momentum of a body or a system of bodies remains conserved.

22. What is gyroscope?

A gyroscope is a device used for measuring or maintaining orientation and angular velocity. It is a spinning wheel or disc in which the axis of rotation (spin axis) is free to assume any orientation by itself.

23. What are the uses of gyroscope?

1. In view of the property of stability, the gyroscope are used as stabilizers in ships, boats and aeroplanes.
2. Due to the inherent stability of the gyroscope, it used as a compass, and a gyro-compass is preferable to the magnetic compass in many respects.
3. Another important application of the directional stability of a rapidly spinning (rotating) body is the rifling of the barrels of the rifles.

24. What is torsional pendulum?

A circular metallic disc suspended using a thin wire that executes torsional oscillation is called torsional pendulum.

25. What are the uses of torsional pendulum?

Torsional pendulum is used to determine

1. Rigidity modulus of the wire
2. Moment of inertia of the disc
3. Moment of inertia of an irregular body.

26. What is double pendulum?

A double pendulum is a pendulum with another pendulum attached to its end.

The pendulum behaves like a linear system for small angles. When the angles are small in the double pendulum, the system behaves like the linear double spring. In this case, the motion is determined by simple sine and cosine functions.

On the otherhand for large angles, the pendulum is non-linear and the phase graph becomes much more complex.

Part - B '16' Marks Questions

1. Define the centre mass of a system of particles. Derive the expression for it.
2. Discuss of centre of mass of continuous bodies.
3. Explain the motion of the centre of mass.
4. Derive an expression for kinetic energy of system of particles.
5. Derive the equation of rotational motion about fixed axis.
6. Derive the relation between rotational kinetic energy and moment of inertia.
7. Derive an expression for the rotational kinetic energy of a rigid body rotating about a fixed axis with an angular velocity ω.
8. State and prove the theorem of parallel axes for the moment of inertia of a rigid body.
9. State and prove the theorem of perpendicular axis for the moment of inertia of a plane lamina.
10. Derive an expression for the moment of inertia of a uniform rod.

    (i) About an axis through its centre and perpendicular to its length.

    (ii) About an axis passing through on end of the rod and perpendicular to its length.
11. Derive an expression for the moment of inertia of thin ring.

    (a) About an axis through its centre and perpendicular to its plane.

    (b) about a diameter.

    (c) about a tangent in the plane of the ring.

12. Derive an expression for the moment of inertia of a thin circular disc
    (a) about an axis through its centre and perpendicular to its plane.
    (b) about a diameter.
13. Derive an expression for the moment of inertia of a solid sphere.
    (a) about diameter
    (b) about a tangent
14. Derive an expression for the moment of inertia of a solid cylinder.
    (a) about an axis passing through the centre and perpendicular to its length
    (b) about the axis of cylinder.
15. Discuss the moment of inertia of a diatomic molecule.
16. Derive an expression for angular momentum of a rigid body.
17. Explain conservation of angular momentum with examples.
18. Discuss the rotational energy states of a rigid diatomic molecule.
19. Describe principle, construction and working of gyroscope. Mention its application in various fields.
20. Derive an expression for time period of torsion pendulum. Explain how it is used to find rigidity modulas of a wire.
21. Write notes on
    (i) double pendulum
    (ii) non-linear oscillations

Problem for Practice

1. A fly wheel of mass 500 kg. and diameter 2 metre makes 500 revolutions per minute. Assuming that whole of the mass is concentrated at the rim, find its angular velocity and moment of inertia. ***[Ans. 52.4 rad / sec,*** $500 \text{ kg} \times \text{m}^2$***]***

2. Five masses each of 0.2 kg are placed on horizontal circular disc of negligible mass which can rotate about a vertical axis passing through its centre. If all the masses be equidistant from the axis and at a distance 5 cms from it; find the moment of inertia of the system.

   [Ans. $25 \times 10^{-4} \text{ kg} \times \text{m}^2$***]***

3. A ring of mass 5 kg, and radius 20 cms. rolls along the ground at the rate of 10 metres per second. Calculate it kinetic energy. ***[Ans. 500 joule]***

4. Calculate the kinetic energy of uniform circular disc of mass 1 kg and diameter 0.1 m, making 100 revolution per minute about its axis. ***[Ans.*** 6.7875×10^{-2} ***joules]***

5. A solid spherical ball rolls on a table. Determine the faction of its total kinetic energy that is rotational. ***[Ans. 2/7]***

6. A solid sphere of mass 0.1 kg and diameter 0.025 m rolls without slipping with a velocity 0.1 m/sec. Calculate its total kinetic energy in joules. ***[Ans.*** 7×10^{-4} ***joules]***

7. Deduce the moment of inertia of HCl molecule about an axis passing through its centre of mass and perpendicular to the bond length. Given internuclear distance = 1.3 Å.

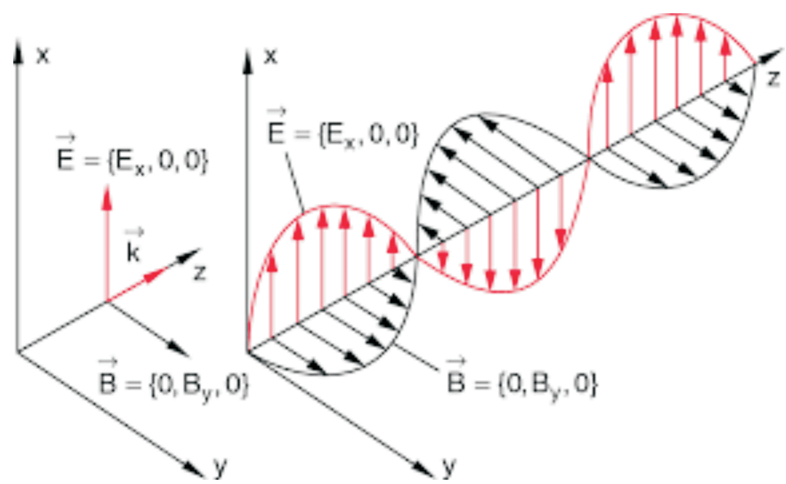
   [Ans. $2.79 \times 10^{-47} \text{ kgm}^2$***]***

8. Calculate the moment of inertia of a uniform disc of mass 0.2 kg and radius 5 cm about an axis passing through its edge and perpendicular to the plane of the disc.

   [Hint. $I = \frac{1}{2} MR^2 + MR^2 = \frac{3}{2} MR^2$

   [Ans. $7.5 \times 10^{-4} \text{ kg} \cdot \text{m}^2$***]***

Unit - II



Electromagnetic Waves

2. Electromagnetic Waves

The Maxwell's equations – wave equation; Plane electromagnetic waves in vacuum, Conditions on the wave field – properties of electromagnetic waves; speed, amplitude, phase, orientation and waves in matter – polarization – Producing electromagnetic waves – Energy and momentum in EM waves. Intensity, waves from localized sources, momentum and radiation pressure – Cell-phone reception. Reflection and transmission of electromagnetic waves from a non-conducting medium-vacuum interface for normal incidence.

Introduction

- The phenomenon of Faraday's electromagnetic induction concludes that a changing magnetic field at a point with time produces an electric field at that point.
- Maxwell in 1865, pointed out that there is a symmetry in nature (i.e) changing electric field with time at a point produces a magnetic field at that point. It means that a change in one field with time (either electric or magnetic) produces another field.
- This idea led Maxwell to conclude that the variation in electric and magnetic fields perpendicular to each other, produces electromagnetic disturbances in space. These disturbances have the properties of a wave and propagate through space without any material medium. These waves are called electromagnetic waves.
- Maxwell unified the theories of electricity and magnetism by way of deducing four very important equations which combine the experimental observations reported by Gauss, Ampere, and Faraday with his concept of displacement current.

2.1 MAXWELL'S EQUATIONS

Faraday laid the foundation of electromagnetism. He suggested the possibility of the propagation of electromagnetic waves through space. The idea was later confirmed and fully developed by **Maxwell** who expressed the laws of electromagnetism in terms of fundamental equations.

He showed that electromagnetic waves can be produced by changing electric and magnetic fields. Maxwell's equations form the foundation of electromagnetic theory.

Maxwell's Equations (Derivation)

There are four Maxwell's equations in electromagnetic theory. The first two equations are known as **steady state equations** and the last two equations are known as **time varying equations**

Maxwell's Equation - I
(From Gauss's law in electrostatics)

Integral form

Gauss's law in electrostatics states that the total electric flux through any closed surface is equal to the charge enclosed by it (Fig. 2.1).

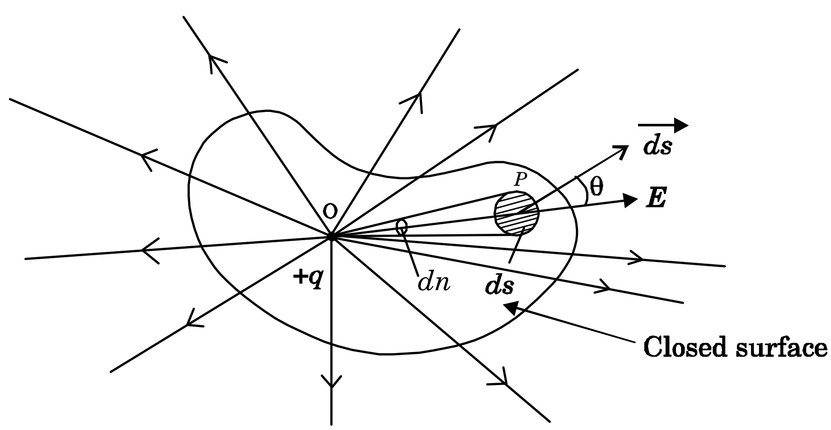


Fig. 2.1

According to Gauss law,

$$\oiint_S \vec{E} \bullet \vec{ds} = \frac{q}{\varepsilon} \quad \text{... (1)}$$

$$\oiint_S \varepsilon \vec{E} \bullet \vec{ds} = q \qquad (\because \vec{D} = \varepsilon \vec{E})$$

$$\oiint_S \vec{D} \bullet \vec{ds} = q \quad \text{... (2)}$$

Now, if ρ be the charge density, (charge per unit volume) then total charge inside the closed surface is given by

$$q = \iiint_V \rho \, dV \quad \text{... (3)}$$

Substituting eqn (3) in eqn (2), we have

$$\boxed{\oiint_S \vec{D} \bullet \vec{ds} = \iiint_V \rho \, dV} \quad \text{... (4)}$$

This is Maxwell's equation in integral form from Gauss law in electrostatics.

Applying Gauss's divergence theorem to LHS of eqn (4), we get

$$\text{ie.,} \quad \oiint_S \vec{D} \bullet \vec{ds} = \iiint_V \vec{\nabla} \bullet \vec{D} \, dV \quad \text{... (5)}$$

on substituting eqn (5) in eqn (4), we get

$$\iiint_V \vec{\nabla} \bullet \vec{D} \, dV = \iiint_V \rho \, dV \quad \text{... (6)}$$

or

$$\boxed{\vec{\nabla} \bullet \vec{D} = \rho}$$

$$\vec{\nabla} \bullet \varepsilon_o \vec{E} = \rho \qquad (\because D = \varepsilon_o \vec{E})$$

$$\varepsilon_o \vec{\nabla} \bullet \vec{E} = \rho$$

$$\vec{\nabla} \bullet \vec{E} = \frac{\rho}{\varepsilon_o} \qquad \dots (7)$$

This is Maxwell's equation from Gauss's law in electrostatics in differential form

Statement

The total electric displacement through the surface enclosing a volume is equal to the total charge within the volume

Maxwell's Equation - II
(From Gauss's law in magnetostatics)

Integral form

It is a well known fact in magnetism that the magnetic lines of force are continuous and do not appear to have the origin or the end. Thus the total magnetic flux through any closed surface in a magnetic field is zero ie.,

$$\oiint_S \vec{B} \bullet \vec{ds} = 0 \qquad \dots (1)$$

This is Maxwell's equation in integral form from Gauss's law in magnetostatics.

Applying Gauss divergence theorem to the L.H.S. of equation (1), We get

$$\oiint_S \vec{B} \bullet \vec{ds} = \iiint_V \vec{\nabla} \bullet \vec{B} \, dV \qquad \dots (2)$$

on substituting eqn (2) in eqn (1), we have

$$\iiint_V \vec{\nabla} \bullet \vec{B}\, dV = 0 \qquad \dots (3)$$

or

$$\boxed{\vec{\nabla} \bullet \vec{B} = 0} \qquad \dots (4)$$

This is Maxwell's equation in differential form from Gauss's law in magnetostatics.

Statement

The net magnetic flux emerging through any closed surface is zero.

Maxwell's Equation - III
(From Faraday's law)

Let us consider C be a closed circuit and S be the surface with C as its boundary (Fig. 2.2). Let B be the magnetic flux density inside the closed circuit.

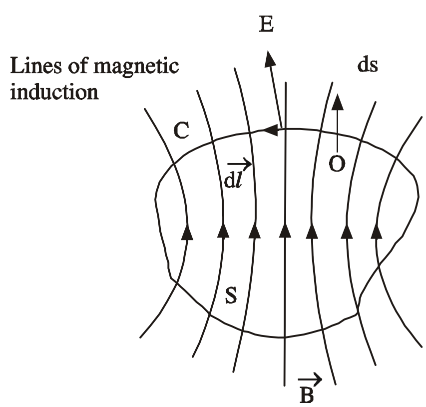


Fig. 2.2

Magnetic flux through a small area $ds = \vec{B} \bullet \vec{ds}$ ... (1)

$\therefore$ Total magnetic flux linked with the circuit $\Big\} = \phi_B = \oint\!\!\oint_S \vec{B} \bullet \vec{ds}$ ... (2)

Faraday's law states that the induced emf e is the rate of change of magnetic flux ϕ_B

$$\therefore \; e = -\frac{d\phi_B}{dt} = -\frac{d}{dt}\left[\oiint_S \vec{B} \bullet \vec{ds}\right] \quad \text{... (3)}$$

$$= \oiint \frac{\partial \vec{B}}{\partial t} \bullet \vec{ds}$$

If $\vec{E}$ be the electric field strength, then

We know that $\vec{E} = \frac{dV}{dl}$

$$dV = \vec{E} \bullet \vec{dl}$$

$$\therefore \; V = \int dV = \int \vec{E} \bullet \vec{dl}$$

$$V = e = \int \vec{E} \bullet \vec{dl}$$

$$e = \oint \vec{E} \bullet \vec{dl} \quad \text{... (4)}$$

Here, the integral is taken over a closed curve C.

Equating eqn (3) and eqn (4), we have

$$\boxed{\oint_C \vec{\boldsymbol{E}} \bullet \vec{\boldsymbol{dl}} = -\iint_S \frac{\partial \vec{\boldsymbol{B}}}{\partial \boldsymbol{t}} \bullet \vec{\boldsymbol{ds}}} \quad \text{... (5)}$$

This is Maxwell's equation in integral form from Faraday's law of electromagnetic induction.

Now, applying Stoke's theorem to L.H.S. of eqn (5), we have

$$\oint_C \vec{\boldsymbol{E}} \bullet \vec{\boldsymbol{dl}} = \iint_S (\vec{\nabla} \times \vec{\boldsymbol{E}}) \bullet \vec{\boldsymbol{ds}} \quad \text{... (6)}$$

On substituting the eqn (6) in eqn (5), we get

$$\iint_S (\vec{\nabla} \times \vec{E})\; \vec{ds} = -\iint_S \frac{\partial \vec{B}}{\partial t} \bullet \vec{ds} \quad \text{... (7)}$$

Since this must be true for all surfaces S, it follows that

$$\boxed{\vec{\nabla} \times \vec{\boldsymbol{E}} = -\frac{\partial \vec{\boldsymbol{B}}}{\partial \boldsymbol{t}}} \qquad \dots (8)$$

Eqn. (8) represents Maxwell's equation from Faraday's law of electromagnetic induction in differential form.

Statement:

The electromotive force around a closed path is equal to the rate of magnetic displacement (flux density) through that closed path.

Conduction current density

This is the current due to flow of electrons through the resistance in the circuit obeying ohm's law

We know that

$$V = I_C R \quad \text{or} \quad I_C = \frac{V}{R} \qquad \dots (1)$$

But $$R = \frac{\rho\, l}{A}$$

$$R = \frac{l}{\sigma A} \qquad \dots (2)$$

where ρ – resistivity of the conductor

l – length of the conductor

A – area of cross-section of conductor

σ – Conductivity of conductor $\left(\sigma = \frac{1}{\rho}\right)$

Substituting for R from eqn (2) in eqn (1), we have

$$I_C = \frac{V}{\frac{l}{\sigma A}}$$

$$I_C = \frac{V \sigma A}{l} \quad \text{... (3)}$$

$$\frac{I_C}{A} = \frac{V \sigma}{l}$$

$$\boxed{\vec{J}_C = \sigma \vec{E}} \quad \text{... (4)}$$

$$\left(\because J_C = \frac{I_C}{A} \text{ and } E = \frac{V}{l} \right)$$

Here, J_C represents the conduction current density

Displacement current density

It is the current that exists across the capacitor in the circuit. When alternating field is applied to the parallel plate capacitor, no charge motion takes place (Fig 2.3).

It is due to vacuum or dielectric medium in capacitor but exchange (displacement) of charge takes place inside the capacitor.

It results in the existence of current on the surface of capacitor. This current is called **displacement current** in the capacitor.

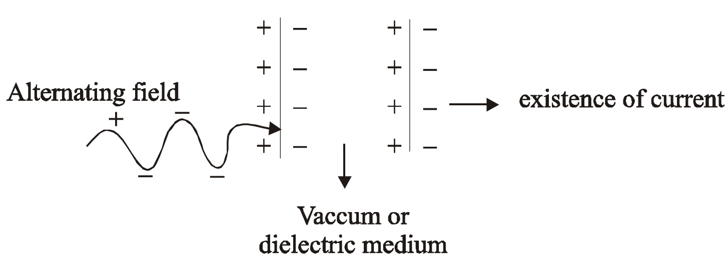


Fig. 2.3

$$I_D = \frac{dq}{dt}$$

But $\quad q = CV$

C – Capacitance of capacitor

V – Potential difference

$$\therefore \quad I_D = \frac{d}{dt}(CV) = C\frac{dV}{dt} \quad \text{... (1)}$$

The capacitance of parallel plate capacitor is given by

$$C = \frac{\varepsilon A}{d} \quad \text{... (2)}$$

where ε – Permittivity of the medium

A – Area of the parallel plate capacitor

d – Distance between two plates

Substituting for C from eqn (2) in eqn (1), we get

$$I_D = \frac{\varepsilon A}{d}\frac{dV}{dt} \quad \text{... (3)}$$

But $\quad V = Ed \quad \left(\because E = \frac{V}{d}\right) \quad$ ... (4)

substituting eqn (4) in eqn (3), we have

$$I_D = \frac{\varepsilon A}{d}\frac{d}{dt}(\vec{E}d)$$

$$I_D = \frac{\varepsilon A}{d}d\frac{d\vec{E}}{dt}$$

$$I_D = \varepsilon A\frac{d\vec{E}}{dt} \quad \text{... (5)}$$

$$\frac{I_D}{A} = \varepsilon \frac{d\vec{E}}{\partial t}$$

$$\vec{J}_D = \varepsilon \frac{d\vec{E}}{\partial t} \qquad \left(\because \frac{I_D}{A} = J_D \right)$$

where J_D displacement current density

$$\text{or} \qquad \vec{J}_D = \frac{d\,\varepsilon\vec{E}}{\partial t} \qquad \text{... (6)}$$

$$\boxed{J_D = \frac{\partial \vec{D}}{\partial t}} \qquad [\because D = \varepsilon E] \quad \text{... (7)}$$

Maxwell's Equation - IV

From Ampere's Circuital law

Ampere's law states that the line integral of magnetic field intensity *H* on any closed path is equal to the current (*I*) enclosed by that path

$$\oint \vec{H} \bullet \vec{dl} = I \qquad \text{... (1)}$$

But, current density $J = \frac{I}{A}$

where A is cross sectional area

$$\text{or } I = JA \qquad \left(\because A = \iint_s ds \right)$$

$$I = J \iint_S ds$$

or $$I = \iint_S \vec{J} \cdot \vec{ds} \qquad \text{... (2)}$$

substituting eqn (2) in eqn (1), we have

$$\oint \vec{H} \bullet \vec{dl} = \iint_S \vec{J} \bullet \vec{ds} \qquad \text{... (3)}$$

Ampere's law is modified by introducing displacement current density

$$\oint \vec{H} \bullet \vec{dl} = \iint_S (\vec{J}_C + \vec{J}_D)\, \vec{ds} \qquad \text{... (4)}$$

substituting for $\vec{J}_C = \sigma \vec{E}$ and $\vec{J}_D = \frac{\partial \vec{D}}{\partial t}$, we have

$$\oint \vec{H} \bullet \vec{dl} = \iint_S \left(\sigma \vec{E} + \frac{\partial \vec{D}}{\partial t} \right) ds$$

$$\oint \vec{H} \bullet \vec{dl} = \iint_S \left(\sigma \vec{E} + \varepsilon \frac{\partial \vec{E}}{\partial t} \right) ds \qquad \text{... (5)}$$

Unless or otherwise it is not specified, J stands for conduction current density alone. (ie., $J = J_C$)

then, $$\boxed{\oint \vec{H} \bullet \vec{dl} = \iint_S \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) ds} \qquad \text{... (6)}$$

$$(\because\ J = \sigma \vec{E} \qquad D = \varepsilon \vec{E})$$

This is Maxwell's equation in integral form from Ampere's circuital law.

Applying Stoke's Theorem to LHS of equation (6), we have

$$\oint \vec{H} \bullet \vec{dl} = \iint_S (\vec{\nabla} \times \vec{H})\, ds \quad \text{... (7)}$$

On substituting eqn (7) in eqn (6) we get

$$\iint_S (\vec{\nabla} \times \vec{H})\, ds = \iint_S \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) ds \quad \text{... (8)}$$

or

$$\boxed{\vec{\nabla} \times \vec{\boldsymbol{H}} = \vec{\boldsymbol{J}} + \frac{\partial \vec{\boldsymbol{D}}}{\partial \boldsymbol{t}}} \quad \text{... (9)}$$

$$\boxed{\vec{\nabla} \times \vec{\boldsymbol{H}} = \sigma \vec{\boldsymbol{E}} + \varepsilon \frac{\partial \vec{\boldsymbol{E}}}{\partial \boldsymbol{t}}} \quad \text{... (10)}$$

Equations (9) and (10) are Maxwell equations in differential form from Ampere's circuital law

Statement

The magnetomotive force around a closed path is equal to the sum of the conduction current and displacement current enclosed by the path.

Maxwell's equations are summarised as follows

	Differential form	**Integral form**
I.	$\vec{\nabla} \bullet \vec{D} = \rho$	$\oiint_S \vec{D} \bullet \vec{ds} = \iiint_V \rho\, dV$
II.	$\vec{\nabla} \bullet \vec{B} = 0$	$\oiint_S \vec{B} \bullet \vec{ds} = 0$
III.	$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$	$\oint \vec{E} \bullet \vec{dl} = -\iint_S \frac{\partial \vec{B}}{\partial t} \bullet \vec{ds}$
IV.	$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$	$\oint \vec{H} \bullet \vec{dl} = \iint_S \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) \vec{ds}$

Maxwell's Equations in Free Space

Four Maxwell's equations in differential form are

$$\vec{\nabla} \bullet \vec{D} = \rho \quad \text{... (1)}$$

$$\vec{\nabla} \bullet \vec{B} = 0 \quad \text{... (2)}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \text{... (3)}$$

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \quad \text{... (4)}$$

In a free space, there is no charges enclosed. There is no conductivity in the medium and conduction current is zero. Thus current density $\vec{J}$ and charge density ρ are zero. (ie., $\rho = 0$ and $J = 0$)

Therefore,

Maxwell's equations reduce to

$$\vec{\nabla} \bullet \vec{D} = 0 \quad \text{... (5)}$$

$(\because \rho = 0)$

$$\vec{\nabla} \bullet \vec{B} = 0 \quad \text{... (6)}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \text{... (7)}$$

$$\vec{\nabla} \times \vec{H} = \frac{\partial \vec{D}}{\partial t} \quad \text{... (8)}$$

$(\because J = 0)$

Maxwell's Equations in Conducting Media

In conducting media,

$\vec{J} = \sigma \vec{E}$ where σ is electrical conductivity of the conducting medium

$\vec{B} = \mu \vec{H}$ where μ is permeability of the medium

$\vec{D} = \varepsilon \vec{E}$ where ε is the permittivity of the conducting medium

By applying above conditions, general Maxwell's equations reduce to

$$\vec{\nabla} \bullet \vec{D} = \rho \quad \text{... (1)}$$

$$\vec{\nabla} \bullet \vec{B} = 0 \quad \text{... (2)}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \text{... (3)}$$

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \quad \text{... (4)}$$

Characteristics of Maxwell's Equation

1. Maxwell's First Equation; $\vec{\nabla} \bullet \vec{E} = \dfrac{\rho}{\varepsilon_o}$

- It explains Gauss's law in electrostatics.
- It is time independent or steady state equation.
- The flux of the lines of electric force depends upon charge density.
- Charge acts as a source or sink for the lines of electric force.

2. Maxwell's Second Equation $\vec{\nabla} \bullet \vec{B} = 0$

- It expresses a well known observation that *isolated magnetic poles do not exist.*
- It states that total magnetic flux entering and leaving a given volume is equal.
- There is no source or sink for lines of magnetic force.
- It is a time independent equation.
- It explains Gauss's law in Magnetostatics.

3. Maxwell's Third Equation $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

- It relates the electric field vector $\vec{E}$ and magnetic induction vector $\vec{B}$.
- It is a time dependent or time varying equation.
- It explains the well known Faraday's laws and Lenz's law of electromagnetic induction.
- $\vec{E}$ is generated by the time variation of $\vec{B}$.

4. Maxwell's Fourth Equation $\vec{\nabla} \times \vec{B} = \mu_o \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right)$

- It gives relation with the magnetic field vector $\vec{B}$ with displacement vector $\vec{D}$ and the current density $\vec{J}$.
- It is also a time dependent equation.
- It explains Ampere's circuital law.
- $\vec{B}$ can be produced by $\vec{J}$ and the time variation of $\vec{D}$

> **Note:** $\vec{B} = \mu \vec{H}$
>
> $\vec{H} = \frac{\vec{B}}{\mu}$ where μ – permeability of medium.

Plane Wave

If a wave is confined to a particular plane (instead of in 3-dimensions) and propagating along particular axis with equal magnitudes of electric and magnetic field vectors, then that wave is called plane wave.

2.2 WAVE EQUATION

Plane Electromagnetic Wave Equation in Vacuum

Maxwell's equations in general form are

$$\vec{\nabla} \cdot \vec{D} = \rho \quad \text{... (1)}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \quad \text{... (2)}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \text{... (3)}$$

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \qquad \text{... (4)}$$

Now, for the **free space** (vacuum) the permittivity and permeability are denoted by ε_o and μ_o respectively. Therefore, $\vec{D} = \varepsilon_o \vec{E}$ and $\vec{B} = \mu_o \vec{H}$.

Also, the conductivity $\sigma = 0$, that is the medium is a perfect insulator. Therefore there is no conduction current in the medium which implies

$$\vec{J} = 0 \quad (\because \vec{J} = \sigma \vec{E}, \text{ and } \sigma = 0)$$

Also there is no charge present in the vacuum therefore $\rho = 0$ and as a result eqn.(1) reduces to

$$\vec{\nabla} \bullet \vec{D} = 0 \text{ or } \vec{\nabla} \bullet \varepsilon_o \vec{E} = 0 \qquad (\because \vec{D} = \varepsilon_o \vec{E})$$

$$\varepsilon_o \vec{\nabla} \bullet \vec{E} = 0$$

$$\boxed{\vec{\nabla} \bullet \vec{E} = 0} \qquad \text{... (5)}$$

Wave equation for electric field vector ($\vec{E}$)

Taking the curl on both sides of equation (3), we get

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla} \times \left(-\frac{\partial \vec{B}}{\partial t} \right)$$

$$= -\frac{\partial}{\partial t} (\vec{\nabla} \times \vec{B})$$

$$= \frac{\partial}{\partial t} (\vec{\nabla} \times \mu_o \vec{H}) \qquad (\because \vec{B} = \mu_o \vec{H})$$

$$\text{or } \vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\mu_o \frac{\partial}{\partial t} (\vec{\nabla} \times \vec{H}) \qquad \text{... (6)}$$

Now from vector calculus identity, we have

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla}(\vec{\nabla} \bullet \vec{E}) - \nabla^2 \vec{E} \qquad \dots (7)$$

But from eqn. (5), $\vec{\nabla} \bullet \vec{E} = 0$ and substituting this in equation (7) we get

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\nabla^2 \vec{E} \qquad \dots (8)$$

substituting eqn (8) in eqn (6)

$$-\nabla^2 \vec{E} = -\mu_o \frac{\partial}{\partial t} (\vec{\nabla} \times \vec{H})$$

Now, substituting for $\vec{\nabla} \times \vec{H}$ from eqn. (4), we get

$$-\nabla^2 \vec{E} = -\mu_o \frac{\partial}{\partial t} \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) \qquad \dots (9)$$

$$\text{or } -\nabla^2 \vec{E} = -\mu_o \frac{\partial}{\partial t} \left[\varepsilon_o \frac{\partial \vec{E}}{\partial t} \right]$$

$$(\because \vec{J} = 0 \text{ and } \vec{D} = \varepsilon_o \vec{E})$$

$$\boxed{\nabla^2 \vec{E} = \mu_o \varepsilon_o \frac{\partial^2 \vec{E}}{\partial t^2}} \qquad \dots (10)$$

This is general electromagnetic wave equation in terms of electric field vector $\vec{E}$ for free space.

Wave equation for magnetic field vector ($\vec{B}$)

Taking curl on both sides of the equation (4), we have

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{H}) = \vec{\nabla} \times \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right)$$

or

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{H}) = \varepsilon_o \frac{\partial}{\partial t} (\vec{\nabla} \times \vec{E}) \quad \text{... (11)}$$

$$[\because \vec{J} = 0 \text{ and } \vec{D} = \varepsilon_o \vec{E}]$$

Now from vector calculus identity, we have

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{H}) = \vec{\nabla}(\vec{\nabla} \bullet \vec{H}) - \nabla^2 \vec{H} \quad \text{... (12)}$$

But from eqn (2), we have

$$\vec{\nabla} \bullet \vec{B} = 0$$

i.e., $$\mu_o (\vec{\nabla} \bullet \vec{H}) = 0 \quad \text{or} \quad (\vec{\nabla} \bullet \vec{H}) = 0 \quad \text{... (13)}$$

$(\vec{B} = \mu_o \vec{H})$

Substituting eqn (13) in eqn (12), we get

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{H}) = -\nabla^2 \vec{H} \quad \text{... (14)}$$

on substituting eqn (14) in eqn (11), we have

Using eqn (14) and eqn (11)

$$-\nabla^2 \vec{H} = \varepsilon_o \frac{\partial}{\partial t} (\vec{\nabla} \times \vec{E}) \quad \text{... (15)}$$

Substituting the eqn (3) in eqn (15)

$$-\nabla^2 \vec{H} = \varepsilon_o \frac{\partial}{\partial t} \left(-\frac{\partial \vec{B}}{\partial t} \right) \qquad (\because \vec{B} = \mu_o \vec{H})$$

$$-\nabla^2 \vec{H} = -\varepsilon_o \frac{\partial^2}{\partial t^2} (\mu_o \vec{H})$$

or $$\boxed{\nabla^2 \vec{H} = \mu_o \varepsilon_o \frac{\partial^2 \vec{H}}{\partial t^2}} \quad \text{... (16)}$$

This general electromagnetic wave equation in terms of $\vec{H}$ for free space.

Discussion

1. The electromagnetic wave equation for $\vec{E}$ and $\vec{H}$ is written as

$$\boxed{\nabla^2 \vec{E} - \mu_o \varepsilon_o \frac{\partial^2 \vec{E}}{\partial t^2} = 0} \quad \text{... (17)}$$

and

$$\boxed{\nabla^2 \vec{H} - \mu_o \varepsilon_o \frac{\partial^2 \vec{H}}{\partial t^2} = 0} \quad \text{... (18)}$$

In one dimension say along x-axis, the wave equations are given by the x-components of the above expression. That is

$$\boxed{\frac{\partial^2 E_x}{\partial x^2} - \mu_o \varepsilon_o \frac{\partial^2 E_x}{\partial t^2} = 0} \quad \text{... (19)}$$

and

$$\boxed{\frac{\partial^2 H_x}{\partial x^2} - \mu_o \varepsilon_o \frac{\partial^2 H_x}{\partial t^2} = 0} \quad \text{... (20)}$$

2.3 SPEED (VELOCITY) OF EM WAVE IN VACUUM

Comparing above equations (19) and (20) with the following general wave equation propagating in x-axis

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 y}{\partial t} = 0 \quad \text{... (21)}$$

where y - instantaneous displacement

c - velocity of wave

We find that the velocity (speed) of the electromagnetic wave is given by

$$\frac{1}{c^2} = \mu_o \varepsilon_o$$

$$c^2 = \frac{1}{\mu_o \varepsilon_o}$$

Note: Magnitude of velocity is called speed.

$$c = \frac{1}{\sqrt{\mu_o \varepsilon_o}} \quad \text{... (22)}$$

For vacuum or free space we have $\mu_o = 4\pi \times 10^{-7}\ \text{H m}^{-1}$ (henry per metre) and $\varepsilon_o = 8.842 \times 10^{-12}\ \text{Fm}^{-1}$ (farad per metre).

substituting these values in eqn (22), we get

$$c = \frac{1}{\sqrt{4\pi \times 10^{-7} \times 8.842 \times 10^{-12}}}$$

$$c = 2.998 \times 10^{8}\ \text{m s}^{-1}$$

Thus, the speed of the waves in free space is a constant and equal to 3 × 10^8 m/s which is identical with the velocity (speed) of light. This reveals that the light is also the electromagnetic waves.

Wave Equations for plane polarized EM wave in free space and their solution

The electromagnetic wave equations for $\vec{E}$ and $\vec{H}$ in free space are given by

$$\nabla^2 \vec{E} - \mu_o \varepsilon_o \frac{\partial^2 \vec{E}}{\partial t^2} = 0 \quad \text{... (1)}$$

and

$$\nabla^2 \vec{H} - \mu_o \varepsilon_o \frac{\partial^2 \vec{H}}{\partial t^2} = 0 \quad \text{... (2)}$$

2.4 CONDITIONS ON THE WAVE FIELD

If the plane polarized waves is propagating along x-axis having electric vector along the y-axis, we have

$$E_y \neq 0, \quad E_z = E_x = 0 \quad \text{and}$$

Similarly for magnetic field vector

$$H_z \neq 0, \quad H_y = H_x = 0$$

Therefore, the wave equations for plane electromagnetic wave reduce to

$$\nabla^2 E_y - \mu_o \varepsilon_o \frac{\partial^2 E_y}{\partial t^2} = 0 \qquad \text{... (3)}$$

and $$\nabla^2 H_z - \mu_o \varepsilon_o \frac{\partial^2 H_z}{\partial t^2} = 0 \qquad \text{... (4)}$$

Now $$\nabla^2 E_y = \frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial y^2} + \frac{\partial^2 E_y}{\partial z^2} \qquad \text{... (5)}$$

$$\nabla^2 H_z = \frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + \frac{\partial^2 H_z}{\partial z^2} \qquad \text{... (6)}$$

But, $$\frac{\partial^2 E_y}{\partial y^2} = 0 \quad \text{and} \quad \frac{\partial^2 E_y}{\partial z^2} = 0$$

Similarly

$$\frac{\partial^2 H_z}{\partial y^2} = 0 \quad \text{and} \quad \frac{\partial^2 H_z}{\partial z^2} = 0,$$

since at the given value of x, E_y and H_z are constant at any instant. Further $\vec{E_y}$ and $\vec{H_z}$ are varying only in x-direction which is the direction of propagation of these fields as shown in fig. 2.4.

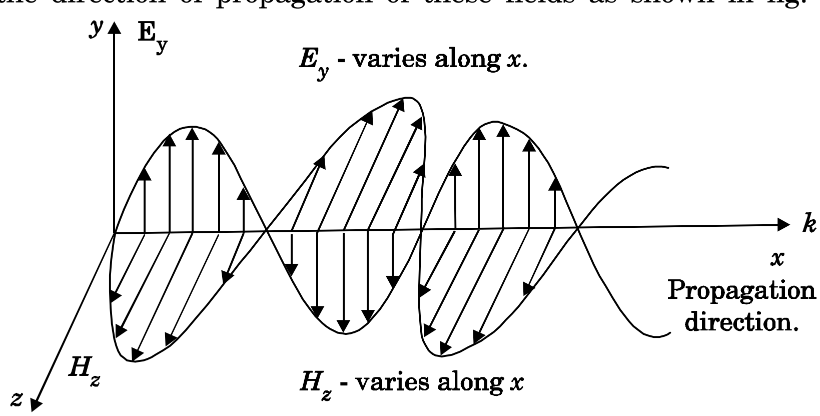


Fig. 2.4

$$\therefore \qquad \nabla^2 E_y = \frac{\partial^2 E_y}{\partial x^2} \qquad \text{... (7)}$$

Similarly $$\nabla^2 H_z = \frac{\partial^2 H_z}{\partial x^2} \qquad \text{... (8)}$$

Substituting eqn (7) in eqn (3) and eqn (8) in eqn (4) we have

$$\boxed{\begin{array}{l} \dfrac{\partial^2 E_y}{\partial x^2} - \mu_o \varepsilon_o \dfrac{\partial^2 E_y}{\partial t^2} = 0 \\[2ex] \dfrac{\partial^2 H_z}{\partial x^2} - \mu_o \varepsilon_o \dfrac{\partial^2 H_z}{\partial t^2} = 0 \end{array}} \qquad \begin{array}{r} \text{... (9)} \\[2ex] \text{... (10)} \end{array}$$

Solutions of the plane Wave Equations

The plane wave equations for electric field and magnetic field are given by

$$\frac{\partial^2 E_y}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 E_y}{\partial t^2} = 0$$

$$\frac{\partial^2 H_z}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 H_z}{\partial t^2} = 0 \qquad \left(\because \mu_o \varepsilon_o = \frac{1}{c^2} \right)$$

c – speed of EM wave

The solutions of the above wave equations of progressive wave are given by

$$\boxed{E_y = E_o \cos(\omega t - kx)} \qquad \text{... (11)}$$

and $$\boxed{H_z = H_o \cos(\omega t - kx)} \qquad \text{... (12)}$$

where, ω - angular frequency

k - wave vector

Here E_o and H_o are the maximum values (amplitudes) of the electric and magnetic vectors respectively.

The general solution of the wave equation is written as

$$\vec{E}_y = E_o e^{i(\omega t - kx)} = E_o e^{ik(ct - x)} \quad \text{... (13)}$$

and

$$\vec{H}_z = H_o e^{i(\omega t - kx)} = H_o e^{ik(ct - x)} \quad \text{... (14)}$$

$(\because \ c = \nu\lambda)$

where $k = \frac{2\pi}{\lambda}$ and $\omega = 2\pi\nu = \frac{2\pi c}{\lambda} = kc$

where c is the wave velocity.

Note: Equation of wave is given by

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} = 0$$

$v -$ velocity

The general solution of above equation is

$$y = f(x, t)$$

$$y = y_o \cos 2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right)$$

$T -$ period, $\lambda -$ wavelength

$$y = y_o \cos\left(\frac{2\pi t}{T} - \frac{2\pi x}{\lambda}\right)$$

$$= y_o \cos\left(2\pi\nu t - \frac{2\pi x}{\lambda}\right)$$

$$y = y_o \cos(\omega t - kx)$$

$\omega = 2\pi\nu$ angular frequency and $k = \frac{2\pi}{\lambda}$, known as wave vector

2.5 PHASE AND ORIENTATION OF EM WAVE IN MATTER

From the equations (13) and (14), it is clear that the phases of electric and magnetic fields are same (ie., $\omega t - kx$). Thus, both fields are in phase with each other.

Relation between electric and magnetic field vectors

We know for electromagnetic waves in free space

$$\vec{E}_y = E_o \, e^{ik\,(ct - x)} \quad \ldots (1)$$

and

$$\vec{H}_z = H_o \, e^{ik\,(ct - x)} \quad \ldots (2)$$

The relation between their time and space variations is given from Maxwell's equation

We know that

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{E} = -\mu_o \frac{\partial \vec{H}}{\partial t} \qquad (\because \vec{B} = \mu_o \vec{H})$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & E_y & 0 \end{vmatrix} = -\mu_o \frac{\partial \vec{H}}{\partial t} \qquad (\because E_x = E_z = 0)$$

or

$$\frac{\partial \vec{E}_y}{\partial x} = -\mu_o \frac{\partial \vec{H}_z}{\partial t} \quad \ldots (3)$$

Substituting E_y and H_z from the equations (1) & (2) in eqn. (3), we get

$$\frac{\partial}{\partial x}\left(E_o e^{i k (ct - x)}\right) = -\mu_o \frac{\partial}{\partial t}\left(H_o e^{ik (ct - x)}\right) \qquad \dots (4)$$

$$-i k E_o e^{i k (ct - x)} = -\mu_o (i k c) H_o e^{i k (ct - x)}$$

$$E_o = \mu_o c H_o \qquad \dots (5)$$

$$\text{But } c = \frac{1}{\sqrt{\varepsilon_o \mu_o}} \qquad \dots (6)$$

Substituting the eqn. (6) in eqn. (5), we have

$$E_o = \mu_o \times \frac{1}{\sqrt{\varepsilon_o \mu_o}} \times H_o = \sqrt{\frac{\mu_o}{\varepsilon_o}} \; H_o$$

or $$\sqrt{\frac{\mu_o}{\varepsilon_o}} = \frac{E_o}{H_o} = \frac{E_o e^{i(\omega t - kx)}}{H_o e^{i(\omega t - kx)}}$$

$$\therefore \quad \boxed{\frac{\vec{E}}{H} = \sqrt{\frac{\mu_o}{\varepsilon_o}}} \qquad \dots (7)$$

This is the relation between the electric field vector and magnetic field vector. It is determined by the μ_o and ε_o.

This ratio of the electric and magnetic vectors is directly proportional to the square root of the ratio of μ_o and ε_o.

$$\frac{\vec{E}}{\vec{H}} = \frac{E_o}{H_o} = \sqrt{\frac{\mu_o}{\varepsilon_o}}$$

Substituting

$\mu_o = 4\pi \times 10^{-7}$ Hm^{-1}, $\quad \varepsilon_o = 8.85 \times 10^{-12}$ Fm^{-1}

$$\frac{E}{H} = \frac{E_o}{H_o} = \sqrt{\frac{\mu_o}{\varepsilon_o}} = \sqrt{\frac{4\pi \times 10^{-7}}{8.85 \times 10^{-12}}}$$

$$= 376.73 \approx 377$$

i.e., $\boxed{E_o \approx 377\, H_o}$ also $\boxed{E \approx 377 \text{ H}}$

It shows that the value of electric field at any instant is about 377 times the value of magnetic vector in electromagnetic wave.

The ratio $\frac{E}{H}$ is having the unit of impedance (Resistance) ie., ohm, Therefore, the quantity $\sqrt{\frac{\mu_o}{\varepsilon_o}}$ has the dimensions of impedance.

$$\sqrt{\frac{\mu_o}{\varepsilon_o}} = \sqrt{\frac{H/m}{F/m}} = \sqrt{\frac{\text{herry / m}}{\text{farad / m}}}$$

$$\sqrt{\frac{\text{herry}}{\text{farad}}} = \sqrt{\frac{\text{ohm} \times \text{sec}}{\text{Coul / volt}}} = \sqrt{\frac{\text{ohm} \times \text{volt}}{\text{Coul / sec}}} = \sqrt{\text{amp} \times \frac{\text{volt}}{\text{ohm}}}$$

$$= \sqrt{\text{ohm} \times \text{ohm}} = \text{ohm}$$

It is known as **intrinsic or characteristic impedance of free space, denoted by Z_o**. It is a constant quantity for free space and having value $\approx 377\Omega$.

The above discussion shows that the vector $\vec{E}$ and the vector $\vec{H}$ are at right angle to each other and the wave propagates in the direction of $\vec{E} \times \vec{H}$. (Fig. 2.5)

Thus if $\vec{E}$ is parallel to Y - axis. The vector $(\vec{E} \times \vec{H})$ is known as **Poynting vector** after **J.S. Poynting** who first investigated its properties.

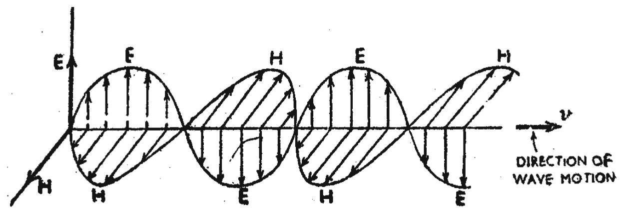


Fig. 2.5

Further, $\vec{E}$ and $\vec{H}$ are always in the same phase, i.e., their relative magnitudes are the same at all points and at all times.

Poynting Vector

The cross product of electric field vector $\vec{E}$ and the magnetic field vector $\vec{H}$ is called poynting vector. It is denoted by $\vec{S} = \vec{E} \times \vec{H}$.

An electro-magnetic wave has electric and magnetic field vectors oscillating perpendicular to each other. The electric and magnetic disturbance travels in a same direction perpendicular to both the electric and magnetic vectors.

Let a plane polarized electromagnetic wave is propagating along the x-axis. The electric vector is directed along the y-axis, then the magnetic vector is directed along the z-axis. Hence,

$$\vec{S} = \vec{E} \times \vec{H} = \hat{j}\, E_y \times \hat{k}\, H_z$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & E_y & 0 \\ 0 & 0 & H_z \end{vmatrix} = \hat{i}\,(E_y H_z)$$

But $E_y H_z$ measures the energy per second (ie power) per unit area. Hence $\vec{S}$ represents the energy propagating along the x-axis in $Js^{-1} m^{-2}$ or Wm^{-2}.

In the other words, **Poynting vector gives the time rate of flow of electromagnetic wave energy per unit area of the medium.**

Further the average Poynting vector for one complete cycle of electromagnetic wave is given by

$$S_{avg} = \frac{1}{2}(\vec{E} \times \vec{H}) = \frac{1}{2}E_o \times H_o = \frac{E_o}{\sqrt{2}} \times \frac{H_o}{\sqrt{2}} = E_{rms} \cdot H_{rms}$$

$$\left(\because E_{rms} = \frac{E_o}{\sqrt{2}} \text{ and } H_{rms} = \frac{H_o}{\sqrt{2}} \right)$$

Maxwell's findings is summarised as below.

1. If there is a varying electric field in vacuum, there is also a varying magnetic field and vice versa. The electric and magnetic fields are perpendicular to each other and also perpendicular to the direction of propagation of the wave.
2. The electric and magnetic fields obey wave equation with identical propagation speeds.
3. The speed of propagation given by $c = \frac{1}{\sqrt{\varepsilon_o \mu_o}}$ is the same as the measured speed of light.
4. The light waves can, therefore, be identified as electromagnetic waves.
5. The electric and magnetic fields in electromagnetic waves oscillate in phase with each other.

2.6 PROPAGATION OF ELECTROMAGNETIC WAVE THROUGH A DIELECTRIC MEDIUM (Non - Conducting Isotropic Medium)

Maxwell's equations are

$$\vec{\nabla} \cdot \vec{D} = \rho$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{dt}$$

and
$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{dt}$$

In an isotropic dielectric (or non-conducting isotropic medium)

$$\vec{D} = \varepsilon \vec{E}, \vec{B} = \mu \vec{H}, \vec{J} = \sigma \vec{E} = 0 \text{ and } \rho = 0.$$

Therefore, Maxwell's equations in this case take the form

$$\vec{\nabla} \cdot \vec{E} = 0 \qquad \text{... (1)}$$

$$\vec{\nabla} \cdot \vec{H} = 0 \qquad \text{... (2)}$$

$$\vec{\nabla} \times \vec{E} = -\mu \frac{\partial \vec{B}}{\partial t} \qquad \text{... (3)}$$

and
$$\vec{\nabla} \times \vec{H} = \varepsilon \frac{\partial \vec{E}}{\partial t} \qquad \text{... (4)}$$

Equation of propagation of magnetic vector, *H*

Taking curl of eqn. (4), we get

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{H}) = \varepsilon \vec{\nabla} \times \left(\frac{\partial \vec{E}}{\partial t} \right)$$

or
$$\vec{\nabla}(\vec{\nabla} \cdot \vec{H} - \nabla^2 \vec{H} = \varepsilon \frac{\partial}{\partial t} (\vec{\nabla} \times \vec{E}) \qquad \text{... (5)}$$

Putting values from the eqns.(2) and (3), we get

$$\nabla^2 \vec{H} = \mu \varepsilon \frac{\partial^2 \vec{H}}{\partial t^2} \qquad \text{... (6)}$$

Equation of propagation of electric vector, *E*

Taking of eqn. (3), we get

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla} \times \left(- \mu \frac{\partial \vec{H}}{\partial t} \right)$$

$$\vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \nabla^2 \vec{E} = - \mu \frac{\partial}{\partial t} (\vec{\nabla} \times \vec{H})$$

Putting values from eqns (1) and (4), we get

$$\nabla^2 \vec{E} = \mu \varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} \qquad \text{... (7)}$$

The eqns (6) and (7) is compared with the general wave equation,

$$\nabla^2 u = \frac{1}{v^2} \frac{\partial^2 u}{\partial t^2}$$

$$\therefore \qquad v = \frac{1}{\sqrt{\mu \varepsilon}}$$ where v is the speed of wave.

This means that the field vectors $\vec{E}$ and $\vec{H}$ are propagated isotropic dielectric as waves with speed v given by

$$v = \frac{1}{\sqrt{(\mu \varepsilon)}} \qquad \text{... (8)}$$

Now, $\frac{1}{\sqrt{\mu_o \varepsilon_o}} = c$, speed of electromagnetic waves in free space.

Refractive index is

$$n = \frac{c}{v} = \sqrt{\frac{\mu \varepsilon}{\mu_o \varepsilon_o}} = \sqrt{\mu_r \varepsilon_r}$$

In a non-magnetic medium $\mu_r = 1$

$$\therefore \quad \boxed{n = \sqrt{\varepsilon_r}}$$

2.7 EM WAVES IN CONDUCTING MEDIUM (Medium with Finite μ, ε and σ)

General Maxwell's equations are

$$\vec{\nabla} \bullet \vec{D} = \rho \qquad \text{... (1)}$$

$$\vec{\nabla} \bullet \vec{B} = 0 \qquad \text{... (2)}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \text{... (3)}$$

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \qquad \text{... (4)}$$

In conducting medium $\sigma \neq 0$ i.e., there is a conduction current in the medium therefore $\vec{J} \neq 0$ but the charge density is zero everywhere (inside the conductor the charge density is always zero). That is $\rho = 0$.

Therefore the eqn. (1) reduces to $\vec{\nabla} \bullet \vec{D} = 0$

$$\text{or} \quad \vec{\nabla} \bullet (\varepsilon \vec{E}) = 0 \Rightarrow \vec{\nabla} \bullet \vec{E} = 0 \qquad \text{... (5)}$$

Here $\vec{D} = \varepsilon \vec{E}$ and ε is permittivity of the medium.

Taking the curl on both sides of eqn. (3), we get

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla} \times \left[-\frac{\partial \vec{B}}{\partial t} \right] \qquad \text{... (6)}$$

From vector calculus identity, we have

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla}(\vec{\nabla} \bullet \vec{E}) - \nabla^2 \vec{E} \qquad \text{... (7)}$$

But from eqn. (5) $\vec{\nabla} \bullet \vec{E} = 0$

Therefore, equation (7) becomes

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\nabla^2 \vec{E} \qquad \text{... (8)}$$

Also
$$\vec{\nabla} \times \left[-\frac{\partial \vec{B}}{\partial t} \right] = -\mu \frac{\partial}{\partial t} (\vec{\nabla} \times \vec{H}) \qquad \text{... (9)}$$

$$(\because \vec{B} = \mu H)$$

Substituting the eqn. (8) and (9) in (6), we get

$$-\nabla^2 \vec{E} = -\mu \frac{\partial}{\partial t} (\vec{\nabla} \times \vec{H})$$

or
$$\nabla^2 \vec{E} = \mu \frac{\partial}{\partial t} (\vec{\nabla} \times \vec{H}) \qquad \text{... (10)}$$

On substituting the value of $\vec{\nabla} \times \vec{H}$ from eqn. (4) in eqn. (10), we have

$$\nabla^2 \vec{E} = \mu \frac{\partial}{\partial t} \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) \qquad \text{... (11)}$$

Since $\vec{J} = \sigma \vec{E}$ and $\vec{D} = \varepsilon \vec{E}$ equation (11) becomes

$$\nabla^2 \vec{E} = \mu \frac{\partial}{\partial t} \left[\sigma \vec{E} + \frac{\partial}{\partial t} (\varepsilon \vec{E}) \right]$$

or
$$\nabla^2 \vec{E} = \mu \sigma \frac{\partial \vec{E}}{\partial t} + \mu \varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

Thus,
$$\boxed{\nabla^2 \vec{E} - \mu \varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} - \mu \sigma \frac{\partial \vec{E}}{\partial t} = 0} \qquad \text{... (12)}$$

This is the general wave equation for the electric vector in an electromagnetic wave propagating in conducting medium.

In a similar way, by taking the curl of the eqn. (4) we obtain the general wave equation for the magnetic vector in a conducting medium as

$$\boxed{\nabla^2 \vec{H} - \mu\,\varepsilon \frac{\partial^2 \vec{H}}{\partial t^2} - \mu\,\sigma \frac{\partial \vec{H}}{\partial t} = 0} \quad \text{... (13)}$$

Wave Equation for Plane Polarized EM Waves

Let us consider that the electromagnetic wave is travelling in the x-direction and the electric vector is directed along the y-axis and the magnetic vector is directed along the z-axis. For such a wave we have

$$E_y \neq 0, E_z = E_x = 0 \quad \text{and} \quad H_z \neq 0, H_y = H_x = 0$$

Therefore, the wave equations from (12) and (13), reduce to

$$\boxed{\frac{\partial^2 E_y}{\partial x^2} - \mu\,\varepsilon \frac{\partial^2 E_y}{\partial t^2} - \mu\,\sigma \frac{\partial E_y}{\partial t^2} = 0} \quad \text{... (14)}$$

and

$$\boxed{\frac{\partial^2 H_z}{\partial x^2} - \mu\,\varepsilon \frac{\partial^2 H_z}{\partial t^2} - \mu\,\sigma \frac{\partial H_z}{\partial t} = 0} \quad \text{... (15)}$$

In the above wave equations $\mu\,\varepsilon = \frac{1}{v^2}$; where v is the velocity of electromagnetic wave in conducting medium. The **product $\mu\,\sigma$ is called magnetic diffusivity.**

Thus, the finite conductivity adds the diffusion term $-\mu\,\sigma\left(\frac{\partial E_y}{\partial t}\right)$ to the wave equation. This is due to the presence of conduction current ($\vec{J} \neq 0$).

For vacuum or perfect insulators ($\sigma = 0$) and so these equations reduce to the expressions corresponding to the free space or dielectric medium.

Solution of the plane Em Wave Equation in conducting medium $(\sigma \neq 0)$

The solution of the equation (14) should be a function of t and x and is of the form

$$\vec{E}_y = E_0 e^{(i\omega t \pm \gamma x)} \quad \text{... (16)}$$

Similarly, the solution of the equation (15) is of the form

$$\vec{H}_z = H_0 e^{(i\omega t \pm \gamma x)} \quad \text{... (17)}$$

Substituting eqn. (16) in eqn. (14)

$$\frac{\partial^2}{\partial x^2}\left(E_0 e^{(i\omega t \pm \gamma x)}\right) - \mu\varepsilon \frac{\partial^2}{\partial t^2}\left(E_0 e^{(i\omega t \pm \gamma x)}\right) - \mu\sigma \frac{\partial}{\partial t}\left(E_0 e^{(i\omega t \pm \gamma x)}\right) = 0$$

$$\gamma^2 E_0 e^{(i\omega t \pm \gamma x)} - \mu\varepsilon (i\omega)^2 E_0 e^{(i\omega t \pm \gamma x)} - \mu\sigma i\omega E_0 e^{(i\omega t \pm \gamma x)} = 0$$

$$\gamma^2 - \mu\varepsilon i^2\omega^2 - \mu\sigma i\omega = 0$$

$$\gamma^2 + \mu\varepsilon\omega^2 - i\mu\sigma\omega = 0 \quad [i^2 = -1]$$

$$\text{or} \quad \gamma^2 = i\mu\sigma\omega - \mu\varepsilon\omega^2 \quad \text{... (18)}$$

For good conductors, we have $\sigma >> \omega\varepsilon$. Therefore, $\mu\varepsilon\omega^2$ can be neglected as compared to $\mu\sigma\omega$. Hence, from equation (18), we have

$$\gamma^2 = i\mu\sigma\omega$$

$$\gamma^2 = \frac{2i\mu\sigma\omega}{2}$$

$$\text{or } \gamma^2 = (1 - 1 + 2i)\frac{\mu\,\sigma\,\omega}{2} = (1 + i)^2\frac{\mu\,\sigma\,\omega}{2}$$

Taking square root on both sides

$$\gamma = \pm (1 + i)\sqrt{\frac{(\mu\,\sigma\,\omega)}{2}} = \pm (1 + i)\,k$$

$$\text{ie., } \gamma = (1 + i)\,k \quad \text{(or)} \quad \gamma = -(1+i)\,k$$

$$\text{where } k = \sqrt{\frac{\mu\,\sigma\,\omega}{2}} \text{ is a constant.}$$

Taking the –ve value of γ which gives the wave propagation in the +ve x-direction and substituting in the eqn. (16) we get

$$\vec{E}_y = E_0\, e^{[i\,\omega\,t\, -\, (1+i)\,kx]}$$

$$\vec{E}_y = E_0\, e^{(i\,\omega\,t\, -\, kx\, -\, ikx)}$$

$$\boxed{\vec{E}_y = E_0\, e^{-kx}\; e^{i\,(\omega t\, -\, kx)}} \qquad \dots (16)$$

This is a progressive wave having amplitude equal to $E_0\, e^{-kx}$. The amplitude of the wave goes on decreasing as the wave propagates deeper into the medium. Also the propagation constant k which is a constant depending upon the value of μ and σ.

Skin Depth (or) Penetration Depth

In conducting medium the amplitude of the electromagnetic wave decreases exponentially with distance of penetration of the wave. Suppose, the amplitude at a depth x is denoted by E_{0x}, then

$$E_{ox} = E_o \, e^{-kx} \quad \text{... (1)}$$

$$\text{where } k = \sqrt{\frac{\mu \sigma \omega}{2}}$$

The decrease in the amplitude or the attenuation of the field vector is quantitatively expressed in terms of a quantity called **skin depth.**

It is defined as the distance inside the conductor from the surface of the conductor at which the amplitude of the field vector is reduced to 1/e times its value at the surface.

Electromagnetic waves

According to Maxwell, an accelerated charge is a source of electromagnetic radiation.

- In an electromagnetic wave, electric and magnetic field vectors are at right angles to each other and both are at right angles to the direction of propagation.

  They possess the wave character and propagate through free space without any material medium. These waves are transverse in nature.

- Fig. 2.6 shows the variation of electric field $\vec{E}$ along Y direction and magnetic field $\vec{B}$ along Z direction and wave propagation in $+X$ direction.

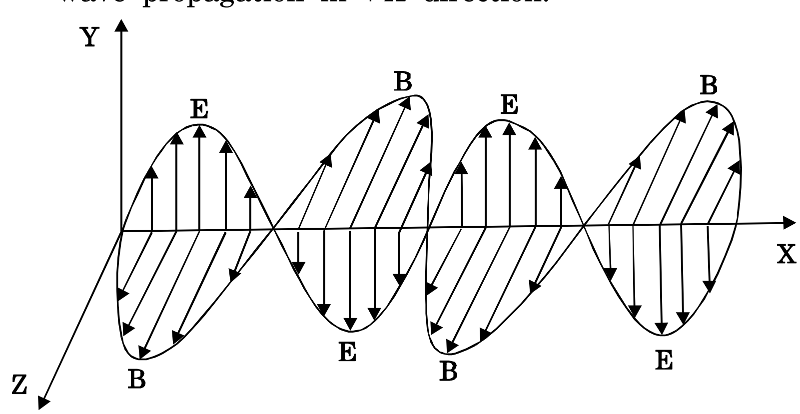


Fig. 2.6 Electromagnetic waves

2.8 PROPERTIES OF ELECTROMAGNETIC WAVES

(i) Electromagnetic waves are produced by accelerated charges.

(ii) They do not require any material medium for propagation.

(iii) In an electromagnetic wave, the electric ($\vec{E}$) and magnetic ($\vec{B}$) field vectors are at right angles to each other and to the direction of propagation. Hence, electromagnetic waves are transverse in nature.

(iv) Variation of maxima and minima in both $\vec{E}$ and $\vec{B}$ occur simultaneously (in phase).

(v) They travel in vacuum or free space with a speed $3 \times 10^8 \text{ ms}^{-1}$ given by the relation $c = \frac{1}{\sqrt{\mu_o \varepsilon_o}}$.

(μ_o – permeability of free space and
ε_o – permittivity of free space)

(vi) The energy in an electromagnetic wave is equally divided between electric and magnetic field vectors.

(vii) The electromagnetic waves being chargeless, they are not deflected by electric and magnetic fields.

2.9 LOCALIZED SOURCES FOR ELECTROMAGNETIC WAVES

Any stationary charge produces only electric field. When the charge moves with uniform velocity, it produces steady current which gives rise to magnetic field (not time dependent, only space dependent) around the conductor in which charge flows.

If the charged particle accelerates, it produces magnetic field in addition to electric field.

Both electric and magnetic fields are time varying fields. Since the electromagnetic waves are transverse waves, the direction of propagation of electromagnetic waves is perpendicular to the planes containing electric and magnetic field vectors.

Any oscillatory motion is also an accelerated motion. So, when the charge oscillates (oscillating molecular dipole) about their mean position (Fig. 2.7), it produces electromagnetic waves.

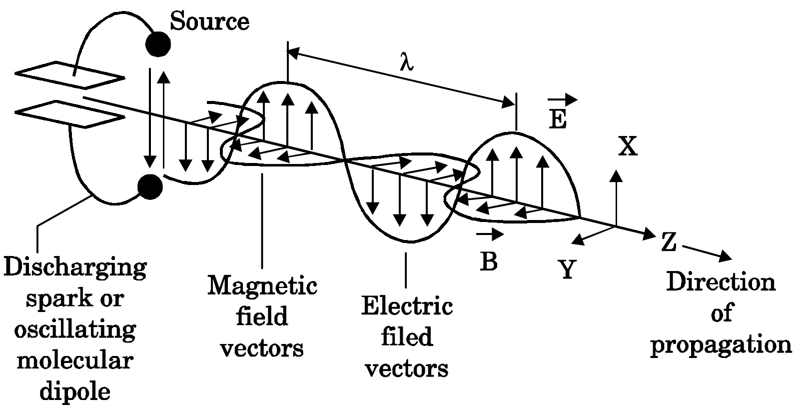


Fig. 2.7 Oscillating charges - Sources of Electromagnetic Waves

It the electromagnetic field in free space propagates along x-direction and if the electric field vector points along y-axis, then the magnetic field vector is mutually perpendicular to both electric field and the direction of wave propagation. Thus

$$\boxed{E_y = E_o \cos(\omega t - kx)}$$

$$\boxed{B_z = B_o \cos(\omega t - kx)}$$

where E_o and B_o are amplitudes of oscillating electric and magnetic field, k is a wave number denotes the direction of propagation of electromagnetic wave. ω is the angular frequency of the wave.

Note both electric field and magnetic field oscillate with a frequency (frequency of electromagnetic wave) which is equal to the frequency of the source (here, oscillating charge is the source for the production of electromagnetic waves).

In free space or in vacuum, the ratio between E_o and B_o is equal to the speed of electromagnetic wave and is equal to speed of light c.

$$\boxed{c = \frac{E_o}{B_o}}$$

In any medium, the ratio of E_o and B_o is equal to the speed of electromagnetic wave in that medium. Thus

$$\boxed{v = \frac{E_o}{B_o} < c}$$

Further, the energy of electromagnetic waves comes from the energy of the oscillating charge.

Note:

$$\frac{E_o}{H_o} = \sqrt{\frac{\mu_o}{\varepsilon_o}}$$

$$\frac{E_o}{\frac{B_o}{\mu_o}} = \sqrt{\frac{\mu_o}{\varepsilon_o}} \qquad (\because B_o = \mu_o H_o)$$

$$\frac{\mu_o E_o}{B_o} = \sqrt{\frac{\mu_o}{\varepsilon_o}}$$

$$\frac{E_o}{B_o} = \sqrt{\frac{\mu_o}{\varepsilon_o}} \times \frac{1}{\mu_o}$$

$$\frac{E_o}{B_o} = \sqrt{\frac{1}{\mu_o \varepsilon_o}}$$

$$\frac{E_o}{B_o} = c$$

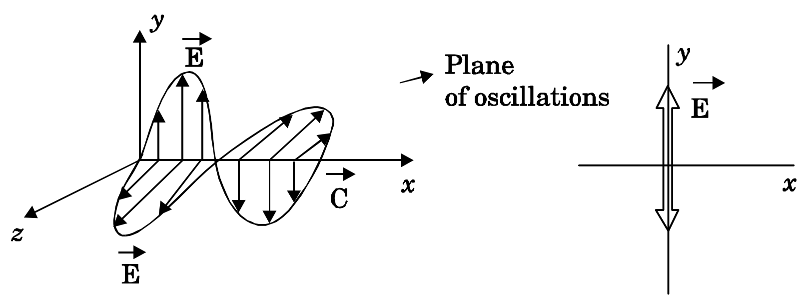
$$\boxed{\boldsymbol{E_o = c B_o}}$$

2.10 POLARIZATION

In an electromagnetic wave, the direction in which the electric field's amplitude vector $\vec{E}_o$ points, specifies the geometrical orientation of the oscillation. The direction of $\vec{E}_o$ is now called the **polarization** of the wave.

Fig. 2.8 shows an electromagnetic wave with its electric field oscillating parallel to the vertical '*y*' axis. The plane containing the $\vec{E}$ vectors is called the **plane of oscillation** of the wave.

Hence the *EM* wave is called as **plane-polarized** parallel to the *y* axis. The wave's polarization can be represented by two arrows, indicating that its electric field oscillates vertically. It is continuously changing between directed up and down along the *y*-axis.



(a) (b)

Fig. 2.8 (a) Plane of oscillations of a polarized EM wave
(b) Polarization representation of the electric field along '*y*' axis

Thus the amplitude vector $\vec{E}_o$ cannot point in arbitrary direction for *EM* waves. This is because as Gauss law for electric field for a system with a varnishing charge density

$$\vec{\nabla} \cdot \vec{E} = 0 \qquad \ldots (1)$$

implies that

$$\frac{\partial E_x}{\partial x} = 0, \quad \frac{\partial E_y}{\partial y} = 0, \quad \frac{\partial E_z}{\partial z} = 0 \qquad \ldots (2)$$

So, if we consider a wave solution for the electric field,

$\vec{E}(x, t) = \vec{E}_o \cos(\omega t - kx)$ cannot have an oscillating amplitude in the x-direction since $\frac{\partial E_x}{\partial x} = 0$. Due to this fact, *EM* waves cannot be polarized in the direction they are travelling. So *EM* waves are never **longitudinally polarized**.

2.11 PRODUCING ELECTROMAGNETIC WAVES

It follows from Maxwell's equations that electromagnetic radiation occurs whenever electric charge accelerates.

Let us consider a capacitor like arrangement as shown in fig. 2.9. Here, *A* and *B* are two conducting rods. They are separated by a distance '*d*'.

If a voltage source is connected across these rods or wires, a corresponding electric field occurs between the rods.

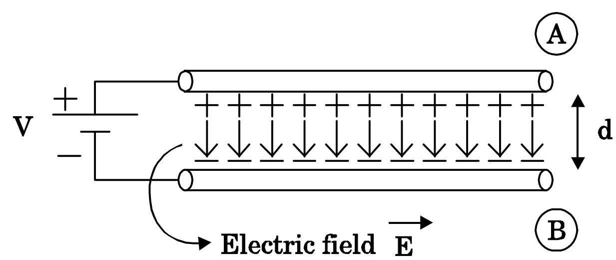


Fig. 2.9

If the two rods are get tilted in the following manner, correspondingly the electric field pattern also changes (Fig. 2.10).

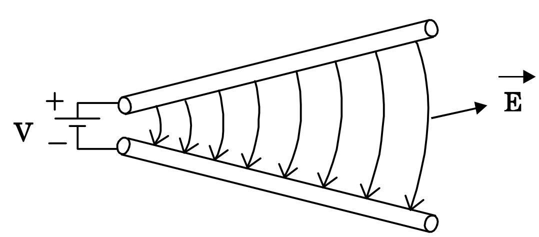


Fig. 2.10

Further if the two rods (or) wires are tilted in the following manner then, the electric field lines are also changed (Fig. 2.11).

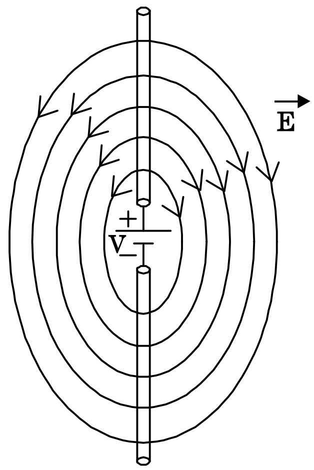


Fig. 2.11

Now by replacing the voltage source as a **AC voltage** source as shown in fig. 2.12, then this arrangement is called as dipole - antenna.

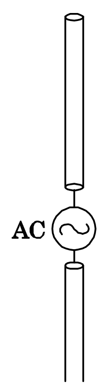


Fig. 2.12 Dipole-Antenna

A basic design of a electromagnetic wave generator involves this dipole - antenna arrangement. A diagram of one such arrangement is shown in fig. 2.13.

In the figure 2.13, the AC voltage source with a particular frequency '*f*' is fed to a LCR circuit. The inductor '*L*' is a transformer. The secondary of the transformer is attached to the two conducting rods (*A*) and (*B*) as shown in the fig. 2.13.

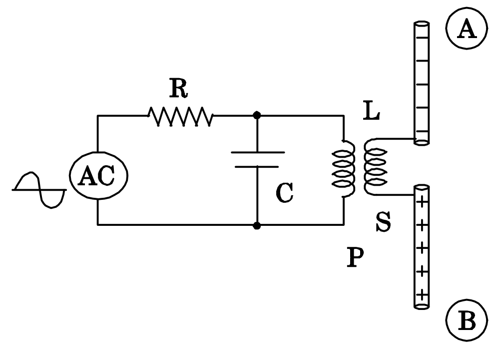


Fig. 2.13 Electro magnetic wave generator

As the voltage of the *AC* source oscillates the electric potentials of the two wires (*A*) and (*B*) also oscillate.

Due to acceleration of charges, the current flowing through the rod is alternating or it periodically changes its direction.

So the electrons in the rods are constantly accelerated and deaccelerated when the direction of the current changes. As the current oscillates, the electric field at the antenna oscillates. So an electromagnetic wave propagates as shown in fig. 2.14.

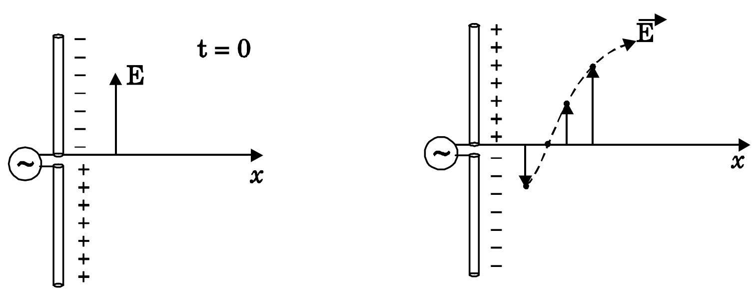


Fig. 2.14

As the charges undergo harmonic motion in the rods or dipole wires at a frequency to the frequency of the AC voltage source $\left(f = \frac{1}{T}\right)$, the frequency of the generated electromagnetic wave is same.

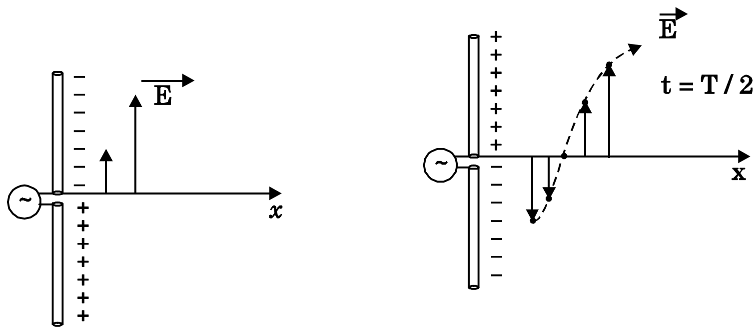


Fig. 2.15

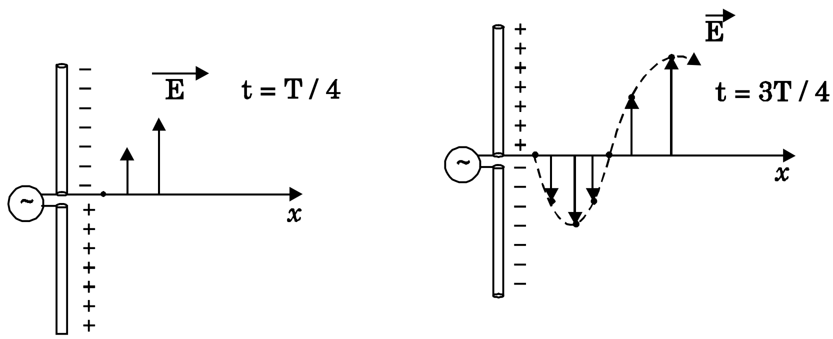


Fig. 2.16

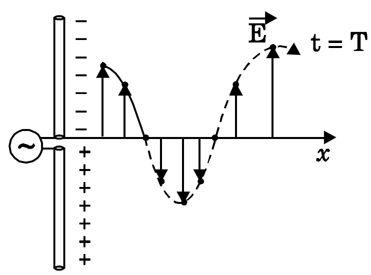


Fig. 2.17

Here 'T' is the time period of oscillations of EM waves.

Thus, whenever an electromagnetic wave will frequency '*f*' is observed, then it must be generated by an electromagnetic oscillator that oscillates at the source frequency '*f*'. Most of the dipole antennas have a total length of $\lambda/2$. Here λ is the wavelength of *EM*.

Production of electromagnetic waves - Hertz experiment

Maxwell's prediction was experimentally confirmed by Heinrich Rudolf Hertz in 1888. The experimental set up used is shown in fig. 2.18.

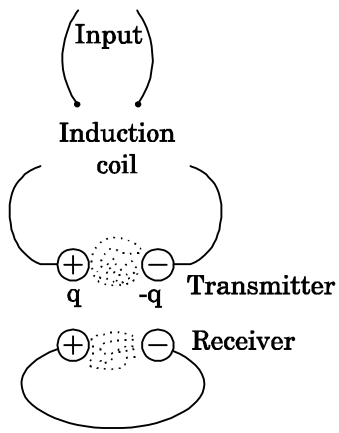


Fig. 2.18 Schematic diagram of Hertz apparatus

It consists of two metal electrodes which are made of small spherical metals. These are connected to larger spheres and the ends of them are connected to induction coil with very large number of turns. This is to produce very high electromotive force (emf).

Since the coil is maintained at very high potential, and air between the electrodes gets ionized and spark (spark means discharge of electricity) is produced.

This discharge of electricity affects another electrode (ring type – not completely closed) which is kept at far distance.

This implies that the energy is transmitted from electrode to the receiver (ring electrode) in the form of waves, known as electromagnetic waves.

If the receiver is rotated by 90°, then no spark is observed by the receiver. This confirms that electromagnetic waves are transverse waves as predicted by Maxwell.

Hertz detected radio waves and also computed the speed of radio waves which is equal to the speed of light ($3 \times 10^8\ ms^{-1}$).

2.12 ELECTROMAGNETIC ENERGY FLOW AND POYNTING VECTOR

Electromagnetic waves transports energy from one region to another. This energy transferred is described by power / unit area for an area perpendicular to the direction of an *EM* wave travel.

Let us consider a stationary plane perpendicular to the x-axis which coincides with the wave front at a certain time t. (Fig 2.19)

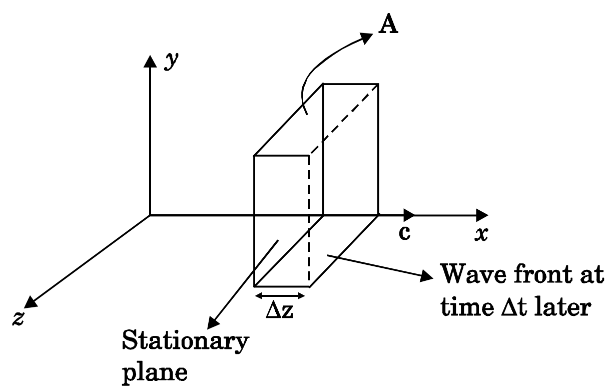


Fig. 2.19 A plane wave front at a time Δt after it passes through the stationary plane with area A and volume V.

At a time Δt after this, the wavefront moves a distance to the right side of the plane.

$$\Delta x = c\,\Delta t \qquad \left(\because c = \frac{\Delta x}{\Delta t}\right)$$

Area of cross section of the stationary plane is A and its volume $\Delta V = A \cdot \Delta x = A \cdot c \cdot \Delta t$.

If ΔU is the available energy in this volume, then then

$$\Delta U = u \, \Delta V = (\varepsilon_o E^2)(A \, c \, \Delta t) \qquad \text{... (1)}$$

Here 'u' is the energy density which is equal to $\varepsilon_o E^2$.

This energy ΔU passes through the area 'A' in time Δt. Therefore, the energy flow / unit time / unit area is

$$S = \frac{\Delta U}{A \cdot \Delta t} = \varepsilon_o E^2 c \qquad \text{... (2)}$$

Since $E = c B$ and $c = \dfrac{1}{\sqrt{\varepsilon_o \mu_o}}$

Eqn. (2) becomes

$$S = \varepsilon_o c^2 B^2 c \qquad \text{... (3)}$$

$$S = \frac{\varepsilon_o c B^2}{\varepsilon_o \mu_o} \qquad \text{... (4)}$$

$$\left(\because c^2 = \frac{1}{\varepsilon_o \mu_o} \right)$$

$$(\because c\vec{B} = \vec{E})$$

$$\text{or} \qquad S = \frac{cB \cdot B}{\mu_o} = \frac{EB}{\mu_o} \quad \text{(in vacuum)} \qquad \text{... (5)}$$

The unit of 'S' is energy per unit area per unit time or power per unit area. The SI unit of 'S' is W/m^2.

In vector quantity, it is represented as

$$\boxed{\vec{S} = \frac{1}{\mu_o} \vec{E} \times \vec{B}} \quad \text{... (6)}$$

or
$$\vec{S} = \vec{E} \times \frac{\vec{B}}{\mu_o} \qquad (\because \vec{B} = \mu_o \vec{H})$$

or
$$\boxed{\vec{S} = \vec{E} \times \vec{H}} \quad \text{...(7)}$$

Eqn. (6) is the Poynting vector in vacuum. Its direction is in the direction of the propagation of the *EM* wave.

2.13 INTENSITY OF AN EM WAVE IN VACUUM

The magnitude of the average value of $\vec{S}$ at a point is called the intensity of radiation at that point. The S.I unit of intensity is W/m^2.

Let us consider the electric and magnetic field solutions

$$\vec{E}(x, t) = E_y \cos(\omega t - kx) \quad \text{... (8)}$$

and

$$\vec{B}(x, t) = B_z \cos(\omega t - kz)$$

From eqn. (6)

$$\vec{S} = \frac{1}{\mu_o} \vec{E} \times \vec{B} \text{ becomes}$$

$$\vec{S}(x, t) = \frac{1}{\mu_o} E_y \cos(\omega t - kx) \times B_z \cos(\omega t - kz) \quad \text{... (9)}$$

The x-component (Direction of propagation) of the poynting vector is given as

$$S_x(x,t) = \frac{E_y B_z}{\mu_o} \cos^2(\omega t - kx) \qquad \text{... (10)}$$

$$= \frac{E_y B_z}{\mu_o} \left(\frac{1 + \cos 2(\omega t - kx)}{2} \right) \qquad \text{... (11)}$$

The time average value of $\cos 2(\omega t - kx)$ is zero. So the average value of the poynting vector is

$$S_{\text{average}} = \overline{S_x}(x,t) = \frac{E_x B_y}{2\mu_o} \qquad \text{... (12)}$$

or simply

$$S_{av} = \frac{E_y B_z}{2\mu_o} = \frac{E_y \cdot E_y}{2\mu_o c} \qquad \text{... (13)}$$

$$= \frac{E_y E_y}{2\mu_o \times \frac{1}{\sqrt{\mu_o \varepsilon_o}}}$$

$$S_{av} = \frac{1}{2}\sqrt{\frac{\varepsilon_o}{\mu_o}}\, E_y^2$$

$$S_{av} = = \frac{1}{2}\sqrt{\frac{\varepsilon_o \varepsilon_o}{\mu_o \varepsilon_o}}\, E_y^2$$

(or) $$S_{av} = \frac{\varepsilon_o}{\sqrt{\mu_o \varepsilon_o}} E_y^2$$

Intensity, $$\boxed{I = S_{av} = \frac{1}{2}\varepsilon_o c E_y^2} \qquad \text{... (14)}$$

This is the intensity of an *EM* wave in vacuum.

Also intensity is represented as for localised sources as $I = \frac{\text{Power}}{\text{Area}} = \frac{P}{4\pi r^2}$

2.14 MOMENTUM AND RADIATION PRESSURE

It is important to note that as *EM* waves carry energy, they also carry momentum. Maxwell proved that wave energy *U* and momentum are related by

$$P = \frac{v}{c} \quad \text{... (1)}$$

where v is energy density and c is the velocity by of light.

As the electromagnetic waves carry momentum, they exert pressure when they are reflected or absorbed at the surface of a body. This is known as radiation pressure.

From Newton's second law, the change in momentum is related to a force by

$$F = \frac{\Delta P}{\Delta t} \quad \text{... (2)}$$

As intensity $\mathrm{I} = \frac{\text{Power}}{\text{Area}} = \frac{\text{energy / time}}{\text{Area}}$

then for a flat surface of area A, which is perpendicular to the path of an *EM* wave radiation, the energy intercepted in a given time Δt is

$$\Delta U = I \cdot A \cdot \Delta t \quad \text{... (3)}$$

So, from eqn. (1), the momentum is

$$\Delta P = \frac{\Delta u}{c} = \frac{I \cdot A \cdot \Delta t}{c} \quad \text{... (4)}$$

and as $$\boldsymbol{F} = \frac{\Delta \boldsymbol{P}}{\Delta \boldsymbol{t}} = \frac{\boldsymbol{I \cdot A}}{\boldsymbol{c}} \quad \text{... (5)}$$

This is the relation for the total absorption of EM radiation.

This is due to 'ΔP' is the momentum change and the direction of momentum change of the object is the direction of the incident *EM* radiation that the object **absorbs**.

If the radiation is completely reflected back by the object along the original path then

$$\boxed{F = \frac{2IA}{c}} \qquad \dots (6)$$

Thus if the radiation is partly absorbed or completely reflected by the object, the magnitude of the force on area A varies between the values $\frac{IA}{c}$ and $\frac{2IA}{c}$

Radiation pressure

The force per unit area on an object due to EM radiation is the radiation pressure P_r. Thus from eqns. (5) and (6) we obtain

$$\text{Radiation pressure } P_r = \frac{F}{A}$$

$$P_r = \frac{I}{c}$$

for total absorption of radiation

and

$$\boxed{P_r = \frac{2I}{c},}$$

for total reflection back along the path

2.15 CELL PHONE RECEPTION

A typical cell phone contains a tiny low-power radio transmitter or antenna. EM signal intensity decreases as the inverse square of the distance from the phone.

The antenna's length is comparable to $\lambda/2$, where 'λ' is the wavelength of the EM signal being emitted by the cell phone.

As λ is short, so the cell phone antenna is also very short. Typically a simple dipole antenna as shown in the fig. 2.20 is used to detect the incoming EM signal in the cell phone.

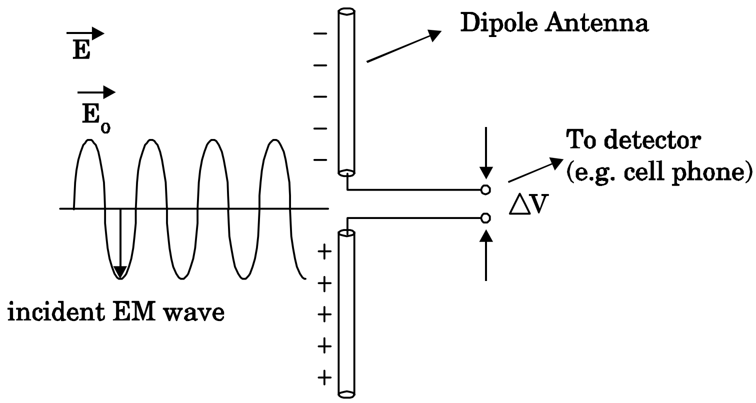


Fig. 2.20 Cell Phone Reception

In this antenna, the incident electric field of the *EM* signal induces a voltage across the wires of the antenna. This induced voltage is then amplified and processed by the circuitary in the cell phone.

The low power signals emitted by the cell phone will be received and transmitted by the cell phone towers. The towers are also another type of antenna. The cell phone transmits on one frequency and receive with other frequency.

2.16 REFLECTION AND TRANSMISSION OF EM WAVES VACUUM - NON - CONDUCTING MEDIUM INTERFACE FOR NORMAL INCIDENCE

Let us consider a monochromatic (single frequency) uniform plane wave that travels through one medium (vacuum) and enters another medium (non-conducting) of infinite extent.

The uniform plane *EM* wave propagating along *x*-direction in a vacuum medium (μ_o, ε_o) **incident normally** on the surface of a flat non-conducting medium permittivity, ($\mu \neq \mu_o$ and permittivity, $\varepsilon \neq \varepsilon_o$).

Here the incoming *EM* wave is called the incident wave, the interface is an infinite plane at $x = 0$, the region to the left of the interface is medium 1 ($x \leq 0$) and the region to the right of the interface is medium 2 ($x \geq 0$).

At the interface, a part of the incident *EM* wave will penetrate the boundary (interface) and continue its propagation in medium 2. Now this wave is called the transmitted wave.

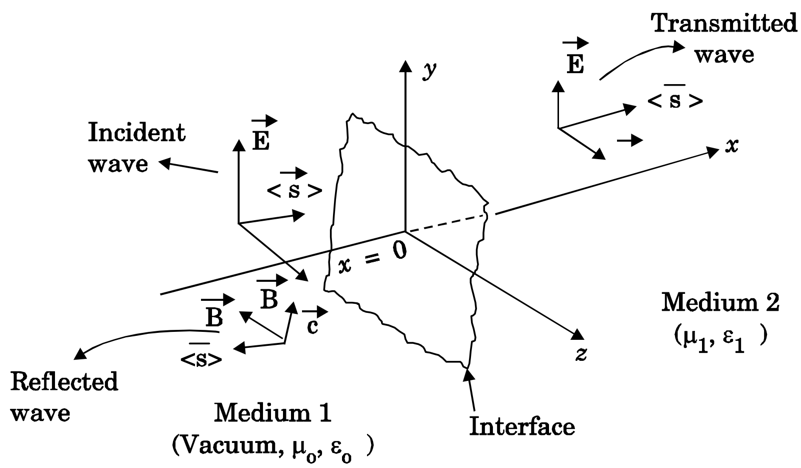


Fig. 2.21 Here $<\vec{s}>$ is the propagation vector (or) poynting vector

Another remainder of the wave is reflected at the interface and then propagates in the negative x direction. This wave is called the reflected wave as shown in fig. 2.21.

Thus both the incident and transmitted waves propagate in $+x$ direction. The reflected wave will propagate in $-x$ direction. So the incident and reflected waves are in medium 1 and the transmitted wave is in medium 2.

Now by considering the electric field $\vec{E}$ of the incident wave which is polarized in y-direction (plane polarized) and has an amplitude E_o at the interface as shown in fig. 2.21.

If $(k_1 = \omega / v_1)$ is the propagation constant of this wave (with angular frequency ω and velocity equal to v_1) in medium-1, then the electric and magnetic field waves are represented as

$$\vec{E}_i(x, t) = E_o \cos(\omega t - k_1 x), \qquad \dots (1)$$

and

$$\vec{B}_i (x, t) = \frac{E_o}{v_1} \cos(\omega t - k_1 x), \quad \text{... (2)}$$

$$\left(\because B_o = \frac{E_o}{v_1} \right)$$

Then, the reflected waves are represented as,

$$\vec{E}_R (x, t) = E_1 \cos (\omega t + k_1 x), \quad \text{... (3)}$$

and $$\vec{B}_R (x, t) = \frac{E_1}{v_1} \cos(\omega t + k_1 x), \quad \text{... (4)}$$

$$\vec{E}_T (x, t) = E_2 \cos (\omega t - k_2 x) \quad \text{... (5)}$$

We know that $E_o = cB_o$

$$\vec{B}_T (x, t) = \frac{E_2}{v_2} \cos(\omega t - k_2 x) \quad \text{... (6)}$$

Here in eqns (3) and (4), the sign is reversed used in the wave number k to denote that this wave is propagating from the interface (boundary) along negative x direction (backward travelling wave). Also the wave numbers k_1 and k_2 are related to

$$\boxed{k_1 = \frac{\omega}{v_1}} \quad \text{... (7)}$$

and

$$\boxed{k_2 = \frac{\omega}{v_2}} \quad \text{... (8)}$$

where v_1 and v_2 are the velocities of *EM* waves in medium-1 and medium-2 respectively.

The total instantaneous electric field $\vec{E}_y$ for any value of x with medium 1 is equal to the sum of the incident and reflected waves, so

$$\vec{E}_y (x, t) = E_o \cos (\omega t - k_1 x) + E_1 \cos (\omega t + k_1 x) \quad \text{... (9)}$$

(or)

$$\vec{E}_y (x, t) = \vec{E}_i (x, t) + \vec{E}_R (x, t) \quad \text{... (10)}$$

The total instantaneous electric field $\vec{E}_y$ for any value of x in the medium-2 is

$$\vec{E}_y (x, t) = E_2 \cos (\omega t - k_2 x) \quad \text{... (11)}$$

At the interface $x = 0$, the boundary conditions require that the tangential components of $\vec{E}$ and $\vec{B}$ fields must be **continuous**.

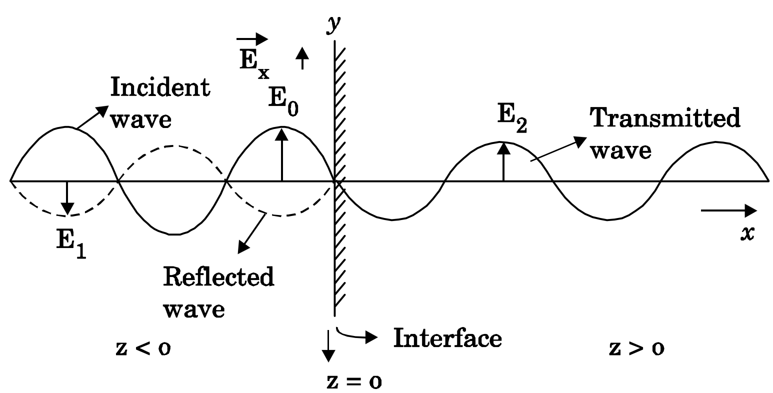


Fig. 2.22 (a) Electric field wave patterns

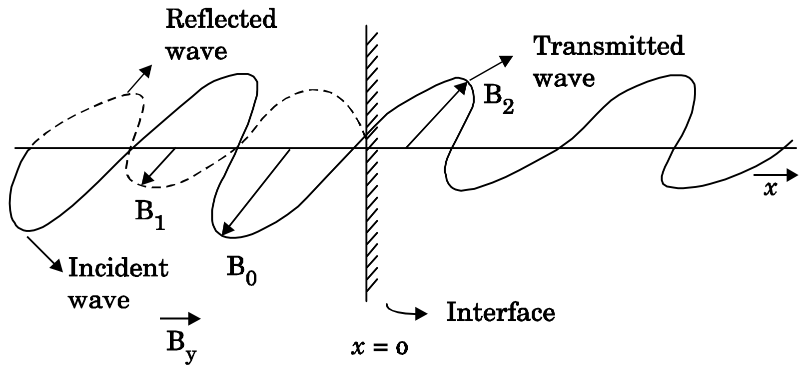


Fig. 2.22 (b) Magnetic field wave patterns

Since the waves are transverse, $\vec{E}$ and $\vec{B}$ fields are entirely tangential to the interface. Hence at $x = 0$, eqn. (9) and (10) are equal, so

$$E_o \cos(\omega t - k_1 x) + E_1 \cos(\omega t - k_1 x) = E_2 \cos(\omega t - k_2 x) \quad (12)$$

as $x = 0$, then

$$E_o \cos(\omega t) + E_1 \cos(\omega t) = E_2 \cos(\omega t)$$

(or)

$$E_o + E_1 = E_2 \quad \text{... (13)}$$

Also at boundary $x = 0$, as tangential components are continuous therefore

$$\frac{dE_i}{dx} + \frac{dE_R}{dx} = \frac{dE_T}{dx} \quad \text{... (14)}$$

which yields

$$-E_o k_1 \sin(\omega t) - E_1 k_1 \sin(\omega t) = E_2 k_2 \sin(\omega t) \quad \text{... (15)}$$

(or)

$$E_o k_1 - E_1 k_1 = E_2 k_2,$$

(or)

$$k_1 (E_o - E_1) = E_2 k_2,$$

(or)

$$E_o - E_1 = E_2 \cdot \left(\frac{k_2}{k_1} \right) \quad \text{... (16)}$$

As $k_1 = \dfrac{\omega}{v_1}$ and $k_2 = \dfrac{\omega}{v_2}$, then eqn. (16) becomes

$$E_o - E_1 = E_2 \cdot \left(\frac{v_1}{v_2} \right) \quad \text{... (17)}$$

Adding eqns (13) and (17) gives

$$2 E_o = E_2 + E_2 \left(\frac{v_1}{v_2} \right)$$

$$= E_2 \left(1 + \frac{v_1}{v_2} \right)$$

$$E_o = \left(\frac{E_2}{2} \right) \left(1 + \frac{v_1}{v_2} \right) \quad \text{... (18)}$$

When medium-1 is vacuum $v_1 = c$, and $v_2 = v$

$$\therefore \quad E_o = \left(\frac{E_2}{2} \right) \left(1 + \frac{c}{v} \right)$$

Subtracting eqn. (17) from eqn. (13) gives

$$E_1 = \left(\frac{E_2}{2} \right) \left(1 - \frac{v_1}{v_2} \right) \quad \text{... (19)}$$

$$\boxed{E_1 = \left(\frac{E_2}{2} \right) \left(1 - \frac{c}{v} \right)}$$

ANNA UNIVERSITY SOLVED PROBLEMS

Problem 2.1:
A circular cross section conductor of radius 2 mm carries a current $I_C = 2.5 \sin (5 \times 10^8 t)\ \mu A$ what is the amplitude of the displacement current density if $\sigma = 35\ M\Omega^{-1}\ m^{-1}$ and $\varepsilon_r = 1$.
(A.U. May 2017)

Given data

$$I_c = 2.5 \sin (5 \times 10^8 t)\ \mu A$$

$$= 2.5 \sin (5 \times 10^8 t) \times 10^{-6}\ A$$

$$\sigma = 35\ MS\ m^{-1} = 3.5 \times 10^6\ S\ m^{-1}$$

$$\varepsilon_r = 1$$

$$\varepsilon_o = 8.85 \times 10^{-12}\ F\ m^{-1}$$

$$\omega = 5 \times 10^8\ rads^{-1}$$

$$\text{radius } a = 2\ mm = 2 \times 10^{-3}\ m$$

Solution

We know that

$$J_C = \sigma E$$

$$J_D = \varepsilon \frac{\partial E_S}{\partial t}$$

$$\text{if } E_S = E \sin \omega t$$

$$\frac{\partial E_s}{\partial t} = E\omega \cos \omega t$$

$$\therefore \quad \vec{J}_D = \varepsilon E \omega \cos t\, \omega t$$

Taking only the amplitude

$$\varepsilon \frac{\partial E_s}{\partial t} = \omega \varepsilon E$$

$$\therefore \quad \frac{J_C}{J_D} = \frac{\sigma E}{\omega \varepsilon E} = \frac{\sigma}{\omega \varepsilon} = \frac{\sigma}{\omega \varepsilon_o \varepsilon_r} \qquad [\because \varepsilon_r = 1]$$

$$= \frac{35 \times 10^6}{(5 \times 10^8) \times 1 \times 8.85 \times 10^{-12}}$$

$$\frac{J_C}{J_D} = 79.09 \times 10^8$$

Further, $J_C = \dfrac{I_C}{\text{Area}}$

Area $= \pi a^2 = \pi \times (2 \times 10^{-3})^2$

taking only amplitude of I_C

$$J_C = \frac{2.5 \times 10^{-6}}{\pi \times (2 \times 10^{-3})^2} = 0.198$$

$$\frac{J_C}{J_D} = 79.09 \times 10^8$$

$$J_D = \frac{J_C}{79.09 \times 10^8}$$

$$J_D = \frac{0.198}{79.09 \times 10^8}$$

$$\text{J}_\text{D} = 2.515 \times 10^{-11} \text{ A / m}^2$$

Problem 2.2:
The conduction current flowing through a wire with conductivity $\sigma = 3 \times 10^7$ S/m and relative permittivity $\varepsilon_r = 1$ is given by $I_C = 3 \sin \omega t$ (mA). If $\omega = 10^8$ rad/sec. Find the displacement current.

(A.U. EE May 2014)

Given data

Conductivity $\sigma = 3 \times 10^7 \text{ Sm}^{-1}$

Conducting current $I_C = 3 \sin \omega t$ mA

$= 3 \times 10^{-3} \sin \omega t$ A

Angular frequency $\omega = 10^8 \text{ rad s}^{-1}$

Solution:

Displacement current density $J_D = \varepsilon \dfrac{\partial E}{\partial t}$

Since I_C is given, E is found from conduction current density as

$$J_C = \sigma E$$

$$\frac{I_C}{A} = \sigma E \qquad \left[\because J_C = \frac{I_C}{A}\right]$$

$$E = \frac{I_C}{\sigma A} = \frac{3 \times 10^{-3} \sin \omega t}{3 \times 10^7 \times A} \quad \text{Vm}^{-1}$$

$$= \frac{1 \times 10^{-10}}{A} \sin \omega t$$

($\because$ ignoring – ve sign and taking only magnitude)

$$\frac{\partial E}{\partial t} = \left(\frac{1 \times 10^{-10}}{A}\right) \omega \cos \omega t$$

Multiplying both sides by ε

$$\varepsilon \frac{\partial E}{\partial t} = \varepsilon \left(\frac{1 \times 10^{-10}}{A} \right) \omega \cos \omega t \qquad \left(\because J_D = \varepsilon \frac{\partial E}{\partial t} \right)$$

$$J_D = \varepsilon\omega \left(\frac{1 \times 10^{-10}}{A} \right) \cos \omega t$$

$$J_D = \varepsilon_o \varepsilon_r \omega \left(\frac{1 \times 10^{-10}}{A} \right) \cos \omega t \qquad (\because \varepsilon = \varepsilon_o \varepsilon_r)$$

$$J_D = 8.84 \times 10^{-12} \times 10^{8} \times \left(\frac{1 \times 10^{-10}}{A} \right) \cos \omega t$$

$$J_D = \frac{8.85 \times 10^{-14}}{A} \cos \omega t$$

$$I_D = J_D A$$

substituting for J_D

$$I_D = \left(\frac{8.85 \times 10^{-14}}{A} \cos \omega t \right) A$$

$$I_D = 8.85 \times 10^{-14} \cos \omega t \quad \text{ampere}$$

Problem 2.3:
A poor conductor is characterised by a conductivity $\sigma = 100$ (S/m) and permittivity $\varepsilon = 4\,\varepsilon_o$. At what angular frequency ω is the amplitude of the conduction current density J_C equal to the amplitude of the displacement current density J_D? **(A.U. EC June 2014)**

Given data

Conductivity of the conductor $\sigma = 100\ \text{Sm}^{-1}$

Permittivity of the conductor $\varepsilon = 4\,\varepsilon_o$

Solution:

$$J_C = \sigma E$$

$$J_D = \frac{\partial \vec{D}}{\partial t}$$

$$= \varepsilon \frac{\partial E_o}{\partial t} \cos \omega t$$

$$= \varepsilon \omega E$$

$$J_D = \varepsilon\omega E \text{ (only amplitude)}$$

Equating conduction current density and displacement current density, we have

$$\sigma E = \varepsilon\omega\, e$$

$$\sigma = \omega\, \varepsilon$$

$$100 = \omega \times 4\, \varepsilon_o$$

$$\omega = \frac{100}{4 \times 8.85 \times 10^{-12}}$$

$$= 2.82 \times 10^{12} \text{ rad s}^{-1}$$

Problem 2.4:
In a material for which $\sigma = 5$ S/m and $\varepsilon_r = 1$ the electric field intensity is $E = 250 \sin 10^{10} t$ V/m. Find the conduction and displacement current densities and the frequency at which they have equal magnitudes. **(A.U. Dec 2015)**

Given data

Conductivity of medium $\sigma = 5 \text{ Sm}^{-1}$

Relative permittivity of medium $\varepsilon_r = 1$

Solution

Conduction current density

$$J_C = \sigma E$$

$$J_C = 5 \times 250 \sin 10^{10} t$$

$$= 1250 \sin 10^{10} t \quad \text{Am}^{-2}$$

Displacement current density

$$J_D = \varepsilon \frac{\partial E}{\partial t}$$

$$= \varepsilon \frac{\partial}{\partial t} 250 \sin 10^{10} t \qquad (\because \varepsilon = \varepsilon_o \varepsilon_r \text{ and}$$

$$= \varepsilon_o \varepsilon_r \frac{\partial}{\partial t} 250 \cos 10^{10} t \times 10^{10} \qquad \varepsilon_r = 1)$$

$$= 8.85 \times 10^{-12} [250 \times 10^{10} \cos 10^{10} t]$$

$$= 22.13 \cos 10^{10} \text{t} \;\; \text{Am}^{-2}$$

For $J_C = J_D$

$$\sigma E = \omega \varepsilon E \qquad \text{[Take the amplitude only]}$$

$$\omega = \frac{\sigma}{\varepsilon} = \frac{\sigma}{\varepsilon_o \varepsilon_r} \frac{5}{8.85 \times 10^{-12} \times 1}$$

$$= 5.65 \times 10^{11} \text{ rad s}^{-1}$$

$$\omega = 2 \pi f$$

$$\therefore f = \frac{\omega}{2\pi} = \frac{5.65 \times 10^{11}}{2 \times \pi} = \frac{5.65 \times 10^{11}}{2 \times 3.14}$$

$$f = 8.997 \times 10^{10} \text{ Hz}$$

Problem 2.5

Find the conduction and displacement current densities in a material having conductivity of 10^{-3} S/m and ε_r = 2.5 if the electric field in the material is $E = 5.0 \times 10^{-6}$ sin $9.0 \times 10^9 t$ V/m. **(A.U. Dec 2017)**

Given data

Conductivity of the medium $\sigma = 10^{-3}\ Sm^{-1}$

Relative permittivity $\varepsilon_r = 2.5$

Solution

Conduction current density

$$J_c = \sigma E = 10^{-3} \times 5 \times 10^{-6} \sin 9 \times 10^9\ t \quad Am^{-2}$$

$$= 5 \times 10^{-9} \sin 9.0 \times 10^9\ t$$

Displacement current density

$$J_D = \frac{\partial D}{\partial t} = \frac{\partial\,(\varepsilon_o\,\varepsilon_r E)}{\partial t}$$

$$= \varepsilon_o\,\varepsilon_r \frac{\partial \vec{E}}{\partial t}$$

$$= 8.85 \times 10^{-12} \times 2.5 \times \frac{\partial}{\partial t}(5 \times 10^{-6} \sin 9 \times 10^9 t)$$

$$= 8.85 \times 10^{-12} \times 2.5 \times 5 \times 10^{-6} (\cos 9 \times 10^9 t) \times 9 \times 10^9$$

$$= 9.96 \times 10^{-7} \cos 9.0 \times 10^9\ t\ .\ Am^{-2}$$

Problem 2.6

A light bulb of 20 W radiates energy isotropically, mostly in infrared region. Assuming it to be a point source find the irradiance (average energy per unit area per unit time or intensity) at a distance 1 m away. Calculate the strength of $\vec{E}$-field associated with the radiation at this distance. **(A.U. EC Dec 2013)**

Given data:

Power radiated by bulb $P = 20$ W

Distance $r = 1$ m

Solution:

Irradiance is average energy per unit area per unit time at a distance '*r*'

$$I = \frac{\text{Power}}{\text{Area}} = \frac{P}{A}$$

$$= \frac{P}{4\pi r^2}$$

$$I = \frac{20}{4\pi \times 1^2} = 1.59 \text{ Wm}^{-2}.$$

Intensity at a point is nothing but poynting vector at that point.

$$\text{ie., } I = S_{\text{avg}}$$

Now, average poynting vector $S_{\text{avg}} = \dfrac{E^2_{\text{rms}}}{c\,\mu_o}$.

$$I = \frac{E^2_{\text{rms}}}{\mu_o\, c}$$

$$E^2_{\text{rms}} = I\,\mu_o\, c$$

$$E_{\text{rms}} = \sqrt{\mu_o\, c\, I}$$

substituting the given values, we have

$$= \sqrt{4\,\pi \times 10^{-7} \times 3 \times 10^{8} \times 1.59}$$

$$= 24.5 \text{ Vm}^{-1}$$

Problem 2.7

The intensity of sunlight reaching the earth's surface is about 1300 W m^{-2}. Calculate the strength of electric and magnetic fields of the incoming sunlight.

(A.U. EC Dec 2017)

Solution:

The time average Poynting's vector is

$$S_{avg} = \frac{1}{2} E_o \cdot H_o = \frac{E_o}{\sqrt{2}} \cdot \frac{H_o}{\sqrt{2}} = E_{\text{rms}} \cdot H_{\text{rms}} .$$

$$(\because \frac{E_o}{\sqrt{2}} = E_{\text{rms}}, \qquad \frac{H_o}{\sqrt{2}} = H_{\text{rms}})$$

But in electromagnetic wave

$$H_{\text{rms}} = \frac{E_{\text{rms}}}{c\,\mu_o}$$

$$S_{avg} = \frac{E_{\text{rms}}^2}{c\,\mu_o}$$

$$E_{\text{rms}}^2 = c\,\mu_o\, S_{\text{avg}}$$

$$\text{or } E_{\text{rms}} = \sqrt{c\,\mu_o \cdot S_{avg}}$$

$$= \sqrt{3 \times 10^{8} \times 4\pi \times 10^{-7} \times 1300}$$

$$= 700 \text{ Vm}^{-1} .$$

from above relation

$$H_{\text{rms}} = \frac{E_{\text{rms}}}{\mu_o\, c}$$

$$\mu_o H_{rms} = \frac{E_{rms}}{c}$$

$$\therefore B_{rms} = \frac{E_{rms}}{c} \qquad (\because B_{rms} = \mu_o H_{rms})$$

$$= \frac{700}{(3 \times 10^8)}$$

$$= 2.33 \times 10^{-6}\ T.$$

Problem 2.8

(i) Show that for a good conductor skin depth $\delta = \lambda_c/2\pi$, where λ_c is the wavelength of electromagnetic waves in the conductor.

(ii) Show that for an electromagnetic wave incident on a good conductor the electric vector reduces to about 1% at a depth of 0.73 λ_c.

(iii) Find the wavelength and the propagation speed in copper for radio waves at 1 MHz. For copper assume $\mu = \mu_o$, $\varepsilon = \varepsilon_o$ and conductivity $\sigma = 5.8 \times 10^7\ \Omega - m^{-1}$

[A.U. EE Dec 2013]

Solution:

(i) Suppose the wave is incident normally on the surface of the conductor along z-axis, then the electric field inside the conductor is given by

$$\vec{E}(x, t) = E_o e^{-kx} \cdot e^{i(\omega t - kx)}$$

k - wave vector

For a good conductor

$$k = \sqrt{\frac{\omega \mu \sigma}{2}}$$

and skin depth

$$\delta = \frac{1}{k} = \sqrt{\frac{2}{\omega \mu \sigma}}$$

The wavelength of the wave in the conductor is

$$\lambda_c = \frac{2\pi}{k} = 2\pi\delta . \qquad \left(\because \alpha = \frac{1}{\delta} \right)$$

$$\therefore \text{ Skin depth } \delta = \frac{\lambda_c}{2\pi}.$$

(ii) Let us write

$$E(x) = E_o e^{-kx} = E_o e^{-k/\delta} = E_o(x)\ e^{-2\pi x/\lambda_c}$$

Now put $\qquad \left(\because k = \frac{2\pi}{\lambda_c} \right)$

$$\frac{E(x)}{E_o(x)} = e^{-2\pi x/\lambda_c} = \frac{1}{100}$$

$$e^{-2\pi x/\lambda_c} = \frac{1}{100}$$

$$e^{2\pi x/\lambda_c} = 100$$

taking ln on both sides

$$\frac{2\pi x}{\lambda_c} = \ln 100$$

$$x = \frac{\lambda_c}{2\pi} \ln 100 = 0.73\,\lambda_c$$

(iii) The required wavelength at 1 MHz $= 1 \times 10^6$ Hz

$$\lambda_c = \frac{2\pi}{k} = 2\pi\sqrt{\frac{2}{\omega \mu \sigma}}$$

$$= 2\,\pi \sqrt{\frac{2}{2\,\pi\nu\,\mu\,\sigma}}$$

$$= 2\,\pi \sqrt{\frac{2}{2\,\pi \times 10^{6} \times 4\pi \times 10^{-7} \times 5.8 \times 10^{7}}}$$

$$= 4.15 \times 10^{-4}\ \text{m}$$

and the propagation speed

$$v = \frac{\omega}{k_c} = \omega \times \frac{\lambda_c}{2\,\pi} = \frac{2\,\pi \times 10^{6}}{2\,\pi} \times 4.15 \times 10^{-4}$$

$$v = 415\ \text{ms}^{-1}$$

Problem 2.9:
Find the depth of penetration of a plane wave in copper at a power frequency of 60 Hz and at microwave frequency 10^{10} **Hz. Given** $\sigma = 5.8 \times 10^{7}$ **mho** m^{-1} **(A.U. June 2017)**

Given data

$$\sigma = 5.8 \times 10^{7}\ \text{mho m}^{-1}.$$

Depth of penetration $\delta = \dfrac{1}{k} = \sqrt{\dfrac{2}{\omega\,\mu\,\sigma}}$

(i) $f = 60$ Hz, $\sigma = 5.8 \times 10^{7}$ mhom^{-1}, $\mu_r = 1$,

$\mu_o = 4\pi \times 10^{-7}\ \text{Hm}^{-1}$

$\omega = 2\,\pi f$

$$\delta = \sqrt{\frac{2}{2\,\pi \times 60 \times 4\pi \times 10^{-7} \times 5.8 \times 10^{7}}}$$

$$= 8.53 \times 10^{-3}\ \text{m}$$

(ii) $f = 10^{10}$ Hz

$$\delta = \sqrt{\frac{2}{2\,\pi \times 10^{10} \times 4\pi \times 10^{-7} \times 5.8 \times 10^{7}}}$$

$$\boxed{\delta = 6.6 \times 10^{-7}\ \text{m}}$$

Problem 2.10:

A sinusoidal plane wave is transmitted through a medium whose electric field is 10 kV/m and relative permittivity of the medium is 4. Determine the mean rms power flow/unit area. (A.U. ECE June 2015)

Solution:

$$E = 10\ \text{KVm}^{-1} = 10 \times 10^{3}\ \text{V m}^{-1}$$

$$\varepsilon_r = 4, \quad \mu_r = 1$$

$$\frac{\vec{E}}{\vec{H}} = \sqrt{\frac{\mu}{\varepsilon}}$$

$$= \sqrt{\frac{\mu_o\,\mu_r}{\varepsilon_0\,\varepsilon_r}}$$

$$= \sqrt{\frac{\mu_o}{\varepsilon_o}}\sqrt{\frac{\mu_r}{\varepsilon_r}} \qquad \left[\because \sqrt{\frac{\mu_o}{\varepsilon_o}} = 120\pi\right]$$

$$= \frac{120\pi}{2} = 60\,\pi \text{ ohm}$$

ie., $\frac{E}{H} = 188.5\,\text{ohm}$

$$\therefore \qquad \vec{H} = \frac{\vec{E}}{188.5}$$

Power flow/unit area is poynting vector

$$S = \vec{E} \times \vec{H}$$

$$= \vec{E} \times \frac{\vec{E}}{188.5}$$

$$= \frac{\vec{E}^2}{188.5}$$

$$= \frac{(10 \times 10^3)^2}{188.5}$$

$$\boxed{S = 5.305 \times 10^5 \text{ W m}^{-2}}$$

Problem 2.11:
Find the conducting behaviour of ground at 1 KHz, 10 MHz and 10 GHz. Given $\varepsilon_r = 10$ *and* $\sigma = 5 \times 10^{-3}$ mho / m.
(A.U. EE May 2016)

Solution:

The ratio of conduction current to displacement current is $\frac{\sigma}{\omega\,\varepsilon}$ which determines conducting behaviour

For $f = 1$ kHz $= 1 \times 10^3$ Hz

$$\omega = 2\,\pi f$$

$$\sigma = 5 \times 10^{-3} \text{ mhom}^{-1}$$

$$\varepsilon_r = 10$$

$$\varepsilon_o = 8.85 \times 10^{-12}\,\text{Fm}^{-1}$$

$$\frac{\sigma}{\omega\,\varepsilon} = \frac{\sigma}{2\,\pi\,f\,\varepsilon_o\,\varepsilon_r}$$

$$= \frac{5 \times 10^{-3}}{2\,\pi \times 1 \times 10^{3} \times 8.85 \times 10^{-12} \times 10}$$

$$= 9 \times 10^{3}$$

$$\frac{\sigma}{\omega\,\varepsilon} >> 1$$

For f = 1 MHz = 1×10^{6} Hz

$$\frac{\sigma}{\omega\,\varepsilon}$$

$$\frac{\sigma}{2\,\pi\,f\,\varepsilon_o\,\varepsilon_r}$$

$$= \frac{5 \times 10^{-3}}{2\,\pi \times 1 \times 10^{6} \times 8.85 \times 10^{-12} \times 10}$$

$$\frac{\sigma}{\omega\,\varepsilon} \approx 1$$

For f = 10 GHz = 10×10^{9} Hz

$$\frac{\sigma}{2\,\pi\,f\,\varepsilon_o\,\varepsilon_r}$$

$$= \frac{5 \times 10^{-3}}{2\,\pi \times 1 \times 10^{9} \times 8.85 \times 10^{-12} \times 10}$$

$$= 9 \times 10^{-4}$$

$$\frac{\sigma}{\omega \varepsilon} << 1$$

At 1 KHz the ground acts like a good conductor and at 10 GHz it acts like an insulator. For 10 MHz ground acts like a quasi-conductor.

Problem 2.12
Find the skin depth δ at a frequency of 1 MHz for copper where $\sigma = 5.8 \times 10^7$ S / m and $\mu = \mu_o$. Also find the value of skin depth at 50 Hz.

(A.U. ECE Dec 2004)

Given data

Conductivity of the medium $\sigma = 5.8 \times 10^7 \ \text{Sm}^{-1}$

Frequency $f = 1 \text{ MHz} = 1 \times 10^6$ Hz

Solution:

$$\mu = \mu_o \mu_r \text{ and } f = 1 \text{ MHz}$$

$$\delta = \sqrt{\frac{2}{\omega \mu \sigma}}$$

$$\delta = \sqrt{\frac{2}{2 \pi f \mu \sigma}} \qquad [\because \omega = 2 \pi f]$$

$$\delta = \frac{1}{\sqrt{\pi f \mu \sigma}}$$

Substituting the given values, we have

$$= \frac{1}{\sqrt{3.14 \times 10^6 \times 4\pi \times 10^{-7} \times 5.8 \times 10^7}}$$

$$= 0.0661 \times 10^{-3} \text{m}$$

$$= 6.61 \times 10^{-5} \text{ m}$$

Also at $f = 50$ Hz,

$$\delta = \frac{1}{\sqrt{\pi \times 50 \times 4\pi \times 10^{-7} \times 5.8 \times 10^{-7}}}$$

$$= 9.348 \times 10^{-3} \text{ m}$$

Problem 2.13:
Find the velocity of a plane wave in a lossless medium having a relative permittivity of 4 and a relative permeability of 1.2. **(A.U. EE Dec 2016)**

Given data

$$\varepsilon_r = 4; \qquad \mu_r = 1.2$$

Solution:

Velocity of EM wave in a medium is,

$$v = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{\mu_o \mu_r \varepsilon_o \varepsilon_r}}$$

$$= \frac{1}{\sqrt{\mu_o \varepsilon_o} \sqrt{\mu_r \varepsilon_r}}$$

$$\because \frac{1}{\sqrt{\mu_o \varepsilon_o}} = c = 3 \times 10^8 \text{ m s}^{-1}$$

$$v = \frac{c}{\sqrt{\mu_r \varepsilon_r}} = \frac{3 \times 10^8}{\sqrt{\mu_r \varepsilon_r}}$$

$$v = \frac{3 \times 10^8}{\sqrt{1.2 \times 4}}$$

$$\boxed{\boldsymbol{v = 1.37 \times 10^8 \text{ ms}^{-1}}}$$

Problem 2.14:
Find the characteristic impedance of the medium whose relative permittivity is 3 and relative permeability is 1.
(A.U. ECE April 2014)

Given data

$\varepsilon_r = 3; \quad \mu_r = 1$

Solution:

Characteristic impedance, $\eta = \sqrt{\dfrac{\mu}{\varepsilon}}$

$$\eta = \sqrt{\frac{\mu_o \, \mu_r}{\varepsilon_o \, \varepsilon_r}}$$

$$\eta = \eta_o \sqrt{\frac{\mu_r}{\varepsilon_r}}$$

where $\eta_o = \sqrt{\dfrac{\mu_o}{\varepsilon_o}} = 120\pi$

$$\therefore \; \eta = 120\pi \sqrt{\frac{1}{3}}$$

$$\boxed{\eta = 217.66 \text{ ohm}}$$

Problem 2.15:
Find the velocity of a plane wave in a lossless medium having a relative permittivity of 5 and relative permeability of unity.
(A.U. ECE May 2015)

Given data

$\varepsilon_r = 5, \quad \mu_r = 1$

Solution:

Velocity of propagation in the medium

$$v = \frac{1}{\sqrt{\varepsilon_o \varepsilon_r \mu_o \mu_r}}$$

$$v = \frac{1}{\sqrt{\varepsilon_o \mu_o}\sqrt{\varepsilon_r \mu_r}}$$

We know that $\frac{1}{\sqrt{\mu_o \varepsilon_o}} = c = 3 \times 10^8 \text{ m s}^{-1}$

$$v = \frac{1}{\sqrt{\varepsilon_o \mu_o}} \times \frac{1}{\sqrt{1 \times 5}} = \frac{3 \times 10^8}{\sqrt{5}}$$

$$\boxed{v = 1.34 \times 10^8 \text{ ms}^{-1}}$$

Part - 'A'
'2' Marks Questions with answers

1. Write Maxwell's equation - I from Gauss's law in electrostatics. **(A.U. Nov 2013)**

$$\oiint_S \vec{D} \bullet \vec{ds} = \iiint_V \rho \, dV$$

Integral form

Gauss's law in electrostatics states that the total electric flux through any closed surface is equal to the charge enclosed by it.

Differential form

$$\vec{\nabla} \bullet \vec{E} = \frac{\rho}{\varepsilon_o}$$

This is Maxwell's equation from Gauss's law in electrostatics in differential form

Statement

The total electric displacement through the surface enclosing a volume is equal to the total charge within the volume.

2. Write Maxwell's equation - II from Gauss's law in magnetostatics. **(A.U. June 2015)**

Statement

The total magnetic flux through any closed surface in a magnetic field is zero ie.,

$$\oiint_S \vec{B} \bullet \vec{ds} = 0$$

This is Maxwell's equation in integral form from Gauss's law in magnetostatics.

$$\vec{\nabla} \bullet \vec{B} = 0$$

This is Maxwell's equation in differential form from Gauss's law in magnetostatics.

Statement

The net magnetic flux emerging through any closed surface is zero.

3. Write Maxwell's equation - III from Faraday laws of electromagnetic induction.

$$\oint_C \vec{E} \bullet \vec{dl} = -\iint_S \frac{\partial \vec{B}}{\partial t} \bullet \vec{ds})$$

This is Maxwell's equation in integral form from Faraday's law of electromagnetic induction.

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

This eqn represents Maxwell's equation from Faraday's law of electromagnetic induction in differential form.

Statement:

The electromotive force around a closed path is equal to the rate of magnetic displacement (flux density) through that closed path.

4. Write Maxwell's equation - IV from Ampere's circuital law. **(A.U. Dec 2013)**

Ampere's law states that the line integral of magnetic field intensity H on any closed path is equal to the current (I) enclosed by that path

$$\oint \vec{H} \bullet \vec{dl} = I$$

then, $$\oint \vec{H} \bullet \vec{dl} = \iint_S \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) ds$$

This is Maxwell's equation in integral form from Ampere's circuital law.

or $$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

$$\vec{\nabla} \times \vec{H} = \sigma \vec{E} + \varepsilon \frac{\partial \vec{E}}{\partial t}$$

The above equations are Maxwell equations in differential form from Ampere's circuital law

Statement

The magnetomotive force around a closed path is equal to the sum of the conduction current and displacement current enclosed by the path.

5. Give the Maxwell's equations in differential form.

(A.U. May 2015)

$$\vec{\nabla} \bullet \vec{D} = \rho$$

$$\vec{\nabla} \bullet \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

6. Give the Maxwell's equations in integral form.

(A.U. Dec 2015)

$$\oiint_S \vec{D} \bullet \vec{ds} = \iiint_V \rho \, dV$$

$$\oiint_S \vec{B} \bullet \vec{ds} = 0$$

$$\oint \vec{E} \bullet \vec{dl} = -\iint_S \frac{\partial \vec{B}}{\partial t} \bullet \vec{ds}$$

$$\oint \vec{H} \bullet \vec{dl} = \iint_S \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) \vec{ds}$$

7. Write the Maxwell's equations for free space.

(A.U. April 2014)

$$\vec{\nabla} \bullet \vec{D} = 0$$

$$\vec{\nabla} \bullet \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{H} = \frac{\partial \vec{D}}{\partial t}$$

8. Write the Maxwell's equations for conducting medium.

$$\vec{\nabla} \bullet \vec{D} = \rho$$

$$\vec{\nabla} \bullet \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

9. What are the characteristics of Maxwell's First equation $\vec{\nabla} \times \vec{E} = \frac{\rho}{\varepsilon_o}$

- It explains Gauss's law in electrostatics.
- It is time independent or steady state equation.
- The flux of the lines of electric force depends upon charge density.
- Charge acts as a source or sink for the lines of electric force.

10. What are characteristics of Maxwell's second equation $\vec{\nabla} \cdot \vec{B} = 0$

- It expresses a well known observation that *isolated magnetic poles do not exist.*

- It states that total magnetic flux entering and leaving a given volume is equal.
- There is no source or sink for lines of magnetic force.
- It is a time independent equation.
- It explains Gauss's law in Magnetostatics.

11. What are characteristics of Maxwell's Third equation $\vec{\nabla} \times \vec{E} = \frac{\partial \vec{B}}{\partial t}$

- It relates the electric field vector $\vec{E}$ and magnetic induction vector $\vec{B}$.
- It is a time dependent or time varying equation.
- It explains the well known Faraday's laws and Lenz's law of electromagnetic induction.
- $\vec{E}$ is generated by the time variation of $\vec{B}$.

12. What are the characteristics of Maxwell's Fourth equation $\vec{\nabla} \times \vec{B} = \mu_o \left(\vec{J} + \frac{\partial \vec{B}}{\partial t} \right)$

- It gives relation with the magnetic field vector $\vec{B}$ with displacement vector $\vec{D}$ and the current density $\vec{J}$.
- It is also a time dependent equation.
- It explains Ampere's circuital law.
- $\vec{B}$ can be produced by $\vec{J}$ and the time variation of $\vec{D}$

13. Write down general electromagnetic wave equation in terms of electric field vector $\vec{E}$ for free space.

$$\nabla^2 \vec{E} = \mu_o \, \varepsilon_o \frac{\partial^2 \vec{E}}{\partial t^2}$$

μ_o – permeability in free space, ε_o – permittivity in free space.

This is general electromagnetic wave equation in terms of electric field vector $\vec{E}$ for free space.

14. Write down general electromagnetic wave equation in terms of magnetic field vector $\vec{H}$ for free space.

$$\nabla^2 \vec{H} = \mu_o \varepsilon_o \frac{\partial^2 \vec{H}}{\partial t^2}$$

μ_o – permeability is free space, ε_o – permittivity is free space.

This general electromagnetic wave equation in terms of $\vec{H}$ **for free space.**

15. Write down the expression for velocity of em wave in free space. **(A.U. May 2013)**

$$c = \frac{1}{\sqrt{\mu_o \varepsilon_o}}$$

For vacuum or free space we have $\mu_o = 4\pi \times 10^{-7}$ H m^{-1} (henry per metre) and $\varepsilon_o = 8.842 \times 10^{-12}$ Fm^{-1} (farad per metre).

16. Write down the general solution of wave equation for plane polarised em wave.

$$E_y = E_o \cos(\omega t - kx)$$

and $$H_z = H_o \cos(\omega t - kx)$$

where, ω - angular frequency

k - wave vector

17. Write down a relation between the electric field vector $\vec{E}$ and magnetic field vector $\vec{H}$. **(A.U. Dec 2013)**

$$\therefore \quad \frac{\vec{E}}{\vec{H}} = \sqrt{\frac{\mu_o}{\varepsilon_o}}$$

This is the relation between the electric field vector and magnetic field vector. It is determined by the μ_o and ε_o.

18. What is intrinsic or characteristic impedance of free space? **(A.U. Dec 2020)**

The ratio $\frac{\vec{E}}{\vec{H}}$ is having the unit of impedance (Resistance) ie., ohm, Therefore, the quantity $\sqrt{\frac{\mu_o}{\varepsilon_o}}$ has the dimensions of impedance.

It is known as **intrinsic or characteristic impedance of free space, denoted by Z_o**. It is a constant quantity for free space and having value $\approx 377\Omega$.

19. What is poynting vector? **(A.U. Dec 2013)**

The cross product of electric field vector $\vec{E}$ and the magnetic field vector $\vec{H}$ is called poynting vector. It is denoted by $\vec{S} = \vec{E} \times \vec{H}$.

20. Write the general wave equation for the electric vector in an em wave in conducting medium.

$$\nabla^2 \vec{E} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} - \mu\sigma \frac{\partial \vec{E}}{\partial t} = 0$$

$\vec{E}$ – electric field vector.

μ – permeability of medium.

ε – permittivity of medium.

This is the general wave equation for the electric vector in an electromagnetic wave propagating in conducting medium.

21. Write the general wave equation for the magnetic vector in an em wave in conducting medium.

In a similar way, by taking the curl of the eqn. (4) we obtain the general wave equation for the magnetic vector in a conducting medium as

$$\nabla^2 \vec{H} - \mu \varepsilon \frac{\partial^2 \vec{H}}{\partial t^2} - \mu \sigma \frac{\partial \vec{H}}{\partial t} = 0$$

$\vec{H}$ – Magnetic field vector.

μ – permeabity of medium.

ε – permittivity of medium.

22. What is skin depth?

It is defined as the distance inside the conductor from the surface of the conductor at which the amplitude of the field vector is reduced to 1/e times its value at the surface.

23. Define intensity of EM wave.

The magnitude of the average value of $\vec{S}$ at a point is called the intensity of radiation at that point. The S.I unit of intensity is W/m^2. It is given by

$$S_{av} = \frac{1}{2} \varepsilon_o c E_y^2$$

ε_o – permittivity of the medium.

c – velocity of light.

24. Define radiation pressure.

The force per unit area on an object due to *EM* radiation is the radiation pressure P_r.

$P_r = \frac{I}{c}$, for totals absorption of radiation

$P_r = \frac{2I}{c}$, for total reflection back along the path

25. Give the Properties of Electromagnetic Waves.

(i) Electromagnetic waves are produced by accelerated charges.

(ii) They do not require any material medium for propagation.

(iii) In an electromagnetic wave, the electric ($\vec{E}$) and magnetic ($\vec{B}$) field vectors are at right angles to each other and to the direction of propagation. Hence, electromagnetic waves are transverse in nature.

(iv) Variation of maxima and minima in both $\vec{E}$ and $\vec{B}$ occur simultaneously (in phase).

(v) They travel in vacuum or free space with a speed $3 \times 10^8 \text{ ms}^{-1}$ given by the relation $c = \frac{1}{\sqrt{\mu_o \varepsilon_o}}$.

(μ_o – permeability of free space and
ε_o – permittivity of free space)

(vi) The energy in an electromagnetic wave is equally divided between electric and magnetic field vectors.

(vii) The electromagnetic waves being chargeless, they are not deflected by electric and magnetic fields.

Part - B '16' Marks Questions

1. Derive Maxwell's equations in differential and integral form. **(A.U. Dec 2013)**
2. Deduce Maxwell's equations for free space. **(A.U. May 2015)**
3. Write Maxwell's equations and explain the characteristics of each equation.
4. Give an account of Maxwell's equation in free space. Apply the equations to deduce the e.m. wave equation and determine the velocity light in vacuum. **(A.U. Dec 2016)**
5. Deduce Maxwell's equation free space and prove that electromagnetic waves are transverse. **(A.U. Dec 2014)**
6. Discuss propagation of EM wave in a conducting medium.

7. Derive wave equation for conducting medium using Maxwell's equation and determine skin depth in a conducting medium.
8. What is meant by poynting vector? What is its significance. **(A.U. Dec 2013)**
9. Derive general field relations for time-varying electric and magnetic fields using Maxwell's equations. **(A.U. Dec 2017)**
10. Derive Maxwell's equations derived from Faraday's law in integral and point forms. **(A.U. Dec 2016)**
11. Write note on (i) Amplitude (ii) Phase (iii) Energy (iv) Intensity (v) Radiation pressure EM wave.
12. Explain reception and transmission of cell phone.
13. Describe production of EM waves.
14. Discuss propagation EM wave from vacuum to a non conducting medium.

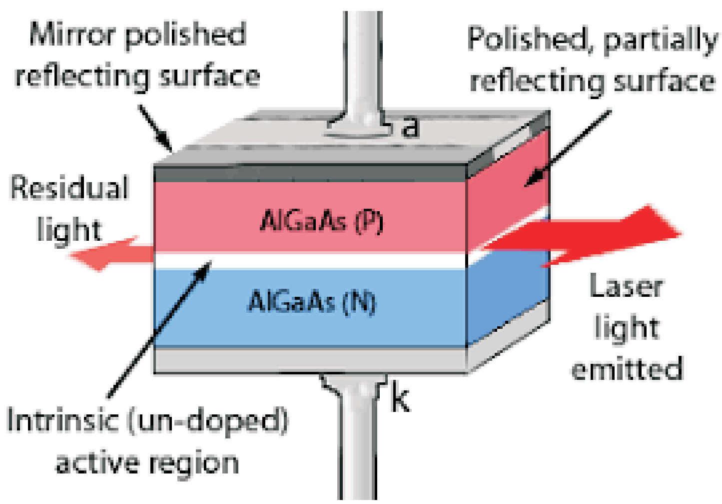
PROBLEMS FOR PRACTICE

1. Find the velocity of a plane in a loss-less medium having a relative permittivity of 8 and relative permeability of unity. **[Ans: 1.06×10^8 ms^{-1}]**
2. Determine the propagation constant for a material having $\mu_r = 1$ $\varepsilon_r = 8$ and $\sigma = 0.3$ n s / m if the wave frequency is 1.5 MHz. **[Ans: 3.76×10^{-2} m^{-1}]**
3. Find the amplitude of the displacement current density inside a capacitor where $\varepsilon_r = 600$ and $D = 3 \times 10^{-6} \sin\left(6 \times 10^6 t - 0.3464\, x\right) u x \, \text{c/m}^2$.
4. Earth is a good conductor when $\frac{\sigma}{w\varepsilon} > 1$. Determine the highest frequency for which earth has $\frac{\sigma}{\omega\,\varepsilon}$ greater than 10. Assume the following constant.

   [Ans: $\sigma = 5 \times 10^{-3}$ S / m $\varepsilon = 10\,\varepsilon_o$]
5. If the magnitude of $\vec{H}$ in a plane wave of 1 amp / metre. Find magnitude of $\vec{E}$ for a plane wave in free space.

   [Ans: E = 376 volts / metre]

Unit - III



Semiconductor Laser

Oscillations, Optics and Lasers

3. Oscillations

Simple harmonic motion – resonance – analogy between electrical and mechanical oscillating systems – waves on a string – standing waves – traveling waves – Energy transfer of a wave – sound waves – Doppler effect.

Introduction

The vibrations or oscillations of any system constitutes one of the most important fields of study in engineering physics. Examples of such oscillations are pendulum of a clock, loaded spring and LC circuit. The knowledge about the resonant behaviour of mechanical and electrical systems is very much essential for all engineers.

Oscillatory Motion

Basic concepts

Motion

When the position of a body (object) in the space changes with time, the body is said to be in motion.

This motion is classified into four types. They are.

(i) **Translational motion**

(ii) **Rotational motion**

(iii) **Periodic motion**

(iv) **Vibrational or Oscillatory motion**

(i) Translational motion

If the position of a body varies linearly with time, then the motion is said to be **translational motion.** A car / train moving on a straight road, a ball moving on the ground are examples of translational motion.

(ii) Rotational motion

If the position of a body as a whole does not change linearly with time but it rotates at the same position about an axis, then the motion is said to be **rotational motion.**

The examples for rotational motion are the motion of earth about its axis, the rotation of a flywheel on its ball bearings.

(iii) Periodic motion

If a body repeats its movement at regular intervals of time, its motion is said to be **periodic.**

Motion of planets round the sun, rotational motion of the earth about its own axis, motion of a pendulum, oscillations of a loaded spring etc are examples for periodic motion.

(iv) Oscillatory motion

An oscillatory motion is one in which a body moves to and fro repeatedly about a mean position.

Motion of a pendulum, oscillations of a loaded spring, to and fro motion of the prongs of tuning fork are examples for oscillatory motion.

Oscillatory motion is a special case of periodic motion in which a body vibrates to and fro about its mean position. Though all oscillatory motions are periodic, all periodic motions are not oscillatory.

Periodic Motion

A motion in which the body describes the same path in same way continuously after equal interval of time is known as the periodic motion. Simple Harmonic Motion (SHM) is a special case of periodic motion with the following characteristics.

- This is a type of periodic motion.
- The acceleration of the body is directly proportional to its displacement from a fixed point and always directed towards the fixed point.
- The restoring force developed in the body tries to bring the particle to its equilibrium position.

3.1 SIMPLE HARMONIC MOTION

An oscillatory motion is harmonic if the displacement can be expressed in terms of sine or cosine function. An oscillator executing harmonic motion is called a **harmonic oscillator**.

Definition

When the acceleration of particle is directly proportional to its displacement from its equilibrium position and it is always directed towards equilibrium position, then the motion of the particle is said to be simple harmonic motion.

Characteristics of simple harmonic motion

A particle executing simple harmonic motion must satisfy the following conditions:

- The motion must be periodic.
- The motion is oscillatory ie., to and fro along a straight line or along a curved path about a mean position.
- The body executing simple harmonic motion is acted upon by a restoring force whose magnitude is proportional to the displacement and its direction is always towards the mean position.
- If there is no air resistance or friction, the motion once started will continue indefinitely.

Example for simple harmonic motion

- **Simple Pendulum**

It consists of an ideally massless inextensible string hanging from a rigid support A with a point mass (i.e., the bob of the pendulum) connected to its other end B.

This is shown in fig. 3.1. When the bob is displaced to a new position P and released, oscillations start and it is in simple harmonic motion.

If θ is the angle made by the string at P with the equilibrium condition of the string AB, m and g are the mass of the bob and the acceleration due to gravity, then the radial component of mg (*i.e.,* $mg \cos \theta$) balances the tension T across the string.

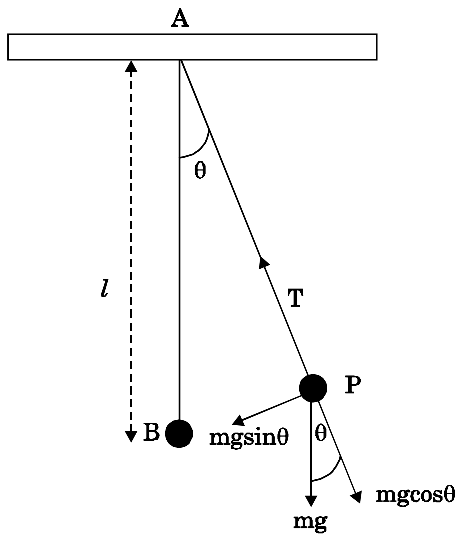
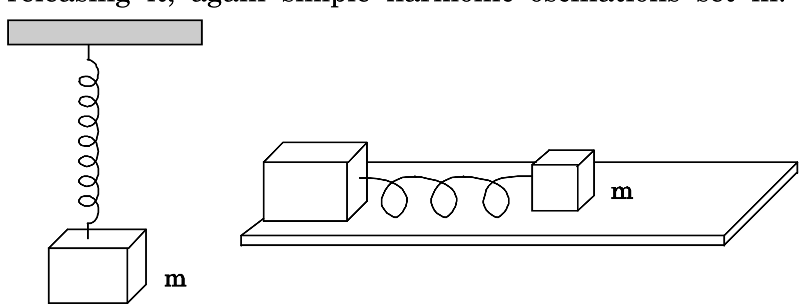


Fig. 3.1 Simple pendulum

The tangential component '$mg \sin \theta$' tries to bring the bob at B. These way oscillations continue till it stops due to air friction.

- **Spring – Mass system**

Here a mass block is connected to a spring either vertically or horizontally from rigid supports as shown in fig.3.2 (a) and (b). On displacing the mass from its equilibrium position and then releasing it, again simple harmonic oscillations set in.



(a) Vertically connected (b) Horizontally connected

Fig. 3.2 SHM by mass block connected with spring

- Vibrations of a tuning fork.
- Vibrations of a sonometer wire.
- Vertical oscillations of the liquid column in a U-tube.
- Angular oscillations of a torsion pendulum.

Types of Simple Harmonic Motion (S.H.M)

The simple harmonic motions are of two types:

(i) Linear Simple Harmonic Motion

If the displacement of a particle executing S.H.M. is linear, the motion is said to be **linear S.H.M**. The examples of linear S.H.M. are motion of simple pendulum, the motion of prongs of vibrating tuning fork, the motion of a point mass attached to a spring.

(ii) Angular Simple Harmonic Motion

If the displacement of a particle executing S.H.M. is angular, the motion is said to be **angular S.H.M**. The example of angular S.H.M. is torsional oscillations of a solid.

Essential conditions for S.H.M

If $\boldsymbol{a}$ is linear acceleration and $\boldsymbol{y}$ is displacement from equilibrium position, then essential condition for linear S.H.M. is

$$\boldsymbol{a} \propto -\boldsymbol{y}$$

If α is angular acceleration and θ angular displacement from equilibrium position, then essential condition for angular S.H.M. is

$$\alpha \propto -\theta$$

Definitions concerning S.H.M

Let a particle execute S.H.M along straight line QOP, about O. (Fig. 3.3)

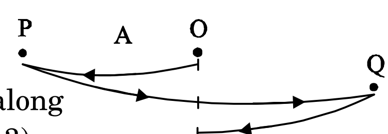


Fig. 3.3

1. Amplitude

The maximum displacement of a particle from mean position is called the amplitude. It is denoted by A. Then,

$$OP = OQ = A.$$

2. Oscillation

When particle moves from mean position O to P, returns from P to Q via O and then comes back from Q to O; then particle is said to complete **one-oscillation** i.e.,

1 oscillation = motion from O to P + from P to Q + from Q to O or motion (from P to Q + from Q to P)

3. Period

The time taken by the particle executing S.H.M to complete one oscillation is called the **period or periodic time**. It is denoted by T.

4. Frequency

The number of oscillations completed by particle in one second is called its frequency. It is denoted by n

$$\text{Frequency } n = \frac{1}{\text{period } (T)}$$

5. Phase

The position and direction of motion of a vibrating particle is different at different instants. The instantaneous position and direction of motion of a vibrating particle is expressed by a physical quantity called the **phase**.

If S.H.M is expressed as $y = A \sin(\omega t + \phi)$; then the quantity $(\omega t + \phi)$ is the phase of vibrating particle.

Differential Equation of S.H.M

A particle executing S.H.M is called a **harmonic oscillator.** Consider a particle of mass m executes S.H.M. along a straight line (Fig. 3.4). Let y be the displacement of particle from mean position at any time t.

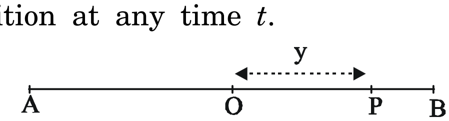


Fig. 3.4

On basic condition of S.H.M. the restoring force F is proportional to displacement y and oppositely directed i.e.,

$$F \propto -y$$

$$\text{or} \quad F = -k\,y \qquad \dots (1)$$

where k is a constant of proportionality and it is called spring factor or force constant. Its unit is newton / metre (N / m).

If $a = \frac{d^2y}{dt^2}$ is acceleration at any instant t, then by Newton's second law of motion $F =$ mass $\times$ acceleration $= ma$

$$F = m\frac{d^2y}{dt^2} \qquad \dots (2)$$

From the eqns. (1) and (2), we have

$$m\frac{d^2y}{dt^2} = -ky,$$

$$\therefore m\frac{d^2y}{dt^2} + ky = 0$$

$$\frac{d^2y}{dt^2} + \frac{k}{m}y = 0$$

i.e., $$\boxed{\frac{d^2y}{dt^2} + \omega^2 y = 0} \qquad \dots (3)$$

where $\omega^2 = \frac{k}{m}$ is a constant and ω is known as angular frequency.

The eqn. (3) represents **the differential equation of SHM.**

A general solution of the differential equation for SHM is given by

$$\boxed{y = A \sin(\omega t + \phi)} \quad \dots (4)$$

where A is the amplitude of the SHM.

ϕ is the initial phase.

Angular harmonic motion

Now consider a particle executing angular harmonic motion. At any instant t, let θ be the angular displacement measured from the equilibrium position of the particle. Then, similar to equation (3) for the linear case, we have

$$\boxed{\frac{d^2\theta}{dt^2} + \omega^2\theta = 0} \quad \dots (5)$$

ω^2 is a constant,

Torsional oscillations of a stretched wire, oscillations of a freely suspended bar magnet in a magnetic field etc. are examples of angular harmonic motion.

Velocity and Acceleration

We have displacement $y = A \sin(\omega t + \phi)$

Differentiating with respect to time t, we get

$$\frac{dy}{dt} = v = A\omega \cos(\omega t + \phi) \quad \dots (6)$$

or $\cos^2(\omega t + \phi) = 1 - \sin^2(\omega t + \phi)$

$$v = A\omega\sqrt{1 - \sin^2(\omega t + \phi)}$$

$(\because \sin^2(\omega t + \phi) + \cos^2(\omega t + \phi) = 1)$

(or) $(\cos^2(\omega t + \phi) = 1 - \sin^2(\omega t + \phi))$

$$v = \omega\sqrt{A^2 - A^2\sin^2(\omega t + \phi)}$$

or $$v = \omega\sqrt{A^2 - y^2} \quad \dots (7)$$

This equation (7) gives velocity of particle at any displacement y. Maximum velocity is obtained by putting $y = 0$.

$$\therefore \quad \boxed{v_{max} = \omega A \text{ (at mean position)}}$$

Thus, velocity is maximum and equal to $A\omega$ at mean position and velocity is zero at $y = \pm A$ (extreme positions).

Acceleration

Differentiating the eqn (6) with respect to time t,

$$\text{acceleration } a = \frac{dv}{dt} = -A\omega^2 \sin(\omega t + \phi)$$

$$\text{or} \qquad \boxed{a = -\omega^2 y} \qquad \text{... (8)}$$

This equation gives acceleration of oscillating particle at any displacement y from the mean position. This equation is the standard equation of S.H.M.

For maximum acceleration at $y = A$ (extreme position)

$\therefore$ Maximum acceleration, $a_{max} = \omega^2 A$ (at extreme position)

Minimum acceleration is obtained by putting $y = 0$

$\therefore \quad a_{min} = 0$ (at mean or equilibrium position).

Period of S.H.M

The time taken by the particle to make one complete to and fro motion is called the time period of the S.H.M. It is also equal to the time required by the particle to trace an angle 2π.

Since ω is the uniform angular velocity.

$$\omega = \frac{2\pi}{T}$$

$$\text{or, Period, } T = \frac{2\pi}{\omega} \qquad \text{... (9)}$$

From eqn (8), we have

$$\omega^2 = \frac{a}{y}$$

$$\omega = \sqrt{\frac{a}{y}}$$

or $$\frac{1}{\omega} = \sqrt{\frac{y}{a}}$$

Substituting this in eqn. (9), we have

$$\text{Time period, } T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{y}{a}} \qquad \text{... (10)}$$

$$\boxed{T = 2\pi\sqrt{\frac{\textbf{Displacement}}{\textbf{Acceleration}}}}$$

Let n be the frequency of the S.H.M, i.e., number of oscillation made by the particle in one second. But, period = Time taken for one oscillation.

$$\text{ie., } T = \frac{1}{n} \qquad \text{... (11)}$$

From eqns. (10) and (11), we have

$$\frac{2\pi}{\omega} = \frac{1}{n}$$

$$\text{or, } \boxed{\omega = 2\pi n} \qquad \text{... (12)}$$

Types of oscillations

There are three types of oscillations. They are

1. **Free oscillations**
2. **Damped oscillations**
3. **Forced oscillations**

(a) Free oscillation

When a body is displaced from its equilibrium position and then released, *it oscillates under the influence of restoring force with a definite frequency i.e., natural frequency*. (Fig. 3.5)

Such vibrations are called free vibrations. The frequency of such vibration is known as natural frequency.

The frequency of vibration depends only on geometric dimensions of the body and its elastic property $\nu_0 = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$

Examples

1. Vibrations in a stretched string
2. A tuning fork struck against a rubber pad
3. Oscillation of a simple pendulum.

Characteristics of free vibration

(i) Amplitude of oscillation is a constant (Fig. 3.5)

(ii) Frequency depends on geometric dimensions and elastic property.

(iii) No dissipation of energy

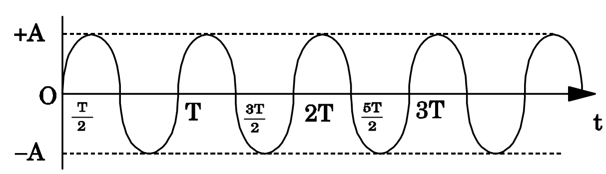
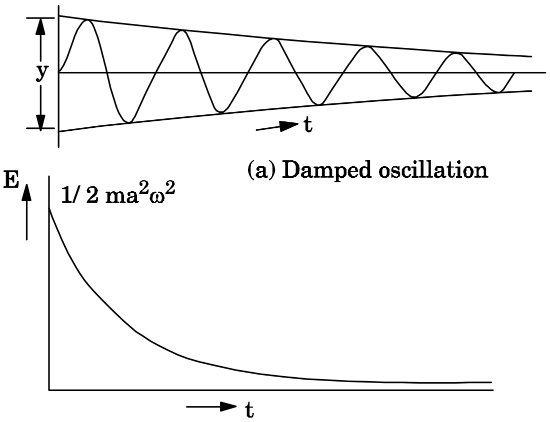


Fig. 3.5 Free vibration

(b) Damped oscillation

Most of the oscillations occur in air or in a medium. Hence, the medium offers some resistive force on the oscillating body. So, a part of the energy is dissipated in overcoming the resistive forces.

Consequently the amplitude of oscillation goes on decreasing exponentially with time and finally becomes zero. **Such oscillations are called damped oscillations** (Fig.3.6(a)). As a result the energy of oscillations decreases with time (Fig.3.6(b)).



(a) Damped oscillation

(b) Energy of damped oscillation

Fig. 3.6

Characteristic of a damped oscillation

(i) Amplitude of oscillation is not a constant

(ii) There is dissipation of energy

(iii) Small changes are produced in the frequency of oscillation

Example

The oscillations of a pendulum in air.

(c) Forced vibration

If some external periodic force is constantly applied on a system of damped oscillations, it is possible to compensate the energy it has lost. Therefore, the system continues to oscillate or vibrate with a constant amplitude.

When a body A is maintained in the state of vibration by a periodic force of frequency ν ***other than its natural frequency*** (ν_0) ***of the body, the vibrations are called forced vibrations***.

Then the body A will vibrate with ν and not with ν_0, such vibrations are called forced vibrations.

External force is the driver and the body A is driven.

Characteristics of forced oscillations are

(i) Amplitude will be a constant

(ii) The frequency of forced vibration is equal to that of the external periodic force.

(ii) The amplitude of the forced vibration is determined by the frequency of the applied force and the natural frequency.

Examples

1. Press the stem of a vibrating tuning fork, against tabla. The tabla suffer forced vibrations
2. Sound boards of stringed instruments suffer forced vibrations.
3. Swing can be kept in motion by giving constant periodic forces. Hence the swing maintains its amplitude while oscillating.

Differential Equation for Forced Oscillations

Consider a particle of mass m connected to a spring (Fig. 3.7). This particle is driven by a periodic force.

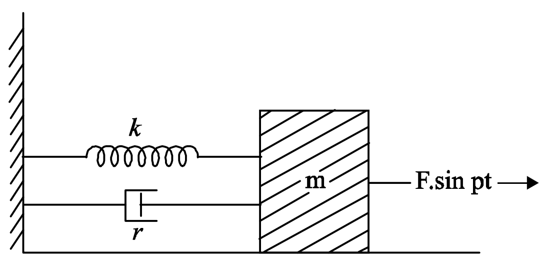


Fig. 3.7 Mechanical forced oscillator with force F sin *pt*

The oscillations are started and the forces acting on the particles are

(i) **a restoring force.** It is proportional to the displacement acting in the opposite direction. It is given by $-ky$ where k is known as the restoring force constant.

(ii) **a frictional force.** It is proportional to velocity but acting on the opposite direction. It is given by $-r\dfrac{dy}{dt}$ where r is the frictional force constant.

(iii) **the external periodic force**, $F \sin pt$ where F is the maximum value of the force and p is its angular frequency. This force opposes the restoring force as well as the frictional force and helps in motion.

Therefore, net force F' acting on the particle

$$F' = -ky - r\frac{dy}{dt} + F \sin pt \qquad \text{...(1)}$$

By Newton's second law of motion, the resultant force acting on the particle

$$F' = \text{mass} \times \text{acceleration} = \text{ma}$$

$$F' = m\frac{d^2y}{dt^2} \qquad \left(\because a = \frac{d^2y}{dt^2}\right) \qquad \text{...(2)}$$

$\therefore$ From the eqns (1) & (2), we have

$$m\frac{d^2y}{dt^2} = -ky - r\frac{dy}{dt} + F \sin pt$$

or $$m\frac{d^2y}{dt^2} + r\frac{dy}{dt} + ky = F \sin pt \qquad \text{...(3)}$$

$$\frac{d^2y}{dt^2} + \frac{r}{m}\frac{dy}{dt} + \frac{k}{m}y = \frac{F}{m}\sin pt$$

$$\frac{d^2y}{dt^2} + 2b\frac{dy}{dt} + \omega^2 y = f \sin pt \qquad \text{... (4)}$$

where $\dfrac{r}{m} = 2b$, $\dfrac{k}{m} = \omega^2$ and $\dfrac{F}{m} = f$

The eqn. (4) is the differential equation of the motion of the forced oscillation of the particle.

The solution of differential eq. (4)

$$y = A \sin (pt - \theta) \qquad \text{... (5)}$$

where A is the steady amplitude of vibrations

We have

$$A = \frac{f}{\sqrt{[(\omega^2 - p^2)^2 + 4 b^2 p^2]}} \qquad \text{... (6)}$$

$$\tan \theta = \frac{2bp}{\omega^2 - p^2}$$

or $$\theta = \tan^{-1}\left(\frac{2bp}{\omega^2 - p^2}\right) \qquad \text{... (7)}$$

The equation (6) gives the amplitude of forced vibration while eqn (7) its phase.

It is clear that the amplitude and phase of the forced oscillations depend upon $(\omega^2 - p^2)$, ie., they depend upon the driving frequency (p) and the natural frequency of the oscillator (ω).

3.2 RESONANCE

It is a special case of forced vibration

The phenomenon of making a body vibrate with its natural frequency under the influence of another vibrating body with the same frequency is called resonance.

If the frequency of the external periodic force is equal to the natural frequency of oscillation of the system, we get oscillations of larger amplitude. This is known as resonance. Amplitude becomes larger if the two frequencies are exactly equal to each other. $\nu = \nu'$ (Fig 3.8)

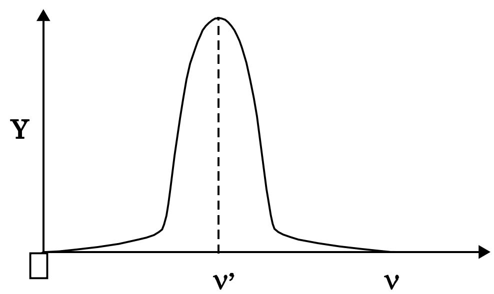


Fig. 3.8 Amplitude of resonance

Theory of resonant vibrations

(a) **Condition of amplitude resonance:** In case of forced vibrations, we have

$$A = \frac{f}{\sqrt{(\omega - p^2)^2 + 4b^2 p^2}} \quad \text{... (1)}$$

The expression (1) shows that the amplitude varies with the frequency of the force p. For a particular value of p, the amplitude becomes maximum. This phenomenon is known as **amplitude resonance**.

The amplitude is maximum when

$$\sqrt{(\omega - p^2)^2 + 4b^2 p^2} \text{ minimum}$$

$$\frac{d}{dp}[(\omega - p^2)^2 + 4b^2 p^2] = 0$$

$$\text{or } 2(\omega - p^2)^2(-2p)^2 + 4b^2(2p) = 0$$

$$\omega^2 - p^2 = 2b^2, \quad p = \sqrt{\omega^2 - 2b^2}$$

Thus the amplitude is maximum when the frequency $\frac{p}{2\pi}$ of the impressed force becomes $\frac{\sqrt{(\omega^2 - 2b^2)}}{2\pi}$. **This is known as the resonant frequency.**

This frequency of the system both in presence of damping, i.e. $\frac{\sqrt{(\omega^2 - b^2)}}{2\pi}$ and in the absence of damping i.e. $\frac{\omega}{2\pi}$.

$$\rho = \omega$$

Condition for Amplitude Resonance

Using equation (1), A is maximum only when,

For negligible damping, $b = 0$ and

$$\boxed{A_{\text{max}} = \frac{f}{2bp_r}}$$

As $b \to 0$, $A_{max} \to \infty$.

Sharpness of Resonance

The rate of change (fall) of amplitude with the change of forcing frequency on each side of resonant frequency is known as sharpness of resonance.

Figure 3.9 shows the variation of A with forcing frequency at different conditions of damping.

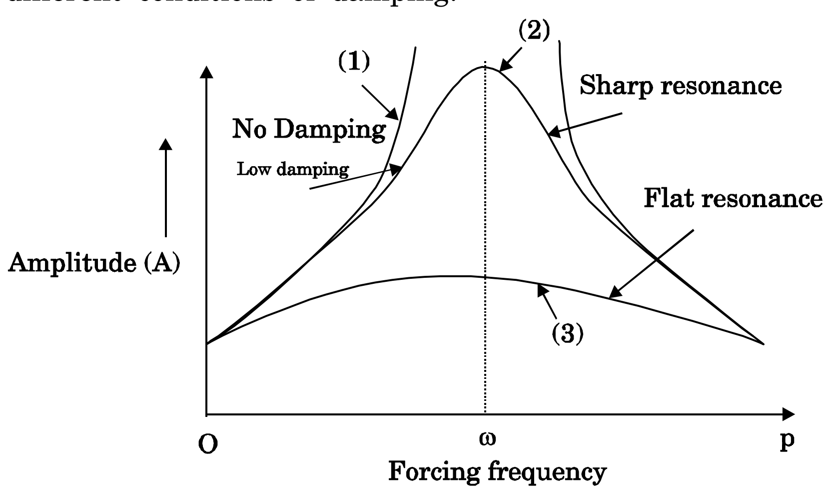


Fig. 3.9 Sharpness of resonance curve

The curve (1) shows the amplitude when there is no damping, i.e., b = 0. In this case the amplitude becomes infinite at $p = \omega$. This case is never attained in practice due to frictional resistance, as slight damping is always present.

The curve (2) and (3) shows the effect of damping on the amplitude. It is observed that the peak of the curve moves towards the left.

Examples

(i) Two tuning forks of same frequency are mounted on a suitable sound boards and arranged as shown in fig.3.10.

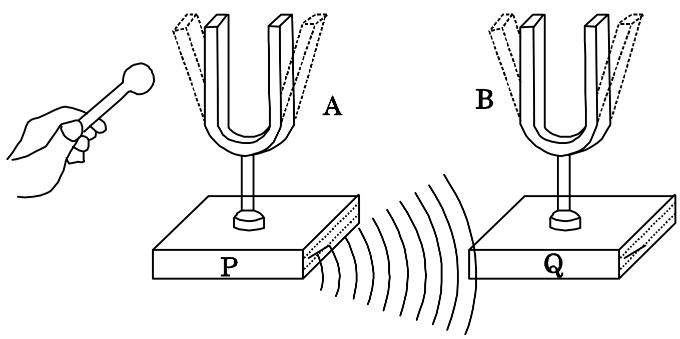


Fig. 3.10 Vibrations in Tuning forks

If one is made to vibrate by striking it with rubber hammer, it is found that the second fork is also set in vibrations.

If the vibrations of the fire fork is stopped by touching with hand. The second fork continue to vibrate and the sound will be heard.

(ii) A column of army marching over a bridge can set forced vibrations of the bridge. If the frequency of foot steps matches the with natural frequency of the bridge, due to resonance, the bridge may vibrate with a larger amplitude.

This may cause damage to the bridge. Hence soldiers are asked to break steps while crossing a bridge.

(iii) Radio receivers

Each radio stations have their own broadcasting (carrier) frequencies. When we tune our radio set, the moment the radio tuner attains exactly the frequency of any broadcasting station, we start hearing the loud and clear sound of that particular radio station due to the resonance.

(iv) Others

There are other examples also. The loud speaker diaphragm vibrates according to the amlifier circuit. Air column in resonance tube vibrates as per the vibrations of the tuning fork.

Simple experiment on Resonance

From an elastic chord a number of pendulums are suspended (Fig. 3.11). If the pendulum P is set into vibrations, other pendulums also vibrate because they are attached to the same chord. But their motions are not regular.

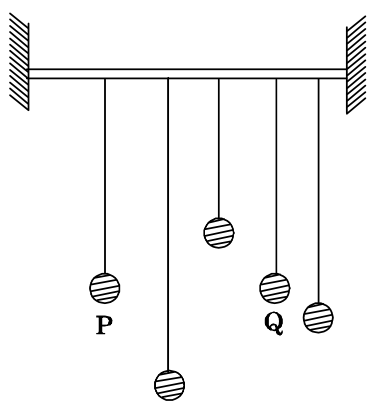


Fig. 3.11 Resonance in pendulum motion

After sometime we will notice that Q which is the same length as P, vibrate with maximum amplitude and of same time period. They are said to be in **resonance.** Other pedulums of different lengths are not in resonance.

3.3 MECHANICAL AND ELECTRICAL ANALOGUES

In a mechanical vibration, the particles have both kinetic energy and potential energy. The total energy is sum of these two energies. The energy of mechanical oscillator is damped by an oppositing force (for example air friction).

The equation of motion is given by $\frac{d^2 y}{dt^2} + \omega^2 y = 0$. When oscillator is free from damping. The mechanical oscillator (pendulum etc) is described by its position, velocity and acceleration.

In the case of spring

$$\boxed{f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}}$$

$$\omega = 2\pi f = \frac{2\pi}{T} = \sqrt{\frac{k}{m}}$$

In an electrical oscillating system has potential energy which is stored as electrostatic energy in a capacitor. Further during electrical oscillation, there is a transfer of energy between electrical and magnetic energy in the case of LC oscillators.

The electrical oscillator system is described by charge q, current $I = \frac{dq}{dt}$ and frequency of oscillator is given by

$$\boxed{\boldsymbol{f = \frac{1}{2\pi}\sqrt{\frac{1}{LC}}}}$$

$$L\,\omega = \frac{1}{C\,\omega}$$

$$\omega^2 = \frac{1}{LC}$$

$$\omega = \frac{1}{\sqrt{LC}}$$

$$\boxed{f = \frac{1}{2\pi\sqrt{LC}}}$$

Now we shall discuss similarity between mechanical system and electrical system. One of the most popular and very important resonant system is an electrical system. It consists of a capacitance (C), an inductance (L) and a resistance (R). It is shown in figure 3.12.

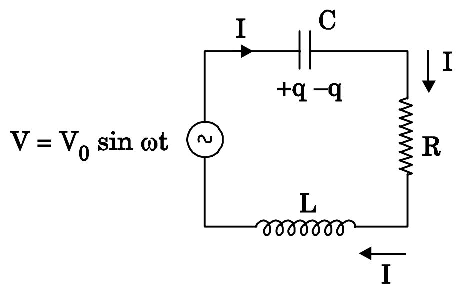


Fig. 3.12 Electrical System

Here $V = V_0 \sin \omega t$ is applied voltage. It is distributed among the three components as,

$$V_C = \frac{q}{C},\ V_R = IR \text{ and } V_L = L\frac{dI}{dt}$$

Here,
$$V_L + V_R + V_C = V$$

$$\text{i.e., } L\frac{dI}{dt} + IR + \frac{q}{C} = V_0 \sin \omega t$$

$$\text{Since, } I = \frac{dq}{dt}$$

$$\textit{i.e., } L\frac{d^2 q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = V_0 \sin \omega t$$

$$\text{or, } \frac{d^2 q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{q}{CL} = \frac{V_0}{L} \sin \omega t \qquad \text{... (1)}$$

The equation (1) is similar to equation of motion for a forced vibration.

$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + \omega_0^2 x = f \sin \omega t \qquad ...(2)$$

where $2k = \frac{r}{m}$, $\omega_0^2 = \frac{S}{m}$ and $f = \frac{F}{m}$

It is noted that the mass m is analogous to self inductance L, r to the electrical resistance R, compliance $\frac{1}{S}$ to the electrical capacitance C, the force F to the voltage V_0, the displacement x to the charge q, and the velocity $\frac{dx}{dt}$ to the electrical current $\left(I = \frac{dq}{dt}\right)$.

Here the mechanical quantities and electrical analogues are given below:

Mechanical Quantities	**Electrical Analogues**
Displacement (x)	Charge (q)
Velocity $\left(\frac{dx}{dt}\right)$	Current $\left(I = \frac{dq}{dt}\right)$
mass (m)	Inductance (L)
Force (F)	Voltage (V_0)
Damping constant (r)	Resistance (R)
Stiffness constant (S)	Reciprocal of capacitance $\left(\frac{1}{C}\right)$
Quality factor, $Q = \frac{\omega' m}{r}$	Quality factor $Q = \frac{\omega' L}{R}$
For mechanical oscillator	For electrical oscillator
Damping co-efficient $k = \frac{r}{2m}$	Damping co-efficient, $k = \frac{R}{2L}$
Mechanical Impedance	Electrical Impedance

Mechanical Quantities	Electrical Analogues
$Z = \sqrt{\left(\frac{S}{\omega} - m\omega\right)^2 + r^2}$	$(Z) = \sqrt{\left(\frac{1}{\omega C} - \omega L\right)^2 + R^2}$
Relaxation time $\left(\frac{1}{k}\right) = \frac{2m}{r}$	Relaxation time $\left(\frac{1}{k}\right) = \frac{2L}{R}$
Potential energy $= \frac{1}{2} Sx^2$	Electrostatic energy $= \frac{q^2}{2C}$
Kinetic energy of moving mass $= \frac{1}{2} m \dot{x}^2$	Electromagnetic energy of moving charge $= \frac{1}{2} L \dot{q}^2$

3.4 WAVES ON STRING

A string is a cord whose length is very long compared to its diameter and which is uniform and flexible. The transverse vibration of strings, is the principle to the working of many stringed instruments like sitar, guitar and violin etc.

When strings are plucked they are set into transverse vibration and under suitable conditions they produce musical note.

Vibrations of Stretched String

"If the string vibrates with nodes at the fixed ends and an antinode at the centre, then it is said to vibrate in fundamental mode.

The frequency corresponding to this mode of vibration is known as frequency of fundamental mode".

Consider a string stretched between two supports. When the string is plucked in the middle, transverse progressive waves are produced. These waves travel towards each end of the wire.

The speed of the waves is given by,

$$\nu = \sqrt{\frac{T}{m}} \quad \text{... (1)}$$

where T is tension in the string and

m is mass per unit length (linear density) of string.

These waves are reflected at both ends of the string. Then, the incident waves and reflected waves transverse stationary waves. (Fig 3.13)

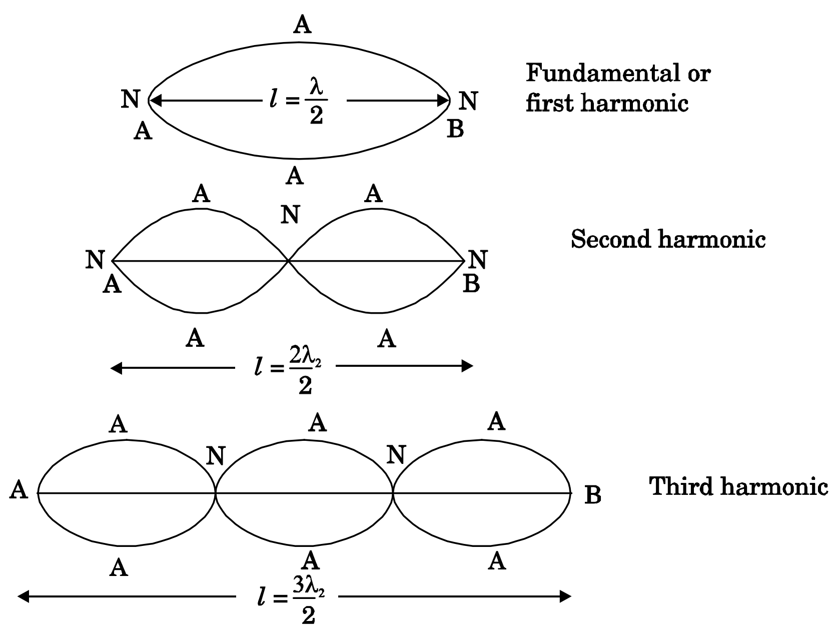


Fig. 3.13

In the stationary pattern, nodes are formed at fixed ends and an antinode in the middle.

The distance between two consecutive nodes is equal to $\frac{\lambda}{2}$, where λ is wavelength.

If l be the length between fixed ends of string, then

$$l = \frac{\lambda}{2} \text{ or } \lambda = 2l \quad \text{... (2)}$$

If ***n*** be the frequency of vibration of string, then

$$\boxed{n = \frac{v}{\lambda} = \frac{v}{2l}} \qquad \dots (3)$$

Substituting for ***v*** from equation (1)

$$n = \frac{1}{2l}\sqrt{\frac{T}{m}}$$

where, T – tension in the string

m – mass perunit length of string

l – length of string

In general,

fundament frequency, $\boxed{(n) = \frac{p}{2l}\sqrt{\frac{T}{m}}}$

where p is number of loops and it takes values as $p = 1, 2, 3, \dots$

For $p = 1$, $n_1 = \frac{1}{2l}\sqrt{\frac{T}{m}}$, called fundamental frequency

$p = 2$, $n_2 = \frac{2}{2l}\sqrt{\frac{T}{m}} = 2n$ called 1st over tone

Laws of Transverse vibrations of stretched strings

The frequency of vibration of the fundamental note of a stretched string is given by,

$$\boxed{n = \frac{l}{2l}\sqrt{\frac{T}{m}}}$$

where T is the tension and m is mass per unit length of string.

(i) Law of length

The fundamental frequency of vibration of a stretched string is inversely proportional to the length, when the tension and the mass per unit length of the string remain constant.

$$\text{i.e., } n \propto \frac{1}{l} \text{ when } T \text{ and } m \text{ are constant}$$

$$\text{or} \quad \boxed{nl = \text{constant}}$$

(ii) Law of tension

The frequency of vibration of a stretched string is directly proportional to the square root of tension when the length and the mass per unit length of the string remain constant.

$$\text{i.e., } n \propto \sqrt{T} \text{ when } l \text{ and } m \text{ are constant}$$

$$\text{or} \quad \boxed{\frac{n}{\sqrt{T}} = \text{constant}}$$

(iii) Law of mass

The fundamental frequency of vibration of a stretched string is inversely proportional to the square root of the mass per unit length when the tension and the length of the string remain constant.

$$\text{i.e., } n \propto \frac{1}{\sqrt{m}} \text{ when } T \text{ and } l \text{ are constant}$$

$$\text{or} \quad \boxed{n\sqrt{m} = \text{constant}}$$

Wave Motion

- An important type of motion that occurs in nature is wave motion.
- A wave motion is a disturbance of some kind which moves from one place to another by means of a medium, such that the medium itself is not transported.

The disturbance transmits both energy and momentum to distant regions with a velocity that depends upon characteristics of the medium.

Types of Wave Motion

There are two types of wave motion

They are

1. **Longitudinal wave motion**
2. **Transverse wave motion**

1. Longitudinal Wave Motion

Wave motion in which the particles of the medium vibrate about their mean position along the same line as propagation of wave is called longitudinal wave motion.

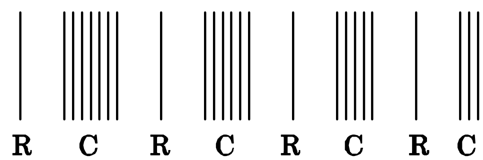


Fig. 3.14 Longitudinal Wave Motion

Example: sound waves.

In longitudinal wave motion region of more particles (compression) and region of less number of particle (rarefaction) are produced.

2. Transverse Wave Motion

Wave motion in which particles of the medium vibrate about their mean position at right angle to the direction of propagation is called transverse wave motion.

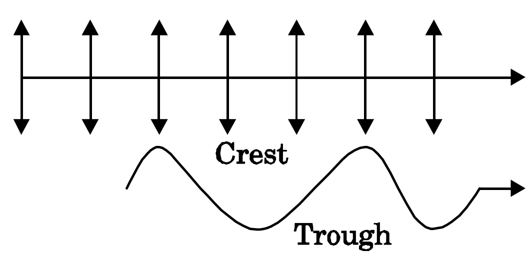


Fig. 3.15 Transverse Wave Motion

Example: waves on the surface of water.

Progressive wave

A progressive wave is defined as the vibratory motion of a body which is transmitted continuously in the same direction from one particle to successive particle of the medium and travel forward through the medium due to its elastic property.

3.5 PLANE PROGRESSIVE WAVES OR TRAVELING WAVES

Progressive wave originating from a point source and propagating through an isotropic medium travel with equal velocity in all directions.

At any instant, the wavefront (locus of all particles vibrating with the same phase) will be spherical in nature.

If the sphere of very large radius is considered, the spherical wave will approximate to a plane surface and the waves are called plane progressive waves.

Relation between Frequency, Wave - Speed and Wavelength

By definition, the distance travelled by the wave in one time-period (T) of vibration of particle = wave length (λ)

ie Distance travelled in T second = λ

$\therefore$ Distance travelled in one second $= \dfrac{\lambda}{T}$

But, distance travelled in one second = wave speed v.

$$\therefore \quad \frac{\lambda}{T} = v \text{ or } \lambda = vT \qquad \text{... (1)}$$

substituting value of T from the relation between frequency (n) and time period T

$$\text{i.e., } T = \frac{1}{n} \text{ in (1), we get } \lambda = \frac{v}{n}$$

$$v = n\lambda$$

ie., speed of the wave $v =$ frequency $(n) \times$ wavelength (λ)

This relation holds for all types of waves.

Wave Equation of a Plane - Progressive Wave

On propagation of wave in a medium, the particles of medium execute simple harmonic motion.

If a plane progressive wave is propagating in a medium along positive $X -$ axis. (i.e. from left to right) (Fig. 3.16) then in fig. 3.12 the positions of particles O, A, B, C, D ... are shown.

As the wave propagates, all the particles of the medium begin to vibrate to and fro about their mean positions.

In fig. 3.16 the instantaneous positions of these particles are shown. **The curve joining these positions represents the progressive wave.**

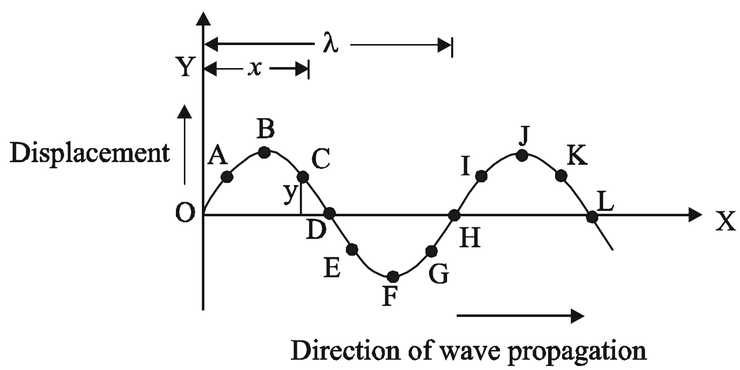


Fig. 3.16

Let the particle begin to vibrate from origin O at time $t = 0$. If y is the displacement of the particle at time t, then equation of particle executing simple harmonic motion about O is

$$y = A \sin \omega t \quad \text{... (1)}$$

where A is amplitude and ω is angular velocity.

If n is frequency of wave, then $\omega = 2 \pi n$.

As the advancing wave reaches the other particles $A, B, C\ldots$ (beyond particle at O), these particles begin to vibrate.

If v is the speed of wave and C is a particle at a distance x from O, then the time taken by wave to reach point C is $\frac{x}{v}$ seconds, therefore the particle will start vibrating $\frac{x}{v}$ seconds after particle at O.

Therefore, the displacement of particle C at any time t will be the same which was of particle O at time $\left(t - \frac{x}{v}\right)$.

The displacement of particle O at time $\left(t - \frac{x}{v}\right)$ can be obtained by substituting $\left(t - \frac{x}{v}\right)$ in place of t in equation (1).

Thus the displacement of particle C at a distance x from origin O at any time t is given by

$$\boxed{y = A \sin \omega \left(t - \frac{x}{v}\right)} \quad \ldots (2)$$

If T is time-period and λ the wavelength of wave, then

$$\omega = \frac{2\pi}{T}$$

$$\therefore \quad y = A \sin \frac{2\pi}{T}\left(t - \frac{x}{v}\right)$$

$$= A \sin 2\pi \left(\frac{t}{T} - \frac{x}{vT}\right)$$

But, $vT = \lambda$

$$\therefore \quad \boxed{y = A \sin 2\pi \left(\frac{t}{T} - \frac{x}{\lambda}\right)} \quad \ldots (3)$$

This equation is expressed as

$$y = A \sin \frac{2\pi}{\lambda} \left(\frac{t\,\lambda}{T} - x \right)$$

$$\therefore \qquad y = A \sin \frac{2\pi}{\lambda} (vt - x) \qquad \ldots (4)$$

$$\left[\because \frac{\lambda}{T} = v \right]$$

$$y = \sin \left(\frac{2\pi vt}{\lambda} - \frac{2\pi x}{\lambda} \right)$$

$$y = \sin \left(\frac{2\pi\, nvt}{v} - \frac{2\pi nx}{v} \right) \qquad \left[\begin{array}{l} \because n\,\lambda = v \\ \frac{n}{v} = \frac{1}{\lambda} \end{array} \right]$$

The equation (2) is also expressed as

$$y = A \sin \left(\omega\, t - \frac{\omega}{v} x \right) \qquad [\because 2\pi\, n = \omega]$$

$$\text{But } \frac{\omega}{v} = \frac{2\pi}{\lambda} = k = \text{propagation constant}$$

$$\boxed{y = A \sin (\omega\, t - kx)} \qquad \ldots (5)$$

Any of the equations (2), (3), (4), (5), represents the equation of a plane progressive wave propagating along positive direction of $X -$ axis. Out of these, the equation (5) is often used.

If the wave is propagating along negative $X -$ axis, then the equation of plane progressive wave may be obtained by substituting $-x$ for x, so that equation (5) takes the form

$$\boxed{y = A \sin (\omega\, t + kx)} \qquad \ldots (6)$$

If ϕ is the phase difference between this wave travelling along positive $X -$ axis and another wave, then the equation of the wave may be expressed as

$$\boxed{y = A \sin [(\omega\, t - kx) + \phi]} \qquad \ldots (7)$$

This is the most general equation of a plane progressive wave travelling along positive direction of X – axis.

Differential equation of wave motion

We have $y = A \sin \frac{2\pi}{\lambda}(vt - x)$

Then, $$\frac{dy}{dt} = \frac{2\pi v A}{\lambda} \cos \frac{2\pi}{\lambda}(vt - x) \quad \ldots (8)$$

Also, $$\frac{dy}{dx} = \frac{-2\pi A}{\lambda} \cos \frac{2\pi}{\lambda}(vt - x) \quad \ldots (9)$$

∴ Particle velocity

$$\frac{dy}{dt} = -v \frac{dy}{dx} \quad \ldots (10)$$

From the eqn. (9)

$$\frac{d^2 y}{dx^2} = -A\left(\frac{2\pi}{\lambda}\right)^2 \sin \frac{2\pi}{\lambda}(vt - x) \quad \ldots (11)$$

From the eqn. (8)

$$\frac{d^2 y}{dt^2} = -A\left(\frac{2\pi}{\lambda}\right)^2 v^2 \sin \frac{2\pi}{\lambda}(vt - x) \quad \ldots (12)$$

This is particle acceleration.

Comparing (11) and (12)

$$\boxed{\frac{d^2 y}{dt^2} = v^2 \frac{d^2 y}{dx^2}} \quad \ldots (13)$$

or $$\boxed{\frac{d^2 y}{dx^2} = \frac{1}{v^2} \frac{d^2 y}{dt^2}}$$

This is the differential equation of wave motion. It can be shown that in case of progressive waves, if, t is increased by δt and x by $v\,\delta t$

$$\text{then } y' = a \sin \frac{2\pi}{\lambda} [v(t + \delta t) - (x + v\,\delta t)]$$

$$y' = a \sin \frac{2\pi}{\lambda} [vt + v\,\delta t - x - v\,\delta t]$$

$$= a \sin \frac{2\pi}{\lambda} (vt - x) = y \qquad \text{... (14)}$$

Thus, in a time δt, the wave advances through $v\,\delta t$.

Hence, v is the velocity of the wave.

Characteristics of progressive wave

1. Each particle of the medium executes vibration about its mean position. The disturbance progresses onward from one particle to another.
2. The particles of the medium vibrate with same amplitude about their mean positions.
3. Each successive particle of the medium performs a motion similar to that of its predecessor along the propagation of the wave, but later in time.
4. The phase of every particle changes from 0 to 2π.
5. No particle remains permanently at rest. Twice during each vibration, the particles are momentarily at rest at extreme positions, different particles attain the position at different time.
6. Transverse progressive waves are characterised by crests and troughs. Longitudinal waves are characterised by compressions and rarefactions.
7. There is a transfer of energy across the medium in the direction of propagation of progressive wave.

8. All the particles have the same maximum velocity when they pass through the mean position.
9. The displacement, velocity and acceleration of the particle separated by $m\lambda$ are the same, where m is an integer.

3.6 STATIONARY WAVES OR STANDING WAVES

When two progressive waves of same amplitude and wavelength travelling along a straight line in opposite directions superimpose on each other, stationary waves are formed.

Analytical method

Let us consider a progressive wave of amplitude a and wavelength λ travelling in the direction of X axis.

$$y_1 = a \sin 2\pi \left(\frac{t}{T} - \frac{x}{\lambda} \right)$$

This wave is reflected from a free end and it travels in the negative direction of X axis, then

$$y_2 = a \sin 2\pi \left(\frac{t}{T} + \frac{x}{\lambda} \right)$$

According to principle of superposition, the resultant displacement is

$$y = y_1 + y_2$$

$$= a \left[\sin 2\pi \left(\frac{t}{T} - \frac{x}{\lambda} \right) + \sin 2\pi \left(\frac{t}{T} + \frac{x}{\lambda} \right) \right]$$

$$= \left[\sin \left(\frac{2\pi t}{T} \right) - \frac{2\pi x}{\lambda} + \sin \left(\frac{2\pi t}{T} - \frac{2\pi x}{\lambda} \right) \right]$$

$$= a \left[2 \sin \frac{2\pi t}{T} \cos \frac{2\pi x}{\lambda} \right]$$

Using trignometry identity

$\sin(A - B) + \sin(A + B) = 2 \sin A \cos B)$

$$\therefore \quad \boxed{y = 2a \cos \frac{2\pi x}{\lambda} \sin \frac{2\pi t}{T}}$$

This is the equation of a stationary wave.

(i) A points where $x = 0, \frac{\lambda}{2}, \lambda, \frac{3\lambda}{2}$, the values of

$$\cos \frac{2\pi x}{\lambda} = \pm 1$$

$\therefore A = \pm 2a$. At these points of the resultant amplitude is maximum. **They are called *antinodes*** (Fig. 3.16).

(ii) At points where $x = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4} \ldots$ the values of

$$\cos \frac{2\pi x}{\lambda} = 0.$$

$\therefore A = 0$. The resultant amplitude is zero at these points. They are called nodes (Fig. 3.17).

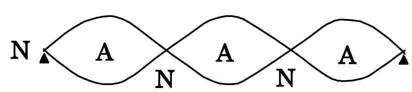


Fig. 3.17 Stationary waves

The distance between any two successive antinodes or nodes is equal to $\frac{\lambda}{2}$ and the distance between an antinode and a node is $\frac{\lambda}{4}$.

(iii) When $t = 0, \frac{T}{2}, T, \frac{3T}{2}, 2T, \ldots$ then $\sin \frac{2\pi t}{T} = 0$, the displacement is zero.

(iv) When $t = \frac{T}{4}, \frac{3T}{4}, \frac{5T}{4}$ etc, ... $\sin \frac{2\pi t}{T} = \pm 1$, the displacement is maximum.

Characteristics of stationary waves

1. The waveform remains stationary.
2. Nodes and antinodes are formed alternately.
3. The points where displacement is zero are called nodes and the points where the displacement is maximum are called antinodes.
4. Pressure changes are maximum at nodes and minimum at antinodes.
5. All the particles except those at the nodes, execute simple harmonic motions of same period.
6. Amplitude of each particle is not the same, it is maximum at antinodes decreases gradually and is zero at the nodes.
7. The velocity of the particles at the nodes is zero. It increases gradually and is maximum at the antinodes.
8. Distance between any two consecutive nodes or antinodes is equal to $\frac{\lambda}{2}$, whereas the distance between a node and its adjacent antinode is equal to $\frac{\lambda}{4}$.
9. There is no transfer of energy. All the particles of the medium pass through their mean position simultaneously twice during each vibration.
10. Particles in the same segment vibrate in the same phase and between the neighbouring segments, the particles vibrate in opposite phase.

Standing waves in strings

In musical instruments like sitar, violin, etc. sound is produced due to the vibrations of the stretched strings.

When a string under tension is set into vibration, a transverse progressive wave moves towards the end of the wire and gets reflected. Thus stationary waves are formed.

Differences between Progressive waves and Stationary waves

S. No.	Progressive waves	Stationary waves
(i)	There is transfer of energy in the direction of propagation of wave.	There is no transfer of energy.
(ii)	No particle of the wave is permanently at rest.	The particles at nodes are permanently at rest.
(iii)	The particles of the medium vibrate with same amplitude about their mean	Amplitude of each particle is not same. It is maximum at antinodes and decreases gradually to zero at the nodes.
(iv)	The phase of vibration varies continuously.	Particles in the same segment vibrate in the same phase.

3.7 ENERGY TRANSFER OF A WAVE

The mechanical erergy is transferred through the vibration of the string.

As a wave propagates in a medium (for example along a string), it transports energy. It means a vibrating string has more energy than a string that is not vibrating.

Let us consider a string under uniform tension T and m the mass / unit length.

If a small element of the string of length dx is considered, then its mass is $m\,dx$.

As the string is vibrating, then the kinetic energy of this small element is (Fig.3.18)

$$dk = \frac{1}{2} m \,(\text{velocity})^2$$

$$dk = \frac{1}{2} m \cdot dx \left(\frac{dy}{dt} \right)^2 \qquad \text{... (1)}$$

where dy is the vertical displacement of the portion of a string and

x is the direction of propagation of the wave in a string.

Now the string is under tension T.

When the string is displaced from equilibrium, a segment associated with interval dx has a length dl (Fig. 3.18).

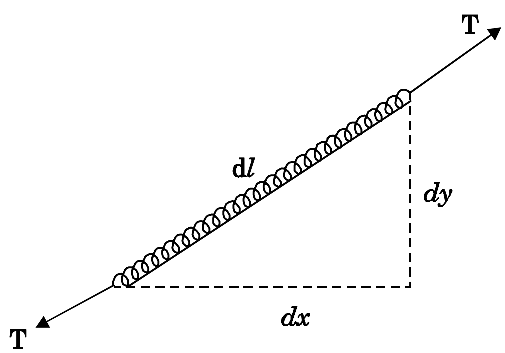


Fig. 3.18 An element of a string under the action of a wave.

$$dl = \sqrt{(dx)^2 + (dy)^2} = dx \sqrt{1 + \left(\frac{dy}{dx} \right)^2}$$

$$\approx dx \left(1 + \frac{1}{2} \left(\frac{dy}{dx} \right)^2 \right) \qquad \text{... (2)}$$

Thus under tension, a small segment of the string has expanded by an amount

$$\Delta l = dl - dx = \frac{1}{2} \left(\frac{dy}{dx} \right)^2 dz \qquad \text{... (3)}$$

So work is done for this expansion and it is stored as potential energy.

If the string is vibrating with displacement.

$$y\,(x, t) = A \cos\,(\omega\, t - k\, x) \qquad \text{... (4)}$$

Therefore, the potential energy is

$$dU = T \cdot \Delta\, l = \frac{1}{2} T \left(\frac{dy}{dx}\right)^2 \cdot dx \qquad \text{... (5)}$$

Substituting (4) in (5) gives

$$dU = \frac{1}{2} Tk^2 A^2 \sin^2\,(\omega\, t - kx) \cdot d\,x$$

$$dU = \frac{1}{2} m\, A^2\, w^2 \sin^2\,(\omega\, t - kx) \cdot dx \qquad \text{... (6)}$$

Where $U = \frac{\omega}{K}$ and for a string $v = \sqrt{\frac{T}{m}}$, then $T = mv^2$

$$Tk^2 = T \cdot \frac{\omega^2}{v^2} = T \cdot \frac{\omega^2}{(T/m)} = m\,\omega^2$$

If we substitute Eq.(4) in (1), then

$$dK = \frac{1}{2} m\, A^2\, \omega^2 \sin^2\,(\omega\, t - kx) \cdot dx \qquad \text{... (7)}$$

Comparing equ (4) and Eq.(6)

$$dU = dK \qquad \text{... (8)}$$

$\therefore$ The total energy is

$$dE = dU + dK = 2dK = m\, A^2\, \omega^2 \sin^2\,(\omega\, t - kx) \qquad \text{... (9)}$$

(or)

$$dE = m\,(A\omega)^2 \sin^2\,(\omega\, t - kx)\, dz \qquad \text{... (10)}$$

The quantity $\frac{dE}{dx}$ is called the linear energy density.

At any point, for example at $x = 0$, the average value of $\sin^2(\omega t)$ over one period is 1/2, so,

$$\left(\frac{dE}{dx}\right)_{\text{Average}} = d\vec{E} = \frac{1}{2} m (A\omega)^2 \qquad \text{... (11)}$$

This relation is called as average energy density $d\vec{E}$

As the average power transmitted by the wave is

$$\vec{P} = \left(\frac{dE}{dt}\right)_{\text{Average}};$$

therefore Eq.(11) becomes

$$(dE)_{\text{Average}} = \frac{1}{2} m A^2 \omega^2 \cdot dx \qquad \text{... (12)}$$

So,

$$\vec{P} = \frac{1}{2} m A^2 \omega^2 \frac{dx}{dt}$$

$$\boxed{\vec{P} = \frac{1}{2} m A^2 \omega^2 \cdot v} \qquad \text{... (13)}$$

where $v = \frac{dx}{dt}$ is the wave velocity.

Eq.(13) states that wave power is directly proportional to the speed or velocity v at which energy moves along the wave.

3.8 SOUND WAVES

Sound is always produced by some vibrating body. The vibrating body excites mechanical waves in the surrounding medium. The mechanical waves propagate in the form of a series of compressions and rarefactions in the air and set the ear drum vibrating. It causes a sensation of hearing.

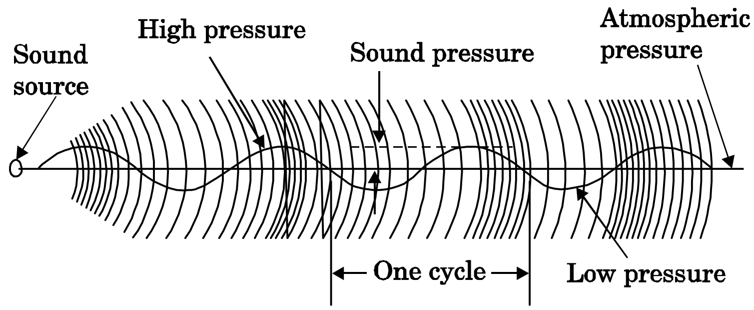


Fig. 3.19 Schematic representation of a sound wave

The propagation of sound requires the presence of an elastic medium. Sound cannot travel in a vacuum.

The following points may be noted about sound waves

(i) Sound waves are longitudinal waves.

(ii) Material medium (solid, liquid or gas) is necessary for the transmission of sound from one place to another.

(iii) The velocity of sound is greater in solids and liquids than in gases.

Audible and Inaudible Sounds

(i) Those sounds which human ear can hear are called *audible sounds*. The range of human hearing is 20 kHz to 20 kHz. In other words, we cannot hear sounds of frequency below 20 Hz or above 20 kHz.

(ii) Those sounds which human ear cannot hear are called *inaudible sounds*. The sounds of frequency below 20 Hz are called *infrasonics* while sounds of frequency above 20 Hz are called *ultrasonics*. Infrasonics and ultrasonics find wide applications in science and engineering.

Velocity of sound in a medium

The velocity of sound in a medium depends upon elasticity and density of the medium. According to Newton's formula, the velocity v of sound is given by

$$v = \sqrt{\frac{E}{\rho}}$$

where, E = modulus of elasticity of the medium

ρ = density of the medium.

(i) For solids, $E = Y$ (Young's modulus of elasticity).

$$\therefore \quad \boxed{v = \sqrt{\frac{Y}{\rho}}}$$

(ii) For liquids, $E = K$ (Bulk modulus of elasticity).

$$\therefore \quad \boxed{v = \sqrt{\frac{K}{\rho}}} \quad \ldots (2)$$

(iii) For gases, $E = K$ (Bulk modulus of elasticity),

$$\therefore \quad \boxed{v = \sqrt{\frac{K}{\rho}}} \quad \ldots (3)$$

Newton assumed that when sound waves travel through a gas (e.g. air), the compressions and rarefactions are formed so slow that temperature of the medium remains the same.

In other words, the propagation of sound waves in a gas takes place under *isothermal conditions*. It can be easily shown that under isothermal conditions, bulk modulus of a gas is equal to the pressure of the gas.

$$\therefore \quad v = \sqrt{\frac{P}{\rho}} \quad \ldots (4)$$

Velocity of sound in air by Newton's formula

According to Newton's formula, the velocity v of sound in air is given by:

$$v = \sqrt{\frac{P}{\rho}}$$

We know that for air at N.T.P:

$$P = 0.76 \times 13{,}600 \times 9.8 \text{ N/m}^2 \; ; \; \rho = 1.293 \text{ kg/m}^3$$

$\therefore$ Velocity of sound in air at N.T.P.

$$v = \sqrt{\frac{0.76 \times 13.600 \times 9.8}{1.293}} = 280 \text{ m/s}$$

This value is about 16% less than the experimental value (332 m/s) for the velocity of sound in air at N.T.P. This large difference in the two values (experimental and theoretical) cannot be attributed to say experimental error.

In 1817, Pierre Laplace, a French scientist, explained the reason for this discrepancy.

The speed of sound in air is commonly take as 344 m/s for normal conditions. This is very less compared to the velocity of light.

Table 3.1 gives the speed of sound in some media. Sound travels faster in liquid media than in gaseous media and much faster in solid media.

Table 3.1 Velocity of sound in various media

	Medium	**Velocity ($\mathbf{ms^{-1}}$)**
Gases	Air 0°C	331
	Air 20°C	343
	Helium	965
	Hydrogen	1284

	Medium	Velocity (ms^{-1})
Liquids	Water 0°C	1402
	Water at 20°C	1482
	Sea water	1522
Solids	Aluminium	6420
	Steel	5921
	Granite	6000

3.9 DOPPLER EFFECT

The pitch or frequency of the whistle of a moving train appears to increase as the sound source approaches a stationary observer. The pitch or frequency decreases as the source of sound recedes away from the stationary observer. **This apparent change in frequency was first observed by Doppler in 1845.**

Definition

The phenomenon of the apparent change in the frequency of the sound due to relative motion between the source of sound and the observer is called Doppler effect.

1. Both source and observer at rest

Suppose S and O are the positions of the source and the observer respectively. Let n be the frequency of the sound and v be the velocity of sound.

In one second, n waves produced by the source travel a distance $SO = v$ as shown in fig. 3.21

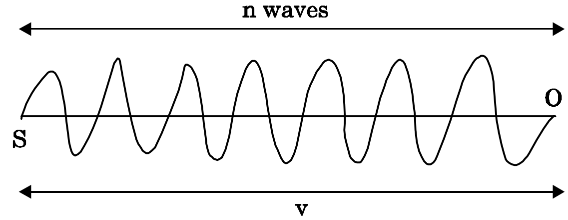


Fig. 3.21 Both source and observer at rest

The original wavelength is $\lambda = \dfrac{v}{n}$ ... (1)

The original frequency $n = \dfrac{v}{\lambda}$ ... (2)

2. When the source moves towards the stationary observer

If the source moves with a velocity v_s towards the stationary observer then after one second the source will reach S' such that $SS' = v_s$.

Now n waves emitted by the source will occupy a distance $(v - v_s)$ only as shown in fig. 3.22

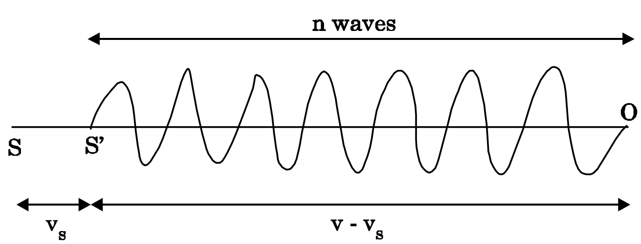


Fig. 3.22 Source moves towards observed at rest

Therefore, the apparent wavelength of the sound,

$$\lambda' = \frac{v - v_s}{n} \quad \text{... (3)}$$

The apparent frequency

$$\boxed{n' = \frac{v}{\lambda'} = \left(\frac{v}{v - v_s}\right) n} \quad \text{... (4)}$$

Comparing equations (2) and (4) we can conclude that $n' > n$, the pitch or frequency of the sound appears to increase.

3. When the sources moves away from the stationary observer

If the source moves away from the stationary observer with velocity v_s

The apparent frequency

$$\boxed{\boldsymbol{n' = \frac{v}{\lambda'} = \left(\frac{v}{v-(-v_s)}\right)n = \left(\frac{v}{v+v_s}\right)n}} \quad \dots (5)$$

Comparing equations (2) and (5) we can conclude that $n' < n$, the pitch or frequency of the sound appears to decrease.

For a moving source, the change in frequency occurs because although the source emits waves at constant rate, the waves emitted in a specified time occupy a shorter or longer length than when the source is stationary. Consequently, there is apparent change in wavelength and hence the frequency changes.

4. Source is at rest and observer in motion

S and O represent the position of source and observer respectively. The source S emits n waves per second having a wavelength $\lambda = \frac{v}{n}$.

Consider a point A such that OA contains n waves which crosses the ear of the observer in one second as shown in figure 3.23 (a).

That is, when the first wave is at the point A, the n^{th} wave will be at O where the observer is situated.

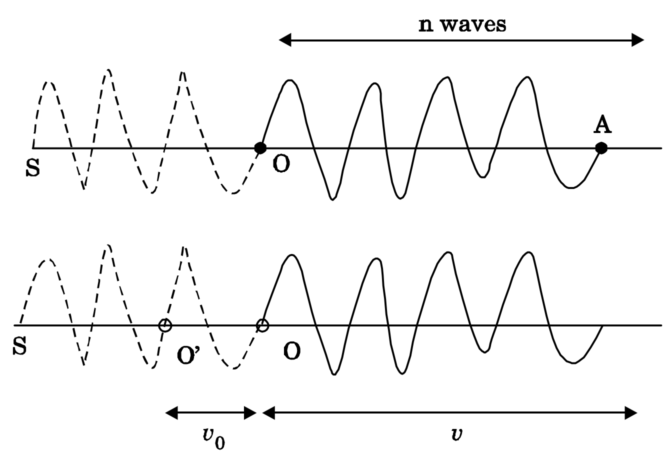


Fig. 3.23 (a) and (b) observer is moving towards a source at rest

5. When the observer moves towards the stationary source

Suppose the observer is moving towards the stationary source with velocity v_o. After one second the observer will reach the point O' such that $OO' = v_o$.

The number of waves crossing the observer will be n waves in the distance OA in addition to the number of waves in the distance OO' which is equal to $\frac{v_o}{\lambda}$ as shown in figure 3.23(b).

Therefore, the apparent frequency of sound

$$n' = n + \frac{v_o}{\lambda} = n + \left(\frac{v_o}{v}\right) n$$

The apparent frequency of sound

$$\boxed{\boldsymbol{n' = \left(\frac{v + v_o}{v}\right) n}} \quad \text{... (6)}$$

Comparing equations (2) and (6) we can conclude that $n' > n$, the pitch of the sound appears to increase.

6. When the observer moves away from the stationary source

Suppose the observer is moving towards the stationary source with velocity v_o.

Therefore, the apparent frequency of sound

$$n' = \left(\frac{v + (- v_o}{v}\right) n$$

The apparent frequency of sound

$$\boxed{\boldsymbol{n' = \left(\frac{v - v_o}{v}\right) n}} \quad \text{... (7)}$$

Comparing equations (2) and (7) we can conclude that $n' < n$, the pitch of the sound appears to decrease.

Similarly for a moving observer the change in frequency occurs because although the source emits at constant rate, the waves emitted in a specified time is heard by the observer or listener receives longer or smaller number of waves than the observer is stationary relative to the source.

Consequently there is apparent change in wavelength and hence the frequency changes.

Note

(a) If the source and the observer move along the same directions, the equation for the apparent frequency becomes

$$\boxed{n' = \left(\frac{v - v_o}{v - v_s}\right) n} \quad \ldots (8)$$

(b) If the source and the observer approaches each other, the equation for the apparent frequency becomes

$$\boxed{n' = \left(\frac{v + v_o}{v - v_s}\right) n} \quad \ldots (9)$$

(c) If the source and the observer are moving away from each other, the equation for the apparent frequency becomes

$$\boxed{n' = \left(\frac{v - v_o}{v + v_s}\right) n} \quad \ldots (10)$$

Suppose the wind is moving with a velocity ω in the direction of propagation of sound, the equation for the apparent frequency becomes

$$\boxed{n' = \left(\frac{v + \omega - v_o}{v + \omega - v_s}\right) n} \quad \ldots (11)$$

Applications of Doppler effect

(i) To measure the speed of an automobile

An electromagnetic wave is emitted by a source attached to a police car. The wave is reflected by a moving vehicle, which acts as a moving source. There is a shift in the frequency of the reflected wave. From the frequency shift using beats, the speeding vehicles are trapped by the police.

(ii) RADAR (Radio detection and ranging)

A RADAR sends high frequency radiowaves towards an aeroplane. The reflected waves are detected by the receiver of the radar station. The difference in frequency is used to determine the speed of an aeroplane.

(iii) SONAR (Sound navigation and ranging)

Sound waves generated from a ship fitted with SONAR are transmitted in water towards an approaching submarine. The frequency of the reflected waves is measured and hence the speed of the submarine is calculated.

(iv) Blood flow meter

Ultrasonic sounds are transmitted towards organs the frequency change in reflected waves used to measure blood flow rate.

(v) Tracking a Satellite

The frequency of radio waves emitted by a satellite decreases as the satellite passes away from the Earth. The frequency received by the Earth station combined with a constant frequency generated in the station gives the beat frequency. Using this, a satellite is tracked.

(vi) Stars moving towards the earth or away from the earth

There is an apparent change in wavelength of spectral lines emitted by a moving star. If the star is moving away from the Earth, there is a shift towards the red end and if it is approaching towards the Earth, a shift towards the violet end of the spectrum is indicated. This spectral shift enables the velocity of star to be computed along the line of sight.

SOLVED PROBLEMS

Problem 3.1

A second source is approaching a stationary observer with a velocity of 166 m / s. The observer hears a sound whose apparent frequency is twice the real frequency. Find the speed of sound in air.

Solution:

If n is real frequency, then due to motion of source, the apparent frequency (n') is given by

$$n' = \frac{v}{v - v_s} n \quad \text{...(1)}$$

where v is velocity of sound in air

Given $n' = 2n, \quad v_s = 166 \text{ m / s}$

$$\therefore \quad 2n = \frac{v}{v - 166} n$$

$$\text{or} \quad 2 = \frac{v}{v - 166}$$

$$2v - 332 = v$$

$$v = 332 \text{ m / s}$$

Problem 3.2

An observer is approaching a stationary sound-source vibrating with frequency 90 vibrations / second with a velocity one-tenth the velocity of sound. What is the apparent frequency heard by observer?

Solution:

If n is real frequency, then apparent frequency due to approaching observe is

$$n' = \frac{v + v_0}{v} n, \text{ where } v \text{ is velocity of sound.}$$

Here velocity of observer $v_0 = \frac{v}{10}$ and $n = 90$ vib/s

$$\therefore\ n' = \frac{v + \left(\frac{v}{10}\right)}{v} \times 90 = \frac{11}{10} \times 90 = 99 \text{ vibrations / s}$$

Problem 3.3

A railway engine moving with a speed of 60 m/s passes in front of a stationary listener. The real frequency of the whistle is 400 Hz. Calculate the apparent frequency heard by the listener.
(i) When the engine is approaching the lister
(ii) When the engine is moving away from the listener.
[velocity of sound = 340 m / s]

Solution:

Given $n = 400$ Hz, $v = 340$ m/s, $v_s = 60$ m/s

(i) When engine is approaching the listener

$$n' = \frac{v}{v - v_s} n$$

$$= \frac{340}{340 - 60} \times 400 = \frac{340}{280} \times 400 = 486 \text{ Hz}$$

(ii) When the engine is moving away from the listener, then

$$n' = \frac{v}{v + v_s} n$$

$$= \frac{340}{340 + 60} \times 400 = \frac{340}{400} \times 340 \text{ Hz}$$

Problem 3.4

A source of sound produces waves of wavelength 60 cm. This source is moving towards north with a speed one-fifth the speed of sound. Find the apparent wavelength of waves in the north and south directions.

Solution:

If sound source is moving with velocity v_s relative to observer, then apparent frequency $n' = \frac{v}{v \pm v_s} n$

Here (+) sign is for receding and (–) sign for approach.

If λ and λ' are real and apparent wavelengths, then

$$n = \frac{v}{\lambda} \text{ and } n' = \frac{v}{\lambda'}$$

$\therefore$
$$\frac{v}{\lambda'} = \frac{v}{v \pm v_s} \cdot \frac{v}{\lambda}$$

i.e.,
$$\lambda' = \frac{v \pm v_s}{v} \lambda$$

Given $v_s = \frac{v}{5}$ and $\lambda = 60$ cm. As source is approaching towards north, therefore for **observer in north-direction.**

$$\lambda' = \frac{v - v_s}{v} \lambda = \frac{v - \left(\frac{v}{5}\right)}{v} \times 60 \text{ cm} = 48 \text{ cm}$$

and for observer in south-direction

$$\lambda' \equiv \frac{v + v_s}{v} \lambda = \frac{v + \left(\frac{v}{5}\right)}{v} \times 60 \text{ cm} = \frac{6}{5} \times 60 \text{ cm} = 72 \text{ cm}$$

Part - 'A'
'2' Marks Questions with answers

1. Define simple harmonic motion.

When the acceleration of particle is directly proportional to its displacement from its equilibrium position and it is always directed towards equilibrium position, then the motion of the particle is said to be **simple harmonic motion**.

2. What are the characteristics of simple harmonic motion?

- The motion must be periodic.
- The motion is oscillatory ie., to and fro along a straight line or along a curved path about a mean position.
- The body executing simple harmonic motion is acted upon by a restoring force whose magnitude is proportional to the displacement and its direction is always towards the mean position.
- If there is no air resistance or friction, the motion once started will continue indefinitely.

3. What are examples of simple harmonic motions?

- Vibrations of a tuning fork.
- Vibrations of a sonometer wire.
- Vertical oscillations of the liquid column in a U-tube.
- Angular oscillations of a torsion pendulum.

4. What are types of oscillation?

1. Free oscillations
2. Damped oscillations
3. Forced oscillations

5. What is resonance?

The phenomenon of making a body vibrate with its natural frequency under the influence of another vibrating body with the same frequency is called resonance.

6. What is sharpness of resonance?

The rate of change (fall) of amplitude with the change of forcing frequency on each side of resonant frequency is known as sharpness of resonance.

7. Define progressive wave.

Progressive wave originating from a point source and propagating through an isotropic medium travel with equal velocity in all directions.

At any instant, the wavefront (locus of all particles vibrating with the same phase) will be spherical in nature.

If the sphere of very large radius is considered, the spherical wave will approximate to a plane surface and the waves are called plane progressive waves.

8. What are the characteristics of progressive wave?

1. Each particle of the medium executes vibration about its mean position. The disturbance progresses onward from one particle to another.
2. The particles of the medium vibrate with same amplitude about their mean positions.
3. Each successive particle of the medium performs a motion similar to that of its predecessor along the propagation of the wave, but later in time.
4. The phase of every particle changes from 0 to 2π.

9. Define standing waves.

When two progressive waves of same amplitude and wavelength travelling along a straight line in opposite directions superimpose on each other, stationary waves are formed.

10. What are the characteristics of standing waves?

1. The waveform remains stationary.
2. Nodes and antinodes are formed alternately.
3. The points where displacement is zero are called nodes and the points where the displacement is maximum are called antinodes.

4. Pressure changes are maximum at nodes and minimum at antinodes.

11. What are differences between progressive waves and stationary waves?

S. No.	**Progressive waves**	**Stationary waves**
(i)	There is transfer of energy in the direction of propagation of wave.	There is no transfer of energy.
(ii)	No particle of the wave is permanently at rest.	The particles at nodes are permanently at rest.
(iii)	The particles of the medium vibrate with same amplitude about their mean	Amplitude of each particle is not same. It is maximum at antinodes and decreases gradually to zero at the nodes.
(iv)	The phase of vibration varies continuously.	Particles in the same segment vibrate in the same phase.

12. State Doppler effect.

The phenomenon of the apparent change in the frequency of the sound due to relative motion between the source of sound and the observer is called Doppler effect.

Part - B '16' Marks Questions

1. Explain simple harmonic motion and discuss its characteristics.
2. Discuss the phenomenon of sharpness of resonance.
3. Deduce the wave equation for progressive wave.
4. Derive the wave equation for standing waves.
5. Discuss analogy between electrical and mechanical oscillating system.

6. Explain Doppler's principle. Calculate the apparent pitch of a note due to the motion of source and the listener.

7. Discuss Doppler's effect in sound and obtain an expression for the apparent frequency of the note when the source and the listener are (i) moving towards each other, and (ii) moving away from each other.

8. Explain Doppler's effect. Explain the apparent frequency of a note when both the source and the listener are in motion, the medium being at rest.

9. Explain Doppler's effect. Find an expression for the ratio of the apparent frequency to the real frequency of the notes emitted when there is relative movement between the source and the listener.

10. State and explain Doppler's effect. Derive an expression for the change in frequency of a note when both the source of sound and the observer are in motion.

11. Explain Doppler's effect. Obtain an expression for the frequency of a note heard by an observer, when both the source and the observer are in motion towards each other.

ASSIGNMENT PROBLEM

Note: Use following physical constants wherever required:
speed of sound in air, $v = 330 \text{ m/s}$
speed of light, $c = 3 \times 10^8 \text{ m/s}$

1. With what velocity should a sound-source travel towards a stationary observer so that the apparent frequency may be double of this actual frequency?

   [Ans.** $v/2$ **where** v **is velocity of sound]

2. With what velocity should a sound source move towards an stationary observer so that the apparent frequency heard be three-times its real frequency. ***[Ans. 280 m / s]***

3. With what velocity should a moving observer go away from a stationary source so that the apparent frequency heard be half of its real frequency. If observer is stationary and source goes away then? **[Ans.** ***165 m/s, 330 m/s]***

4. A motor-car is approaching towards a crossing with a velocity 75 km / h. The traffic man standing at crossing hears the frequency of sound of horn to be 260 Hz. What is the real frequency of sound of horn? **[Ans.** ***244 Hz]***

4. Optics

Reflection and refraction of light waves – total internal reflection – interference Michelson interferometer – Theory of air wedge and experiment

Introduction

Light is a form of energy to which our eye is sensitive. While light itself is **'invisible'** to our eye, it makes objects visible, mainly by the process of scattering.

Light energy exhibits dual nature, that of a particle and a wave. The wave nature of light can be understood on the basis Young's double slit experiment. The wave character of light manifests itself in phenomena such as interference, diffraction, polarization etc.

On an atomic scale when light energy interacts with matter it exhibits particle nature. The particle aspect of light can be understood on the basis of **Quantum theory of radiation** proposed in 1900 by the German physicist **Max Planck**. According to this theory of light energy is emitted and absorbed by matter in packets called **Quantum of radiation**.

"Optics is the branch of physics which deals with the source of light, nature of light, its properties and effects".

Optics is broadly classified into two branches:

(a) **Geometrical optics** (b) **physical optics.**

Geometrical optics deals with the ray concept of light. It deals with the optical phenomena such as reflection, refraction and their applications to optical instruments.

Physical optics deals with the wave nature of light. It deals with the phenomena such as interference, diffraction and polarization etc.

4.1 REFLECTION OF LIGHT WAVES

(i) **The phenomenon where the incident light falling from one medium on a surface of another medium is sent back to the same medium is known as reflection.**

(ii) The angle between the incident ray and the normal to the surface is known as angle of incidence.

(iii) The angle between the reflected ray and the normal of the surface is known as angle of reflection. (Fig 4.1)

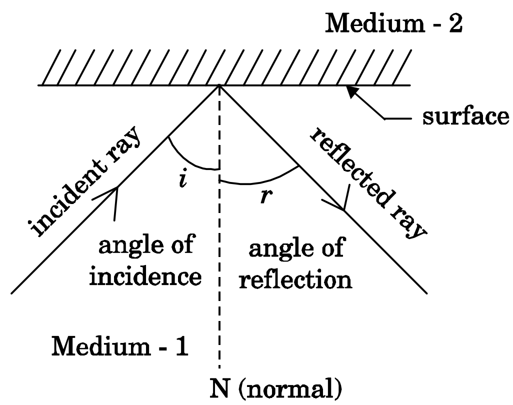


Fig. 4.1 Reflection of Light Waves

Laws of reflection

(i) **Incident ray, normal and reflected ray lie in the same plane.**

(ii) **The angle of incidence is equal to the angle of reflection i.e. $\angle i = \angle r$**

4.2 REFRACTION OF LIGHT WAVES

Refraction is the phenomenon in which light travels from one medium (say air) to another medium (say glass). The direction of light changes due to change in medium.

If a ray of light passes from a rarer medium (air) into denser medium (glass) then the ray of light bends towards the normal. (Fig 4.2)

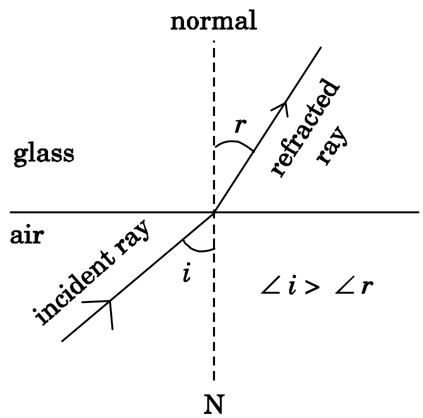


Fig. 4.2

If a ray of light passes from denser medium into rarer medium then the ray of light bends away from the normal. (Fig 4.3)

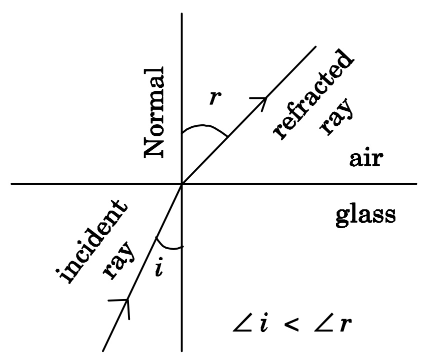


Fig. 4.3

Note: The speed of light in rarer medium is faster than that of denser medium

Laws of refraction

(i) **The incident ray, the refracted ray and the normal at a point of separation of two media lie in the same plane.**

(ii) For any two medium, the ratio of sine of angle of incidence to sine of angle of refraction is constant. It is known as **Snell's law**.

$$\text{Therefore, } \frac{\sin i}{\sin r} = \text{constant}$$

where this constant is called as refractive index (μ) of medium

$$\therefore \quad \boxed{\mu = \frac{\sin i}{\sin r}}$$

Significance of refractive index

The ratio of velocity of light in vacuum to velocity of light in medium, is called as refractive index.

$$\boxed{\text{ie., Refractive index, } \mu = \frac{\text{Velocity of light in vacuum } (c)}{\text{Velocity of light in medium } (v)}}$$

$$\mu = \frac{c}{v}$$

(i) When light travels from rarer medium (air) to denser medium (glass), then the $\angle i$ is greater than $\angle r$. Therefore, refractive index is always greater than 1.

(ii) When light travels from denser medium (glass) to rarer medium (air), then the $\angle i$ is less than $\angle r$. Therefore, refractive index is always less than 1.

(iii) Refractive index for vacuum is unity (1)

4.3 TOTAL INTERNAL REFLECTION

(i) When light passes from denser medium to rarer medium, then the refracted ray bends away from the normal.

(ii) Consider a ray AB incident at $\angle i_1$ and refracted at $\angle r_1$, as angle of incidence increases, angle of refraction also increases.

(iii) For a particular value of angle of incidence $\angle i_2$, the refracted ray travels along the surface of separation between the two medium i.e. $\angle r$ becomes 90°, then the angle of incidence is called as critical angle θ_c.

(iv) For the angle of incidence greater than critical angle ($\angle i_3 > \theta_c$), the ray cannot pass into second medium but completely gets reflected in the same medium. (Fig 4.4)

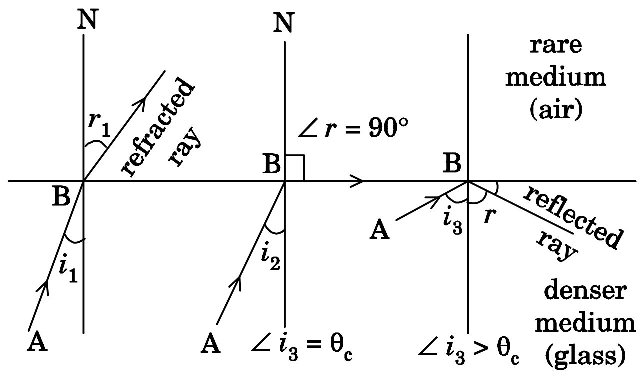


Fig. 4.4

(v) Thus, a ray travelling from denser medium to a rarer medium is reflected into denser medium if angle of incidence is more than the critical angle of medium.

Critical angle

Definition

The angle of incident at which the refracted ray just graze surface between denser and rarer media is called critical angle.

When light travels from denser to rarer medium, from Snells law.

$$n_1 \sin i = n_2 \sin r$$

n_1 – refractive index of denser medium

n_2 – refractive index of rarer medium

i – angle of incident and r-angle refraction

$i = \theta_c$ when $r = 90°$

$$\therefore \qquad n_1 \sin\theta_c = n_2 \sin 90° \qquad (\because \sin 90° = 1)$$

$$\sin\theta_c = \frac{n_2}{n_1}$$

or
$$\boxed{\theta_c = \sin^{-1}\left(\frac{n_2}{n_1}\right)}$$

Definition

***When a ray of light within a denser medium (e.g. water) approaches the surface at an angle of incidence greater than the critical angle, the ray of light is reflected back into the same medium (i.e. water). This phenomenon is known as total internal reflection*. (Fig 4.5)**

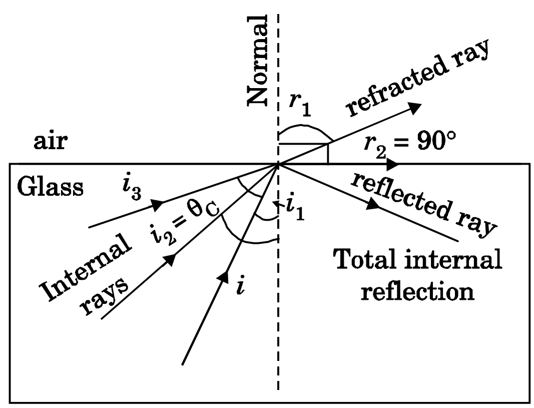


Fig. 4.5

Necessary conditions for total internal reflection

(a) The light should incident from denser medium to rarer medium.

(b) The angle of incidence i in denser medium should be greater than critical angle θ_c.

Applications of total internal reflection

There are a large number of practical applications of the phenomenon of total internal reflection. However, only two applications are briefed below by way of illustration.

(i) **Mirage**. During the day time in the desert, it is seen that sand at some distance from the observer looks like a pond of water. This illusion is called mirage and it is caused due to total internal reflection of light. (Fig 4.6)

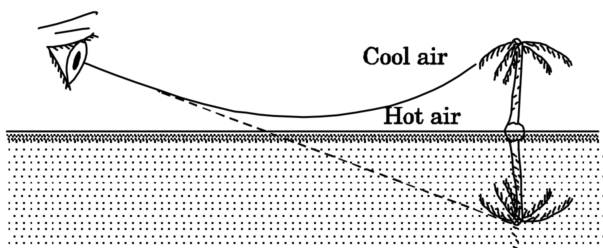


Fig. 4.6

(ii) **Optical fibre**. An optical fibre is a transparent fibre used to conduct light through the phenomenon of total internal reflection. Fig. 4.7 shows a typical optical fibre cable.

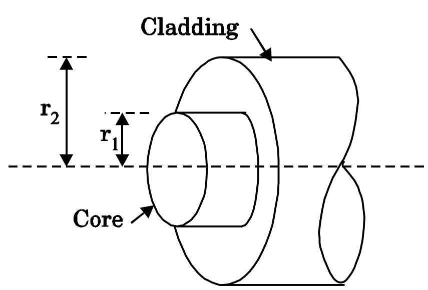


Fig. 4.7

It consists of fibre glass core surrounded by fibre glass cladding. The cable is so designed such that the refractive index of core is more than that of the cladding.

The core performs the function of transmitting the light waves by total internal reflection while the purpose of cladding is to minimise surface losses and to guide the light waves.

Fig. 4.8 shows how optical fibre cable works. The ray of light entering one end of the cable undergoes a series of total internal reflections.

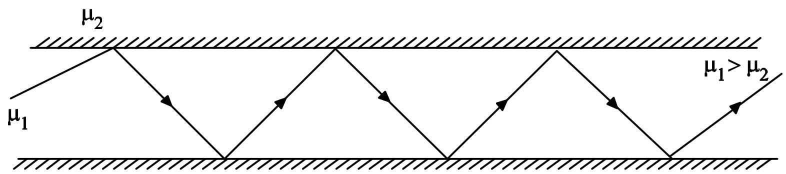


Fig. 4.8

The light emerging from the other end of the optical fibre cable has no loss of intensity during its journey. **Optical fibres are being used in a variety of medical and engineering applications.**

4.4 INTERFERENCE OF LIGHT WAVES

When two light waves of the same frequency and having constant phase difference traverse simultaneously in the same region of a medium and cross each other then there is a modification in the intensity of light, in the region of superposition.

This modification or change of intensity of light resulting from the superposition of two or more waves of light is called interference.

At some points, where the crests of one wave falls on the crest of the other, resultant amplitude is maximum. Hence, the intensity of light is maximum. At certain other points, crest of one wave falls on trough of the other, therefore resultant amplitude becomes minimum and hence intensity of light is minimum.

At the points, where the resultant intensity of light is maximum, the interference is said to be **constructive.** At the points where the resultant intensity of light is minimum, the interference is said to be **destructive.**

In 1801, **Thomas Young** demonstrated successfully the phenomenon of interference of light by the double slit experiment, called **Young's experiment**.

Theory of Interference Fringes

Consider a narrow monochromatic source S and two pin holes S_1 and S_2 equidistant from S. The pin holes provide the two point sources vibrating in the same shades. The two sources are separated at a distant $2d$.

Let a screen XY be placed at a distant D parallel to $S_1 S_2$. The point O on the screen is equidistance from S_1 and S_2. Consider a point P at a distance x from O (Fig 4.9).

We shall now consider the conditions for a bright or dark fringe at this point.

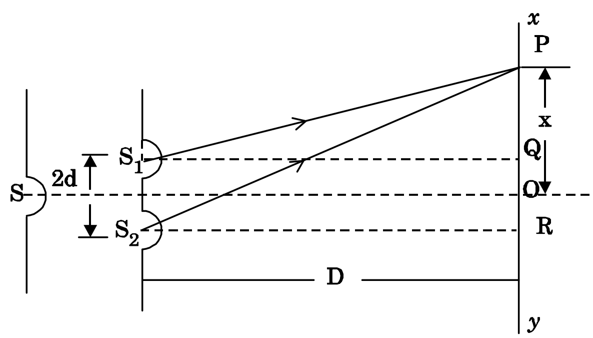


Fig. 4.9

From the right angled triangle $S_1 QP$

$$(S_1 P)^2 = (S_1 Q)^2 + (QP)^2$$

or $$(S_1 P)^2 = D^2 + (x-d)^2 \quad [\because QP = (x-d)] \quad \ldots (1)$$

Similarly in right angle triangle $S_2 RP$

$$(S_2 P)^2 = (S_2 R)^2 + (RP)^2$$

or $$(S_2 P)^2 = D^2 + (x+d)^2 \quad [\because RP = (x+d)] \quad \ldots (2)$$

$$\therefore (S_2 P)^2 - (S_1 P)^2 - (S_1 P)^2 = D^2 + x^2 + d^2 + 2dx - D^2 - x^2 + 2dx - d^2$$

$$(S_2 P)^2 - (S_1 P)^2 = 4\,x\,d$$

(or) $$(S_2P - S_1 P)\,(S_2 P + S_1 P) = 4\,x\,d$$

In Young's experiment, D is some thousand times greater than $2d$ or x so that if $(S_2 P + S_1 P)$ is replaced by $2D$, the error is not more than a fraction of one percent.

Hence,

$$(S_2P - S_1 P)\,2D = 4\,x\,d$$

$$\text{or } (S_2P - S_1 P) = \frac{4\,x\,d}{2D} = \frac{2\,x\,d}{D} \qquad \dots (1)$$

Position and Spacing of fringes

Now, we shall consider the following two cases

1. **Bright Fringes:** The point P is bright when the path difference is a whole number multiple of wavelength λ, i.e.

$$S_2P - S_1 P = n\,\lambda \text{ where } n = 0, 1, 2$$

Substituting the value of $(S_2P - S_1 P)$ from equation (1) we have

$$\frac{2xd}{D} = n\,\lambda$$

$$\text{or } \boxed{x = \frac{n\,\lambda\,D}{2d}} \qquad \dots (2)$$

Equating (2) gives the distances of the bright fringes from point O. At O, the path difference is zero hence there is a bright fringe. The next bright fringes are formed when $n = 1, 2, 3$ …… and so on i.e.

$$\text{when } n = 1,\ x_1 = \frac{\lambda D}{2d}$$

$$n = 2,\ x_2 = \frac{\lambda D}{2d}$$

$$n = 3,\ x_3 = \frac{3\lambda D}{2d}$$

$$n = n,\ x_n = \frac{n \lambda D}{2d}$$

The distance between any two consecutive bright fringes is,

$$x_2 - x_1 = \frac{2\lambda D}{2d} - \frac{\lambda D}{2d} = \frac{D \lambda}{2d} \qquad \dots (7)$$

2. **Dark Fringes:** The point P is dark when the path difference is an odd number multiple of hall wavelength i.e.

$$(S_2P - S_1 P) = (2n + 1)\frac{\lambda}{2}$$

$$\text{where } n = 0, 1, 2, 3 \ldots\ldots$$

$$\frac{2\,x\,d}{D} = \frac{(2n+1)\,\lambda}{2}$$

$$x = \frac{(2n+1)\,\lambda D}{4d} \qquad \dots (8)$$

The equation (8) gives the distances of the dark fringes from point O. The dark fringes are formed as follows.

$$\text{when } n = 0,\ x_0 = \frac{\lambda D}{4d}$$

$$n = 1,\ x_1 = \frac{3\lambda D}{4d}$$

$$n = 2, \; x_2 = \frac{5\lambda D}{4d}$$

$$n = n \;\; x_n, = \frac{(2n+1)\lambda D}{4d}$$

The distance between any two consecutive dark fringes is,

$$x_2 - x_1 = \frac{5\lambda D}{4d} - \frac{3\lambda D}{4d} = \frac{2\lambda D}{4d} = \frac{\lambda D}{2d} \qquad \text{... (9)}$$

Hence the spacing between any two consecutive maximum or minima is the same. This is expressed by $\beta \left(\beta = \frac{\lambda D}{2d} \right)$ and **it is known as fringe width**.

It is obvious that the spacing is directly proportional to and inversely proportional to 2d.

Conditions for Interference of light

(i) Conditions for sustained interference

(a) The sources should be coherent.

(b) The sources should emit continuous waves of the same wavelength and time period.

(ii) Conditions for observations

(a) The separation between the two sources (2d) should be small.

(b) The distance 'D' between two sources and the screen should be large.

(c) The background should be dark.

(iii) Conditions for good contrast

(a) The amplitudes of the interfering waves should be equal or nearly equal.

(b) The sources must be narrow, i.e., they must be extremely small.

(c) The sources should be monochromatic.

Type of Interference

There are normally two methods for obtaining interference pattern.

1. Interference by Division of Wavefront

Here the original wave is divided into two parts by using slits, prism or mirror, and the two wavefronts thus produced are brought together to produce the interference pattern. It requires a narrow source of light.

Examples: Fresnel's Biprism, Lloyd's Mirror.

2. Interference by Division of Amplitude

In this case the original amplitude of the wave is divided into two parts by reflection or refraction and the divided parts are finally brought close together to produce the interference pattern. It requires a wide source of light.

Examples: Air wedge, Newton's Rings, Michelson Interferometer.

Thin film interference

Light reflected from soap bubbles, oil films etc., shows the interference effect. These very thin films have a thickness just a few times the wavelength of light, and hence it is called **thin film interference**.

A plane light wave that falls on the film is reflected from the upper and lower surfaces, and the reflected beams interfere with each other. Thus, thin film interference is due to multiple reflections.

Newton and Hooke observed and developed the interference phenomenon due to multiple reflections from the surface of thin transparent materials.

Thin layers of oil spread on water or any surface, heated metallic surfaces, soap bubbles etc. exhibit brilliant colours when light is incident on them. These colours of thin films are due to the interference of light between the light reflected from the front and back surfaces of the film.

It is observed that interference in the case of thin film takes place due to

1. **Reflected Light and**
2. **Transmitted Light**

Let *ABCD* (Fig.4.10) represent a thin transparent film (thickness t and refractive index μ) with the parallel boundary surfaces *AB* and *CD*. Let *PQ* be a light ray incident *AB*. The ray is partly reflected as *RS* and partly refracted as $Q_1 Q_2$ at Q_1.

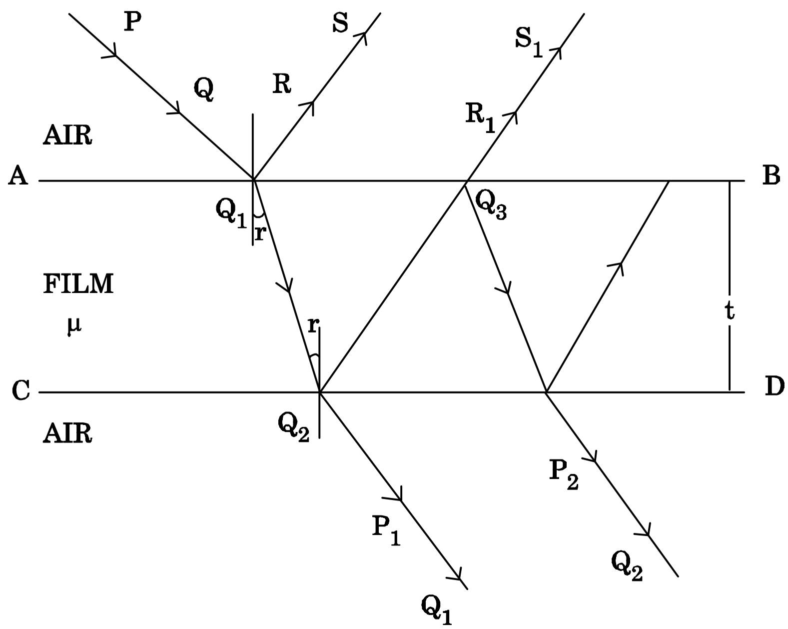


Fig. 4.10

The ray $Q_1 Q_2$ is again split up into a refracted ray $P_1 Q_1$ and a reflected ray $Q_2 Q_3$. The ray $Q_2 Q_3$ suffers reflection and refraction at Q_3 and so on.

1. Interference due to Reflected light

If the thickness of the film is small, the directly reflected ray RS and the refracted ray $R_1 S_1$ from Q_3 superimpose on one another. Terefore the waves of RS and $R_1 S_1$ are in a fit condition to interfere producing either brightness or darkness depending upon the path difference.

The correct path difference between two reflected waves in this case is given by

$$\boxed{x = 2\mu t \cos r - \lambda/2} \quad \text{... (1)}$$

where, μ – refractive index of the medium

t – thickness of the thin film

r – angle of refraction

λ – wavelength of the light

1. If the path difference $x = n\lambda$ where $n = 0, 1, 2, 3, 4$ … etc., constructive interference takes place and the film appears bright

$$2\mu t \cos r - \lambda/2 = n\lambda$$

$$\text{or} \quad 2\mu t \cos r = (2n+1)\lambda/2 \quad \text{... (2)}$$

2. If the path difference $x = (2n+1)\lambda/2$ where $n = 0, 1, 2$ … etc., destructive interference takes place and the film appears dark

$$2\mu t \cos r - \lambda/2 = (2n+1)\lambda/2$$

$$\text{or} \quad 2\mu t \cos r = (n+1)\lambda \quad \text{... (3)}$$

Here n is an integer only, therefore $(n+1)$ can also be taken as n

$$\boxed{2\mu t \cos r = n\lambda} \quad \text{... (4)}$$

where $n = 0, 1, 2, 3, 4$ … etc

2. Interference due to transmitted light:

When the thickness of the film is small, the transmitted rays $P_1 Q_1$ and $P_2 Q_2$ (which are parallel) superimpose and interference of light occurs. The type of fringes (bright or dark) depends upon the path difference between these two rays.

1. The film will appear bright or constructive interference will form if the path difference $x = n\lambda$

$$\text{or} \quad \boxed{2\mu t \cos r = n\lambda} \qquad \text{... (1)}$$

where $n = 0, 1, 3, \ldots\ldots\ldots$ etc.

2. The film will appear dark or destructive interference will form if the path difference

$$x = (2n + 1)\lambda/2$$

$$2\mu t \cos r = (2n+1)\lambda/2 \qquad \text{... (2)}$$

where $n = 0, 1, 2, 3, \ldots\ldots\ldots$ etc.

Colours in thin films:

With monochromatic light alternate bright and dark fringes are obtained. With white light the fringes obtained are coloured because the path difference $2\mu t \cos r$ depends upon μ, t and r.

1. If t and r are constant, the path difference varies with μ or the wavelength of light. Let light is composed of various colours. Therefore, these colours will appear in the order of violet, blue etc. as the wavelength λ increases.
2. If the angle of incidence changes, r also changes and hence the path difference also changes. Therefore, in various directions different colours will be seen with white light.
3. When the thickness of the film varies the film passes through various colours for the same angle of incidence.

4.5 THEORY OF AIR WEDGE AND EXPERIMENT

Air-wedge arrangement is used to find the thickness of a thin sheet or a wire. It is also used to test the planeness of the glass plate.

Definition

A wedge shaped (V-shaped) air film enclosed in between two glass plates is called air wedge.

Theory of air wedge experiment

When two optically plane glass plates (*A* & *B*) are inclined at a very small angle θ, a **wedge shaped thin air film** is formed between the surfaces as shown in fig. 4.11. The thickness of the air film increases outwards from the line of contact 'O' of the glass plates.

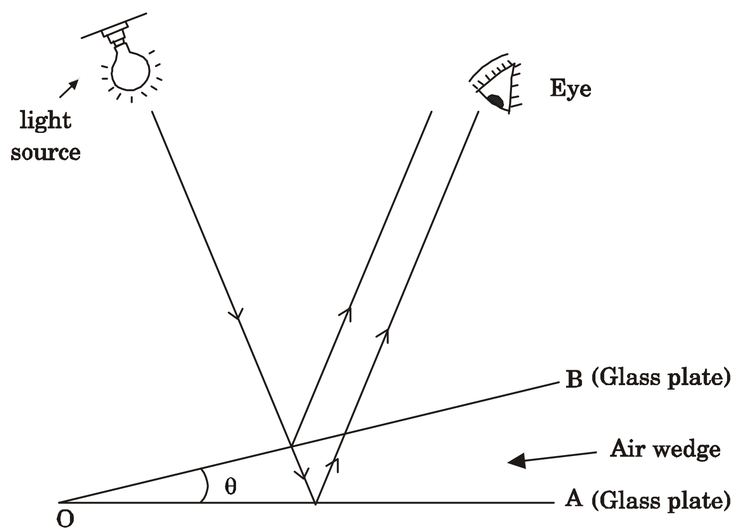


Fig 4.11 Air wedge

The light rays from a monochromatic light source is made to fall perpendicularly on the film.

The incident rays of light is partially reflected from the upper surface of the air film and partially reflected from the lower surface of the air film.

These two reflected rays will interfere and a large number of **straight alternative bright and dark fringes** are formed.

If t is the thickness of the air film corresponding to the n^{th} dark band with wedge angle θ at a distance of x metre from the edge of contact, then the path difference between the two reflected rays (Fig 4.12)

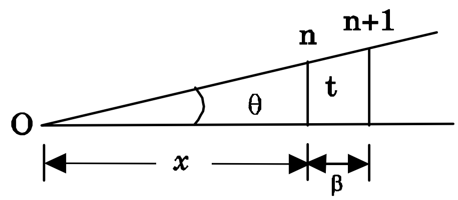


Fig 4.12

$$2\,\mu t \cos r = n\lambda \qquad \text{... (1)}$$

For air film, refractive index of the film $\mu = 1$

$\cos r = 1$, since angle of incidence is very small, so angle of refraction is also very small ie., $r = 0$; $\cos 0 = 1$

$$\text{Now,} \qquad 2t = n\lambda \qquad \text{... (2)}$$

where λ – wavelength light

Since x is the distance of the n^{th} dark band from the edge of contact O,

$$\frac{t}{x} = \tan\theta \text{ from fig. 4.12.}$$

$$\frac{t}{x} = \theta \quad (\because \theta \text{ is very small } \tan\theta \approx \theta$$

$$\therefore \qquad t = x\,\theta \qquad \text{... (3)}$$

substituting equation (3) in equation (2), for the n^{th} dark band

$$2\,x\,\theta = n\lambda \qquad \text{... (4)}$$

Similarly, for the next dark band ie., $(n + 1)^{th}$ dark band

$$2(x + \beta)\,\theta = (n + 1)\,\lambda \qquad \ldots (5)$$

where β is the fringe width

subtracting equation (4) from equation (5), we have

$$2\,\beta\,\theta = \lambda$$

or $$\boxed{\beta = \frac{\lambda}{2\,\theta}} \qquad \ldots (6)$$

The same relation is obtained if we consider the bright fringe.

Thickness of a thin wire and very thin foil

The given wire whose thickness d is to be measured is placed inbetween the two glass plates to form a wedge shaped air film.

Now if l is its distance from the edge of contact (length of the wedge), then from fig 4.13.

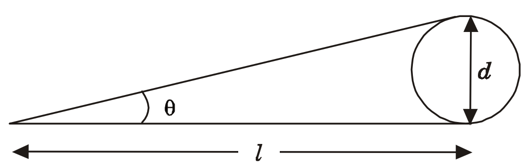


Fig. 4.13 Thickness of a thin wire

$$\tan\theta = \frac{d}{l}, \quad (\therefore\ \tan\theta \approx \theta)$$

$$\text{or}\quad \theta = \frac{d}{l}, \qquad \ldots (7)$$

Substituting eqn (7) in (6)

$$\beta = \frac{\lambda}{\frac{2d}{l}} = \frac{\lambda\,l}{2d}$$

or $$d = \frac{\lambda l}{2\beta}$$

Thus, thickness of very thin specimen can be determined by using the interferance technique in wedge shaped film.

Applications of air-wedge

Determination of diameter (thickness) of a wire or thickness of a thin sheet of paper (Experiment)

An air wedge is formed by keeping two optically plane glass plates in contact along one of the edges and a thin wire near the other end, parallel to the contact edges of the glass plates.

Therefore, glass plater are inclined at a very small angle θ (one end of these two glass plates may be tied using a thread or a rubber band). This is called *airwedge arrangement*

Description

This arrangement is kept on the bed of the travelling microscope (Fig 4.14).

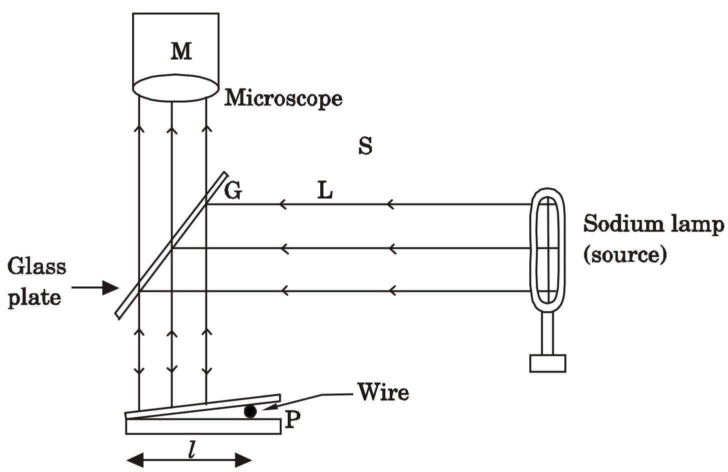


Fig. 4.14 Airwedge arrangement

A parallel beam of monochromatic light from a light source is reflected down on the air wedge by a glass plate kept inclined at angle 45° to the horizontal.

Interference takes place between the light reflected at the top and bottom surfaces of the air film between the two glass plates.

Experiment

Interference pattern (Fig 4.15) consisting of a series of bright and dark bands of equal width is viewed by a travelling microscope arranged above the air wedge.

Microscope is focussed on these fringes and the vertical cross wire is made to coincide with n^{th} bright band near the edge of contact of the glass plates.

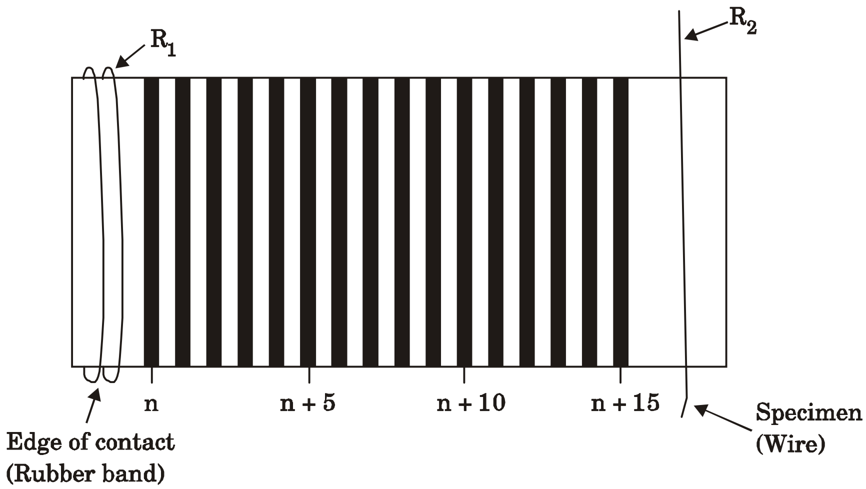


Fig. 4.15 Interference pattern

The reading on the horizontal scale of the microscope is noted. The cross wire is made to coincide with successive 5^{th} fringes ($n + 5, n + 10 \ldots n + 40$) and the corresponding microscope readings are noted. The readings are recorded in the table 4.1.

From the table, the average fringe width β is determined. Using the microscope, the distance l between the edge of the contact and the wire is also measured.

Knowing the wavelength of the monochromatic light source, the thickness of the wire is found out using the formula.

$$d = \frac{l\lambda}{2\beta} \quad \text{metre}$$

Table 4.1

S. No.	Order of the fringes	Microscope reading x 10^{-2} m	Width of 10 fringes m	Band Width β m
1	n			
2	$n + 5$			
3	$n + 10$			
4	$n + 20$			
.	…			
	$n + 40$			

Mean = ……..

ANNA UNIVERSITY SOLVED PROBLEMS

Problem 4.1

Light of wavelength 6000 Å falls normally on a thin wedge shaped film of refractive index $\mu = 1.4$ forming fringes that are 2 mm apart. Find the angle of the wedge? **(A.U. April 2014)**

Given data:

Wavelength of the light

$$\lambda = 6000\text{Å} = 6000 \times 10^{-10} \text{ metre}$$

Fringe width, $\beta = 2\text{mm} = 2 \times 10^{-3}$ metre

Refractive index of the film $\mu = 1.4$

Solution

We know the formula for fringe width in wedge shaped film

$$\beta = \frac{\lambda}{2\,\theta\,\mu} \quad \text{or} \quad \theta = \frac{\lambda}{2\,\mu\,\beta}$$

Substituting the given values, we have

$$\theta = \frac{6000 \times 10^{-10}}{2 \times 1.4 \times 2 \times 10^{-3}} \text{ radian}$$

Angle of the wedge $\theta = 1071.4 \times 10^{-7}$ radian

Problem 4.2

In an air wedge experiment, the distance between successive fringes is 1.09 × 10^{-4} m. Calculate the thickness of the object kept between the two optically plane glass plates forming the air wedge, if the length of the wedge is 3.7 × 10^{-2} m and the wavelength of the monochromatic light illuminating the wedge is 5.983 × 10^{-7} metre. **(A.U. Jan 2015)**

Given data: $\beta = 1.09 \times 10^{-4}$ m **;** $l = 3.7 \times 10^{-2}$ m

$\lambda = 5.893 \times 10^{-7}$ m ; $d = ?$

Solution

We know that thickness of the object

$$d = \frac{l\lambda}{2\beta}$$

Substituting the given values, we have

$$d = \frac{3.7 \times 10^{-2} \times 5.893 \times 10^{-7}}{2 \times 1.09 \times 10^{-4}}$$

Thickness of the object $d = 10^{-4}$ m

Interferometer

An interferometer is an instrument for measuring small changes in length. It is based on the principle of interference.

The study of interferometer is called ***interferometry***.

Michelson originally designed an interferometer which is used to find the wavelength of monochromatic light source and thickness of thin strips.

4.6 MICHELSON'S INTERFEROMETER

Principle

Two interfering beams are formed by splitting the light from a source into two parts by partial reflection and refraction. These beams are sent in two perpendicular directions, and they are finally brought together after reflection from plane mirrors to produce interference fringes.

Construction

Michelson interferometer is shown in fig 4.16. The apparatus consists of two highly polished plane mirrors M_1 and M_2. The mirrors are mounted vertically on two arms perpendicular to each other.

The mirror M_1 is mounted on a carriage C so that it can be moved forward and backward. The distance moved by this mirror can be accurately measured with the help of a graduated drum fitted on the micrometer screw M. The mirror M_2 is fixed.

There are two plane parallel glass plates, G_1 and G_2 of same thickness placed at an angle of 45° to the incident beam. The plate G_1 is semi-silvered on its back side and acts as a beam splitter, i.e., a beam incident on it is partially reflected and partially transmitted.

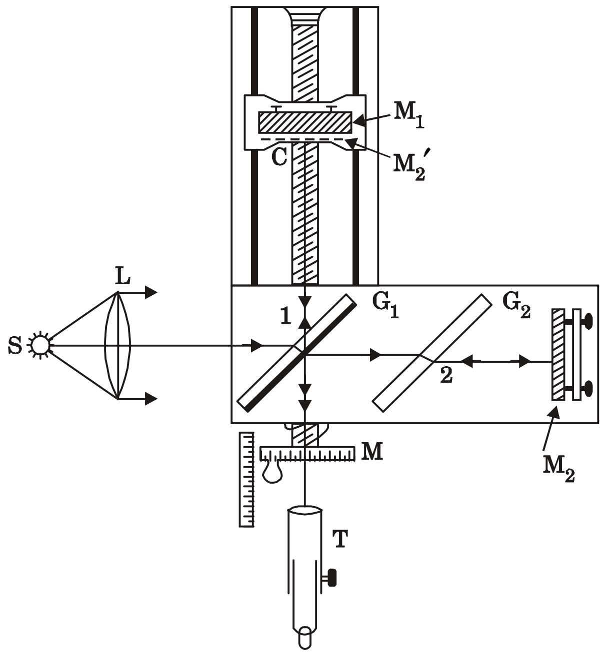


Fig. 4.16 Michelson interferometer

The plate G_2 is a compensating glass plate. S is a monochromatic light source. A telescope (T) is positioned perpendicular to M_1 to receive rays reflected from both M_1 and M_2.

Working

Light from source S is rendered parallel by means of a collimating lens L. This light is made to fall on a semi-silvered glass plate G_1. The light beam is divided into two parts.

One part of the light is reflected to travel towards mirror M_1 (ray 1) while the other part of the light is transmitted towards M_2 (ray 2).

These light rays fall normally on mirrors M_1 and M_2 and are reflected back along its original paths.

The light reflected by mirror M_1 passes through G_1 to reach the telescope T. The ray reflected by mirror M_2, on reaching G_1, gets reflected at its semi-silvered surface to reach the telescope.

A path difference can be introduced between the two reflected rays by moving mirror M_1.

If we look through the telescope in the direction of mirror M_1, we can see mirror M_1 directly together with a virtual image of M_2, denoted by M'_2, formed close to M_1 by reflection in the glass plate G_1. Hence, the rays reaching the telescope appear to travel from M_1 and M'_2.

Thus, Michelson interferometer system is optically similar to the interference from an air film enclosed between M_1 and M'_2.

The interference fringes may be straight, circular or parabolic etc., depending upon

- path difference and
- angle between mirror M_1 and virtual mirror M'_2.

Function of the compensating plate G_2

The ray reflected at M_1 passes through glass plate G_1 twice and reaches the telescope, while the ray reflected at M_2 does not pass even once.

That is why a second plate (G_2) made with same material, thickness and inclination as G_1 is introduced along the path of the ray travelling towards M_2. Thus, the function of plate G_2 is just to equalise the optical paths travelled by both rays.

Formation of fringes

- One of the interfering beams comes from M_1 and the other appears by reflection from the virtual image of mirror M_2, i.e., M'_2.

- An air film is enclosed between the two mirrors M_1 and M'_2.
- The two interfering beams appear from two virtual images S_1 and S_2 of the light source S as shown in fig. 4.17.
- Let S_1 and S_2 be two virtual sources. The distance between the virtual sources is $2d$. As ray 2 coming after reflection from M_2, suffers reflection at the silvered surface of glass plate G_1, an additional path difference of $\lambda/2$ is introduced between the two rays.
- If we observe in the direction making an angle 'θ' with the normal to the mirrors, the total path difference is $2d\cos\theta + \lambda/2$ (Fig. 4.17).

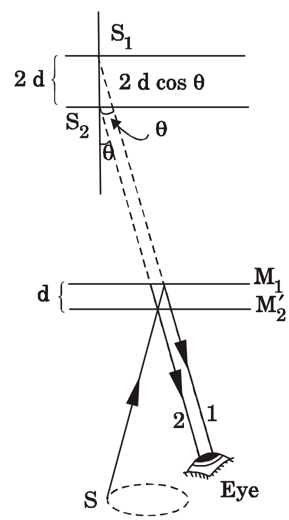


Fig. 4.17 Formation of interference fringes

- For maximum intensity in the fringes,

$$2d\cos\theta + \lambda/2 = n\lambda$$

where $n = 0, 1, 2, \ldots$

Types of Fringes

Case 1:

When M'_2 coincides with M_1 i.e., the paths are exactly equal, the path difference is only $\lambda/2$. Therefore, the field of view is perfectly dark as shown in fig 4.18(a).

Case 2:

M_1 is moved either forward or backward parallel to itself. Now, mirror M_1 is exactly perpendicular to mirror M_2, i.e., mirror M_1 and virtual mirror M'_2 are parallel.

When an air film of constant thickness is enclosed between mirrors M_1 and M'_2, we observe sufficiently spaced circular fringes as shown in fig. 4.18(b).

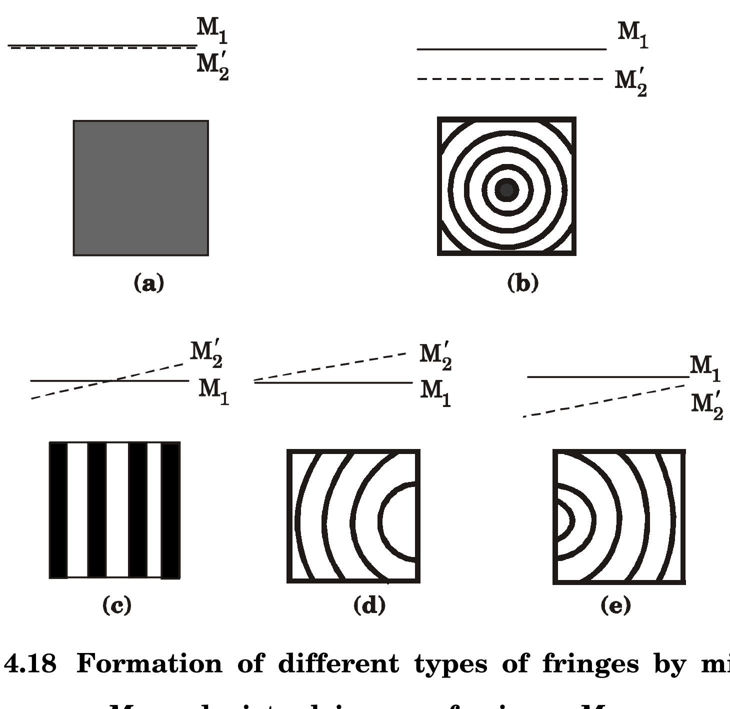


Fig. 4.18 Formation of different types of fringes by mirror M_1 and virtual image of mirror M_2.

Case 3:

When mirror M_1 intersects the virtual image M'_2, the air film enclosed is wedge shaped, and straight line fringes are produced as shown in fig. 4.18(c).

If virtual image M'_2 lies just above the plane of mirror M_1 and they are not perfectly parallel, the fringes produced are curved as shown in fig. 4.18(d).

If the plane of mirror M_1 lies above the plane of virtual image M'_2 and they are not perfectly parallel, the fringes formed are slightly curved as shown in fig. 4.18(e).

Applications of Michelson's Interferometer

It is used to find

(i) the wavelength of a given light source.

(ii) the refractive index and thickness of a transparent material

(iii) the resolution of wavelengths

(iv) the standardisation of metre

Wavelength Determination

Using monochromatic light of unknown wavelength λ, Michelson's interferometer is adjusted for circular fringes. The position of mirror M_1 is adjusted till a particular bright fringe coincides with the cross wire. Now, micrometer screw reading is noted.

When mirror M_1 is slowly moved, the number of fringes n that moved across the field view of the telescope is counted. This may be in the order of 20 or 30 fringes. Again micrometer screw reading is noted.

The distance d moved by the mirror is given by the difference between initial and final micrometer screw readings.

When mirror M_1 is moved through a distance $\frac{\lambda}{2}$, the path difference changes by λ. The position of a particular bright fringe across the cross wire is replaced by the next bright fringe.

For one fringe shift, the distance moved by the mirror M_1 is $\frac{\lambda}{2}$ (half of the wavelength).

Let mirror M_1 move through distance d and the number of fringes that crosses the field of view be n.

For the shift of n fringes,

$$\text{distance moved by mirror } M_1 = d = \frac{n\lambda}{2}$$

$$\text{or} \quad \boxed{\lambda = \frac{2d}{n}}$$

Knowing d and n, the wavelength of monochromatic light λ can be calculated.

Determination of Thickness of a Thin Transparent Sheet

For this purpose, a thin film of refractive index μ is introduced in the path of one of the interfering beams (Fig. 4.19).

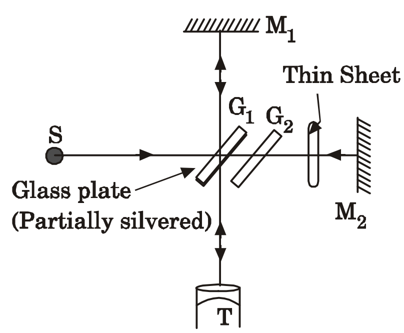


Fig. 4.19

If the thickness is t, then the path of the beam is increased by (μ – 1). Therefore, the path difference between the beams becomes 2 (μ – 1) t. Suppose due to this path difference, n fringes move across the field of view.

Then, $$2(\mu - 1)\,t = n\lambda$$

or $$\boxed{t = \frac{n\lambda}{2(\mu - 1)}}$$

If μ, n and λ are known, thickness t can be calculated.

ANNA UNIVERSITY SOLVED PROBLEMS

Problem 4.4

By moving one of the mirrors in a Michelson interferometer through a distance of 0.1474 mm, 500 fringes cross the centre of the field of view. What is the wavelength of light? ***(A.U. Jan 2015)***

Given data:

Number of fringes that cross the centre of the field of view, $n = 500$

Distance moved by mirror M_1,

$$d = 0.1474 \text{ mm} = 0.1474 \times 10^{-3} \text{ m}$$

Solution:

$$\text{We know that } d = \frac{n\lambda}{2} \quad \text{or} \quad \lambda = \frac{2d}{n}$$

Substituting the given values, we have

$$\lambda = \frac{2 \times 0.1474 \times 10^{-3}}{500}$$

$$= 5896 \times 10^{-10} \text{ m}$$

$$\therefore \quad \lambda = 5896 \text{ Å}$$

Problem 4.5

In a Michelson interferometer if one of the mirrors is moved by 0.04 mm, 125 fringes cross the field of view. Calculate the wavelength of light used. ***[A.U. April 2013]***

Given data

Distance moved by mirror M_1,

$$d = 0.04 \text{ mm} = 0.04 \times 10^{-3} \text{ m}$$

Number of fringes that crossed the field of view,

$$n = 125$$

Solution:

We know that $$\lambda = \frac{2d}{n}$$

Substituting the values, we have

$$\lambda = \frac{2 \times 0.04 \times 10^{-3}}{125}$$

$$= 640 \times 10^{-9} \text{ m}$$

$$\lambda = 640 \text{ nm}$$

Problem 4.6

In a Michelson interferometer, 200 circular fringes crossed the field of view when the movable mirror is displaced through a distance 0.0589 mm. Calculate the wavelength of monochromatic source used. ***[A.U. May 2013]***

Given data:

Number of fringes that crossed the field of view,

$$n = 200$$

Distance moved by the movable mirror,

$$d = 0.0589 \text{ mm} = 0.0589 \times 10^{-3} \text{ metre}$$

Solution:

Wavelength of the monochromatic source,

$$\lambda = \frac{2d}{n}$$

Substituting the given values,

$$\lambda = \frac{2 \times 0.0589 \times 10^{-3}}{200}$$

$$= 0.00589 \times 10^{-5} \text{ mm}$$

$$\therefore \quad \lambda = 589 \times 10^{-9} \text{ m}$$

Problem 4.7

When a thin film of glass of refractive index 1.5 is interposed in the path of one of the interfering beams of the Michelson interferometer, a shift of 30 fringes of sodium light is observed to cross the field of view. If the thickness of the air film is 0.018 mm, calculate the wavelength of the light used. ***(A.U. May 2014)***

Given Data:

Refractive index of the thin film, $\mu = 1.5$

Number of fringes crossed the centre of the field of view, $n = 30$

Thickness of the film,

$$t = 0.018 \text{ mm} = 0.018 \times 10^{-3} \text{ m}$$

Solution:

From the relation $2(\mu - 1)\, t = n\,\lambda$, we have

$$\lambda = \frac{2(\mu - 1)\, t}{n}$$

Substituting the given values, we have

$$\lambda = \frac{2\,(1.5 - 1) \times 0.018 \times 10^{-3}}{30}$$

$$\lambda = 6 \times 10^{-7}\ \text{m}$$

Part - A '2' Marks Question with Answers

1. State laws of reflection.

(i) Incident ray, normal and reflected ray lie in the same plane.

(ii) The angle of incidence is equal to the angle of reflection i.e. $\angle i = \angle r$

2. State laws of refraction.

(i) The incident ray, the refracted ray and the normal at a point of separation of two media lie in the same plane.

(ii) For any two medium, the ratio of sine of angle of incidence to sine of angle of refraction is constant. It is known as **Snell's law**.

$$\text{Therefore, } \frac{\sin i}{\sin r} = \text{constant}$$

3. Define refractive index of the medium.

The ratio of velocity of light in vacuum to velocity of light in medium, is called as refractive index.

$$\text{ie., Refractive index, } \mu = \frac{\text{Velocity of light in vacuum } (c)}{\text{Velocity of light in medium } (v)}$$

$$\boxed{\mu = \frac{c}{v}}$$

4. What is total internal reflection?

When a ray of light within a denser medium (e.g. water) approaches the surface at an angle of incidence greater than the critical angle, the ray of light is reflected back into the same medium (i.e. water). This phenomenon is known as total internal reflection.

5. Define critical angle.

The angle of incident at which the refracted ray just graze surface between denser and rarer media is called critical angle.

6. Give conditions of total internal reflection.

(a) The light should incident from denser medium to rarer medium.

(b) The angle of incidence i in denser medium should be greater than critical angle θ_c.

7. Write expression for critical angle.

$$\theta_c = \sin^{-1}\left(\frac{n_2}{n_1}\right)$$

n_1 – refractive index of denser medium

n_2 – refractive index of rarer medium

8. Mention a few applications of total internal reflection.

(i) **Mirage**. During the day time in the desert, it is seen that sand at some distance from the observer looks like a pond of water. This illusion is called mirage and it is caused due to total internal reflection of light. (Fig 4.6)

(ii) **Optical fibre**. An optical fibre is a transparent fibre used to conduct light through the phenomenon of total internal reflection.

9. What is interference?

This modification or change of intensity of light resulting from the superposition of two or more waves of light is called interference.

10. What is air wedge?

A wedge shaped (V-shaped) air film enclosed in between two glass plates is called air wedge.

11. What is the expression for the fringe width in air wedge experiment?

$$\text{Fringe width } \beta = \frac{\lambda}{2\theta}$$

λ – wavelenth of the light source

θ – Angle of wedge

12. What is the expression for the thickness of the wire in air wedge experiment?

$$d = \frac{\lambda l}{2\beta}$$

λ – wavelenth of the light source

l – Distance from the edge of contact

β – Fringe width

13. What is Michelson interferometer?

An interferometer is an instrument for measuring small changes in length. It is based on the principle of interference.

Michelson originally designed an interferometer which is used to find the wavelength of monochromatic light source and thickness of thin strips.

14. What are the applications of Michelson interferometer?

It is used to find

(i) the wavelength of a given light source.

(ii) the refractive index and thickness of a transparent material

(iii) the resolution of wavelengths

(iv) the standardisation of metre

Part - B '16' Marks

Anna University Questions:

1. Explain the formation of interference fringes in an air-wedge shaped film. How is the thickness of the wire determined by this method?

2. Describe Michelson interferometer and explain how the fringes form in it. How can this be used for measuring the wavelength of monochromatic light. Derive the formula.

   (A.U. May 2013, 2018)

3. How will you use Michelson's interferometer to determine the thickness of a thin transparent film or plate. ***(A.U. Jan 2015)***

4. Explain the construction, types of fringes and applications of Michelson interferometer. ***(A.U. May 2016)***

5. (i) Describe the construction of a Michelson's interferometer and discuss the different types of interference fringes formed in it.

   (ii) How will you use it to determine the wavelength of a monochromatic source? ***(A.U. Dec 2012)***

Assignment Problem

1. Monochromatic light emitted by a broad source of wavelength 580 nm, falls normally on two plates of glass enclosing a wedge shaped film. The plates touch at one end and are separated at a point 15 cm from that end by a wire of 0.05 mm diameter. Find the fringe width.

   (Ans. $\beta = 8.7 \times 10^{-4}$ m)

2. Two pieces of plane glass are placed together with a piece of paper between the two at one edge. Find the angle in seconds of the wedge shape air film between the plates, if on viewing the film normally with monochromatic light of wavelength 480 nm, $\beta = 0.0555$ cm

   (Ans. $= 432 \times 10^{-6}$ radian $= 1'29''$)

3. Fringes of equal thickness are observed in a thin glass wedge of refractive index 1.52. The fringe spacing is 1 mm and wavelength of light is 589.3 nm. Calculate the angle of wedge in seconds of arc. ***(Ans. = 40 seconds of arc)***

4. In a Michelson's interferometer 200 fringes cross the field of view when the movable mirror is displaced through 0.0589 mm. Calculate the wavelength of monochromatic light is used. **(Ans. $= 589 \times 10^{-9}$ m = 589 nm)**

5. The movable mirror of Michelson's interferometer is moved through a distance of 0.02603 mm. Find the number of fringes shifted across the cross-wire of eyepiece of the telescope, if a wavelength of 520.6 nm is used. **(Ans. $N = 100$)**

6. When a thin film of glass of refractive index 1.5 is interposed in the path of one of the interfering beams of the Michelson's interferometer, a shift of 30 fringes of sodium light is observed across the field of view. If the thickness of the air film is 0.018 mm, calculate the wavelength of the light used.

   [**Hint.** $2(n - 1)\, t = m\, \lambda$] **(Ans. 600×10^{-9} m)**

7. A transparent film of glass of refractive index 1.5 is introduced normally in the path of he interfering beams of Michelson's interferometer which is illuminated with light of wavelength 480 nm. This causes 500 dark fringes to sweep across the field. Determine the thickness of the film. **(Ans. 0.00024 m)**

5. Lasers

Theory of laser – characteristics – Spontaneous and stimulated emission – Einstein's coefficients – population inversion – Nd-YAG laser CO_2 laser – Semiconductor laser – Basic applications of lasers in industry.

Introduction

LASER stands for **L**ight **A**mplification by **S**timulated **E**mission of **R**adiation.

Laser is a device which emits a powerful, monochromatic, collimated beam of light. The emitted light waves from laser source are coherent in nature.

Two light waves are said to coherent, when they have the same phase or constant phase difference.

Laser is an artificial light source which exhibits so many superior features than the conventional light source.

Laser light emerges as a narrow beam which can travel long distance without much loss of intensity and energy. Actually, the laser amplifies the light waves.

The first laser, ruby laser was invented by **Dr. T.H. Maiman** in the year 1960.

Since then, the development of lasers is extremely rapid with laser action is being demonstrated in many solids, liquids, gases and semiconductors.

The discovery of laser made an enormous impact on scientific and engineering applications particularly in the fields of communication, metrology, biology, medicine and computers.

5.1 THEORY OF LASER

Interaction of light radiation with materials

Consider an assembly of atoms in a material which is exposed to light radiation (a stream of photons with energy $h\nu$).

In general, three different processes occur when light radiation interacts with a material. They are

1. **Stimulated absorption**
2. **Spontaneous emission**
3. **Stimulated emission**

Process - 1 **Stimulated absorption**

An atom in ground state with energy E_1 absorbs an incident photon of energy $h\nu$ and is excited to higher energy state with energy E_2 (Fig. 5.1).

This process is known as stimulated or induced absorption.

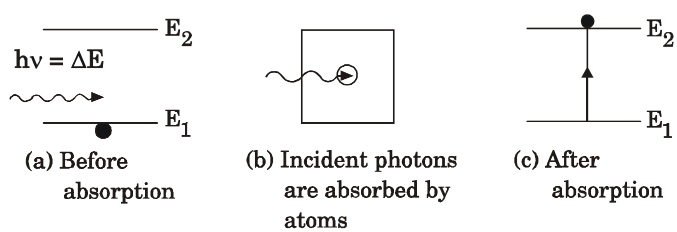


Fig. 5.1 Stimulated absorption

It occurs only when the incident photon energy $h\nu$ is equal to the energy difference between excited state and ground state $(E_2 - E_1)$.

For each such a transition, a certain amount of energy $(h\nu)$ is absorbed from the incident light beam.

The excited atoms do not stay in the higher energy state for a longer time. It is the tendency of atoms in excited state to come to the lower energy state.

Thus, the atoms in excited state quickly return to ground state by emitting a photon of energy $h\nu$.

The emission of photons takes place in two ways, namely

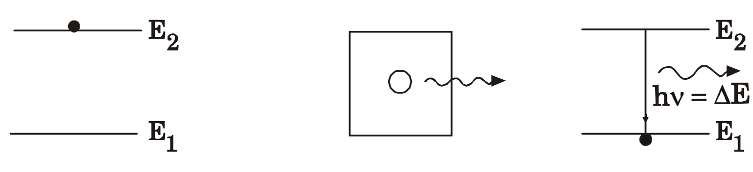
(a) spontaneous emission

(b) stimulated emission

5.2 SPONTANEOUS AND STIMULATED EMISSIONS

Process - 2 Spontaneous emission

The atom in the excited state E_2 (higher energy state) returns to ground state E_1 (lower energy state) by emitting a photon of energy $h\nu$ ($\Delta E = E_2 - E_1$) without the influence of any external agency (Fig. 5.2).



(a) Before emission (b) Random emission of photons (c) After emission

Fig. 5.2 Spontaneous emission

Such emission of light radiation which is not triggered by any external influence is called spontaneous emission.

It is a random and also uncontrollable process.

Process - 3 **Stimulated emission**

Einstein suggested that there must be another mechanism by which an atom in excited state can return to ground state.

He found that there is an interaction between the atom in excited state and a photon. During this interaction, the photon triggers the excited atom to make transition to ground state E_1 (Fig. 5.3).

This transition produces a second photon which is similar to triggering photon with respect to frequency, phase and propagation direction.

Such kind of forced emission of photons by the incident photons is called *stimulated emission*. It is also known as induced emission. It plays a key factor for the working of a laser.

Table 5.1

Differences between spontaneous and stimulated emission

	Spontaneous emission	**Stimulated emission**
1.	Emission of light radiation is spontaneous ie., without any external agency.	Emission of light radiation is stimulated (triggered) by incident photons.
2.	Emitted photons travel randomly in all directions.	Emitted photons travel in particular direction.
3.	Emitted photons cannot be controlled. They are not coherant	Emitted photons can be controlled and they are coherent.
4.	This process is a key factor for ordinary light emission.	This process is a key factor for laser operation.

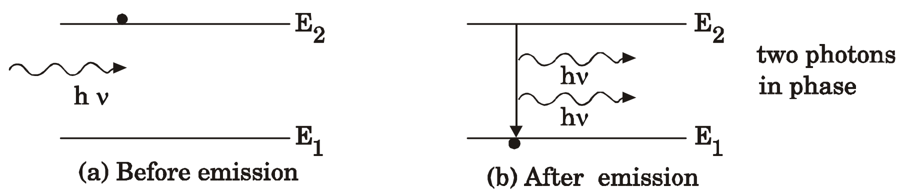


Fig. 5.3 Stimulated emission

5.3 EINSTEIN'S A and B COEFFICIENTS (Derivation)

Consider an assembly of atoms with different energy states at an absolute temperature T.

When light radiation is incident on these atoms, three different processes take place. They are

(a) stimulated absorption

(b) spontaneous emission

(c) stimulated emission

Stimulated absorption

The atom in the lower energy state E_1 absorbs radiation and is excited to the higher energy level E_2. This process is called **stimulated** or **induced absorption** (Fig. 5.4).

The rate of stimulated absorption is directly proportional to number of atoms (N_1) in energy state E_1 and the energy density (Q) of incident radiation (number of photons incident per unit area per unit time).

$$N_{ab} \propto N_1 Q$$

Therefore, the number of stimulated absorption transitions occuring per unit time is given by

$$\boxed{N_{ab} = B_{12} N_1 Q} \quad \text{... (1)}$$

where B_{12} is a proportionality constant.

This process is an **upward transition**. (Fig 5.4)

The atoms in excited state return to lower energy state E_1 (ground state) by emitting a photon of energy $h\nu$ in two ways.

(a) **Spontaneous emission**

(b) **Stimulated emission**

Spontaneous emission

The atoms in the excited state E_2 return to lower energy state E_1 by emitting a photon of energy *energy* $h\nu$ without the influence of any external agency. This emission of light radiation is known **spontaneous emission**. (Fig. 5.4)

The rate of spontaneous emission is directly proportional to the number of atoms in the excited energy state (N_2).

$$N_{sp} \propto N_2$$

Hence, the number of transitions per second is given by

$$\boxed{N_{sp} = A_{21} N_2} \quad \text{... (2)}$$

Here, A_{21} is a proportionality constant.

This process is a **downward transition**. (Fig. 5.4)

Stimulated emission

If the light photon is incident on the atom in the excited energy state, the photon triggers the excited atom to make transition to lower energy energy E_1 along with emission of photons. This kind of emission of light radiation is stimulated emission. (Fig. 5.4)

The rate of transition is directly proportional to the number of atoms in the upper energy level (N_2) and the energy density of incident radiation (Q).

$$N_{st} \propto N_2Q$$

The number of transitions per second.

$$\boxed{N_{st} = B_{21}N_2Q} \quad \text{... (3)}$$

where B_{21} is a proportionality constant.
This process is also **downward transition**.

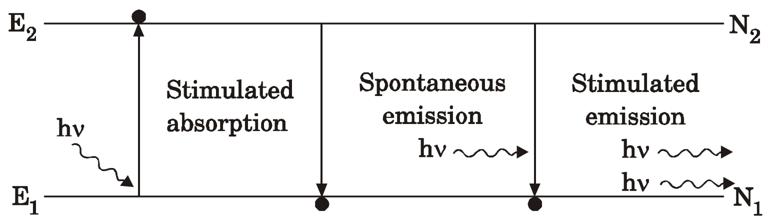


Fig. 5.4

The proportionality constants $\mathbf{A_{12}, B_{12}}$ **and** $\mathbf{B_{21}}$ are known as **Einstein's coefficients A and B.**

Under equilibrium condition, the number of downward and upward transitions per second are equal.

i.e., $$N_{sp} + N_{st} = N_{ab} \quad \text{... (4)}$$

Substituting from the eqns (1), (2) and (3), in eqn (4), we have

$$\boxed{A_{21} N_2 + B_{21} N_2 Q = B_{12} N_1 Q} \quad \text{... (5)}$$

Rearranging the eqn (5), we have

$$B_{12} N_1 Q - B_{21} N_2 Q = A_{21} N_2$$

$$Q (B_{12} N_1 - B_{21} N_2) = A_{21} N_2$$

$$Q = \frac{A_{21} N_2}{B_{12} N_1 - B_{21} N_2} \quad \text{... (6)}$$

Dividing numerator and denominator by $B_{21} N_2$, we have

$$Q = \frac{\dfrac{A_{21}\,N_2}{B_{21}N_2}}{\dfrac{B_{12}\,N_1}{B_{21}\,N_2} - \dfrac{B_{21}\,N_2}{B_{21}\,N_2}}$$

$$\boxed{Q = \frac{A_{21}}{B_{21}} \frac{1}{\left(\dfrac{B_{12}}{B_{21}}\right)\dfrac{N_1}{N_2} - 1}} \qquad \ldots (7)$$

On substituting $\dfrac{N_1}{N_2} = e^{h\nu/kT}$ (from Boltzmann distribution equation) in eqn (7), we have

$$\boxed{Q = \frac{A_{21}}{B_{21}} \frac{1}{\left(\dfrac{B_{12}}{B_{21}}\right)e^{h\nu / kT} - 1}} \qquad \ldots (8)$$

Planck's radiation formula for energy distribution is given by

$$\boxed{Q = \frac{8\,\pi h\nu^3}{c^3} \frac{1}{e^{h\nu / kT} - 1}} \qquad \ldots (9)$$

Comparing the eqns (8) and (9), we have

$$\frac{B_{12}}{B_{21}} = 1$$

$$\boxed{B_{12} = B_{21}} \qquad \ldots (10)$$

and $$\boxed{\frac{A_{21}}{B_{21}} = \frac{8\,\pi h\nu^3}{c^3}} \text{ or } = \frac{8\pi\, h}{\lambda^3} \qquad \ldots (11)$$

Since $B_{12} = B_{21}$, Einstein's coefficients are termed as *A* and *B* coefficients.

Conclusion

- The spontaneous emission is more predominant than the stimulated emission. The laser light is due to stimulated emission. Therefore, stimulated emission should be greater than spontaneous emission. To achive this, population inversion is required.
- The equation (11) gives the relation between spontaneous emission and stimulated emission coefficients. Since this ratio is proportional to ν^3, the probability of spontaneous emission increases with the energy difference between the two states.

If N_0 is the number of atoms in ground state, then, the number of atoms in the excited energy state E is given by **Maxwell - Boltzmann's distribution law**,

$$N = N_0 \, e^{(-E/kT)}$$

where k - Boltzmann's constant.

If N_1 and N_2 are the number of atoms in the states of energies E_1 and E_2, then from Maxwell - Boltzmann's law, we have

$$\frac{N_1}{N_2} = \frac{N_0 \, e^{(-E_1/kT)}}{N_0 \, e^{(-E_2/kT)}} = e^{(E_2 - E_1)/kT}$$

$$\frac{N_1}{N_2} = e^{h\nu/kT}$$

$$[\because E_2 - E_1 = h\nu]$$

Note: The students are not expected to write the informations given in the box in the examination.

Concept of Laser

The photon emitted during stimulated emission has the same energy, phase, frequency and direction as that of the incident photon.

Thus, we have two coherent photons. Now, these two photons incident on two other atoms in the state E_2. This results in induced emission of two more photons.

Now, there are four coherent photons of same energy. These four photons induce further transitions with four other atoms in the energy state E_2. This gives stimulated emission of eight coherent photons of same energy (Fig. 5.5).

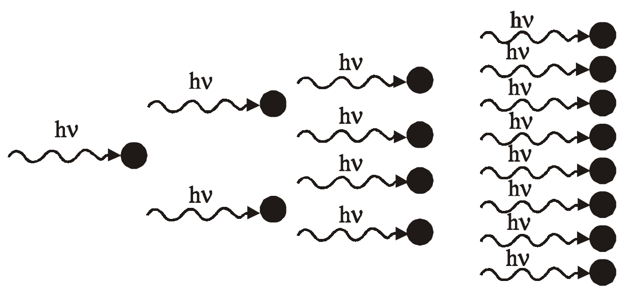


Fig. 5.5. Amplification due to stimulated emission of radiation

If the process continues in a chain, ultimately, this gives rise to increase in intensity of coherent radiation enormously.

Stimulated emission is multiplied through a chain reaction. This multiplication of photons through stimulated emission leads to coherent, powerful, monochromatic, collimated beam of light. This light is known as laser light.

Thus for laser action, stimulated emission is most important. It is achieved by population inversion.

5.4 POPULATION INVERSION

It is a situation in which the number of atoms in higher energy state is more than that in lower energy state.

Usually at thermal equilibrium, the number of atoms N_2 (population of atoms) at higher energy state is much less than the number of atoms (population of atoms) at lower energy state N_1. i.e., $N_1 > N_2$. (Fig. 5.6)

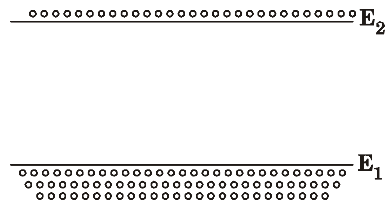


Fig. 5.6 Normal condition

The state of achieving more number of atoms in higher energy state than the that of lower energy state (i.e., $N_2 > N_1$) is known as population inversion (Fig. 5.7)**.**

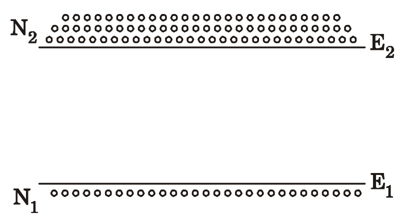


Fig. 5.7 After population inversion

Conditions for population inversion

- There must be atleast two energy levels (E_1 and E_2).
- There must be a source to supply the energy to the medium.
- The atoms must be continuously raised to the excited state.

Active Medium or Material

A medium in which population inversion above threshold inversion density is achieved is known as active medium. It is also called active material.

The inversion density which is just enough to compensate for the losses in the medium is called thershold inversion density.

Pumping action

The process to achieve population inversion in the medium is called pumping action.

It is an essential requirement for producing a laser beam.

Methods for pumping action

The methods commonly used for pumping action are

(i) ***Optical pumping (excitation by photons)***

(ii) ***Electrical discharge (excitation by electrons)***

(iii) ***Direct conversion***

(iv) ***Inelastic collision between atoms.***

(i) Optical pumping

When the atoms are exposed to light radiation of energy $h\nu$, atoms in the lower energy state absorb these radiation and go to an excited state (Fig. 5.8). **This is known as optical pumping**.

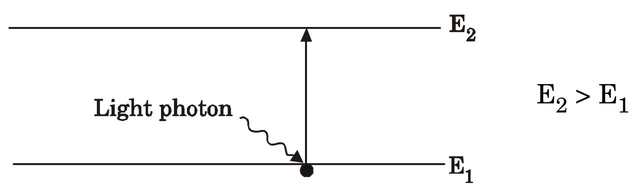


Fig. 5.8 Optical pumping

This type of pumping is used in solid state lasers like ruby and Nd-YAG lasers.

(ii) Electrical discharge (excitation by electrons)

In this method, the electrons are produced in an electrical discharge tube. These electrons are accelerated to high velocities by a strong electrical field. Now, accelerated electrons collide with the gas atoms.

During the collision, the energy of the electrons is transferred to gas atoms. Thereby atoms gain energy and go to excited state (Fig. 5.9). This results in population inversion. **This is known as electrical discharge.**

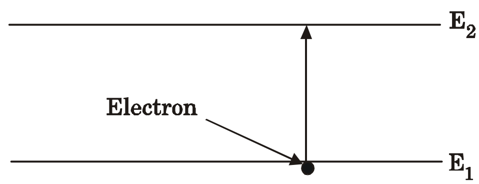


Fig. 5.9 Electrical discharge method

The energy transfer is represented by the equation

$$A + e^{*} \rightarrow A^{*} + e$$

where A – Gas atom (or molecule) in ground state

A^{*} – Same gas atom in excited energy state

e^{*} – Electron with more kinetic energy

e – Same electron with less energy.

This method of pumping is used in gas lasers like argon and CO_2 lasers.

(iii) Direct conversion

In this method, the electrical energy is applied to a direct band gap semiconductor like GaAs. The recombination of electrons and holes takes place. During the recombination process, the electrical energy is directly converted into light energy (Fig. 5.10).

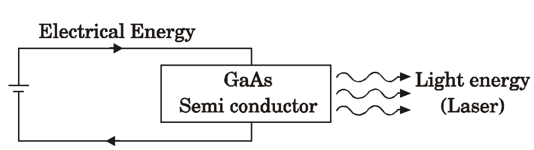


Fig. 5.10 Direct conversion

This method of pumping is used in semiconductor diode laser (e.g. GaAs laser).

(iv) Inelastic collision between atoms

In this method, a combination of two gases (say A and B) is used. The excited energy levels of gases of A and B nearly coincides each other. (Fig. 5.11).

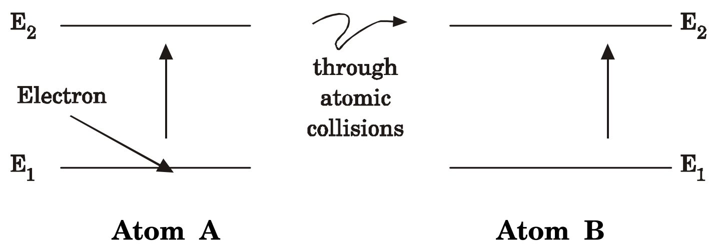


Fig. 5.11 Inelastic atom–atom collision

During the electrical discharge, atoms of gas A are excited to higher energy states A^* due to collision with the electrons.

$$A + e^* \rightarrow A^* + e$$

e^* – Electron with more kinetic energy

e – Same electron with less energy.

Now A^* atoms at higher energy state collide with B atoms in lower energy states. Due to this inelastic collision, B atoms gain energy and excited to higher state B^*. Hence, A atoms lose energy and return to lower state.

$$A^* + B \rightarrow A + B^*$$

Thus, population inversion in the energy states of B is achieved. This method is used in He - Ne laser.

5.5 CHARACTERISTICS OF LASER LIGHT

Laser is basically a light source. Laser light has the following important characteristics

(i) High directionality

(ii) High intensity

(iii) Highly monochromatic

(iv) Highly coherent

(i) High directionality

An ordinary light source emits the light in all directions. But, a laser source emits light in only one direction. The divergence of laser beam is very small (Fig. 5.12). So, laser light has high directionality.

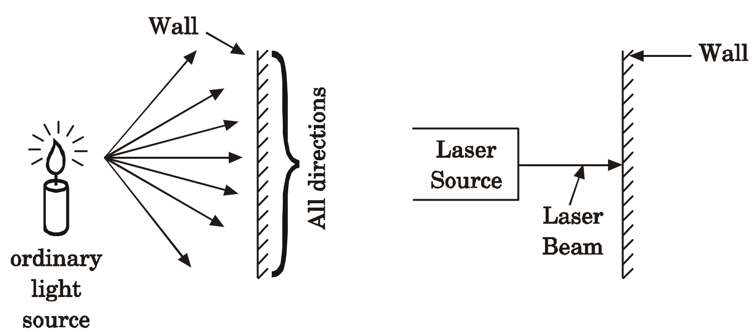


Fig. 5.12. High directionality

(ii) High Intensity

Laser source emits light as a narrow beam and its energy is concentrated in a small region (spot). This concentration of energy gives a high intensity to the laser light (Fig. 5.13).

(iii) Highly monochromatic

Ordinary light spreads over a wavelength range of the order of 100 nm.

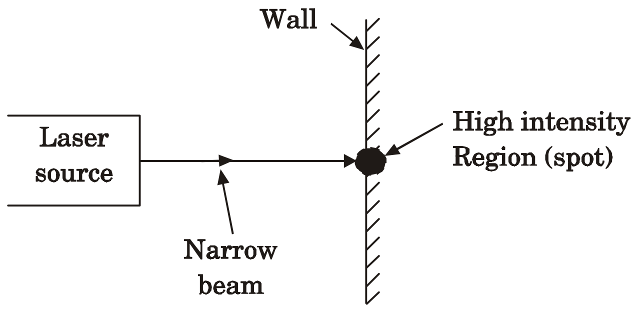


Fig. 5.13. High intensity

But a laser beam has very little spreading of the order of 1 nm. (Fig. 5.14). Thus, laser beam is highly monochromatic. ie., it emits only one colour of light.

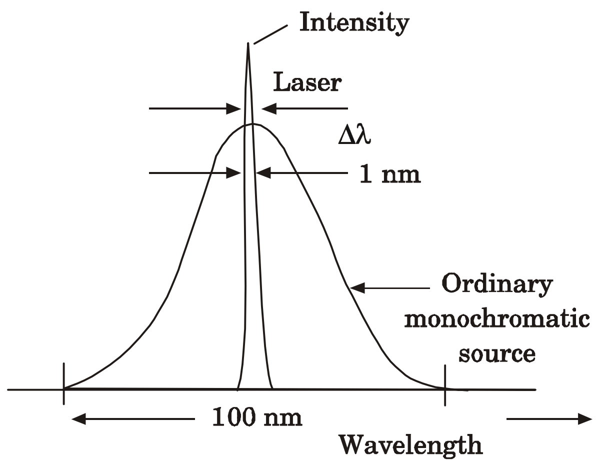


Fig. 5.14 Spectral width of laser

(iv) Highly Coherent

The light emitted from a laser source consists of wave trains. These wave trains have same frquency, phase and direction. So, they are coherent (Fig. 5.15).

Laser light has a high degree of coherence. The coherence of laser emission results in extremely high intensity and hence more power.

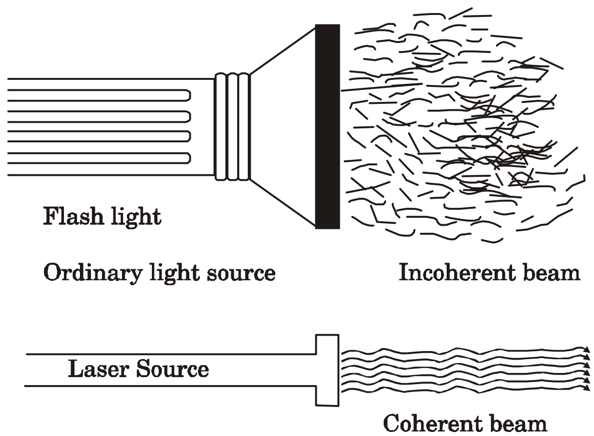


Fig. 5.15. Coherence

These important properties make the laser light superior to other conventional light sources such as flame, Sunlight, ordinary electric bulbs, CFL, etc.

Table 5.2
Differences between ordinary light and Laser light

S.No.	Ordinary Light	Laser Light
1.	Light emitted is not monochromatic.	Light emitted is highly monochromatic.
2.	Light emitted does not have high degree of coherence	Light emitted has high degree of coherence.
3.	Emitted light spreads in all directions (not directional)	Emitted light spreads only in one direction (directional)
4.	Light is less intense and bright	Laser light is more intense and bright.

Basic components of a laser system

A laser system consists of three important components. They are

(a) Active medium or material

(b) Pumping source

(c) Optical resonator

(a) Active medium or material

It is a medium in which atomic transitions take place to produce laser action. The active medium may be a solid, liquid, gas, dye or semiconductor.

(b) Pumping system

It is a system used to produce population inversion in the active medium.

(c) Optical resonator

An optical resonator consists of a pair of reflecting surfaces in which one is fully reflecting (R_1) and the other is partially reflecting (R_2). (Fig. 5.16). The active medium is placed in between these two reflecting surfaces.

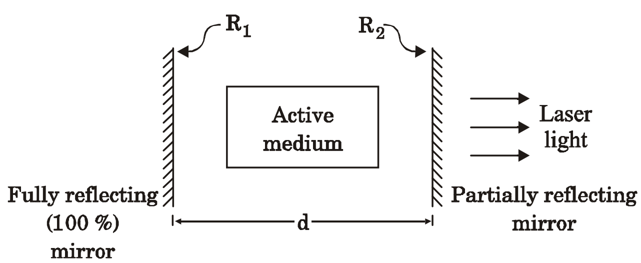


Fig. 5.16 Optical resonator

The photons generated due to stimulated emission are bounced back and forth between these two reflecting surfaces.

This induces more and more stimulated transition leading to laser action.

Types of lasers

Based on the type of active medium, the laser systems are broadly classified into the following types.

Sl.No.	Type of Laser	Examples
1.	Solid state laser	Ruby, Nd:YAG lasers
2.	Gas laser	He - Ne, CO_2, Argon lasers
3.	Liquid laser	$SeOCl_2$, Europium chelate lasers
4.	Dye laser	Rhodamine 6G , Coumarin dye lasers
5.	Semiconductor laser	GaAs, GaAsP, GaAlAs, InP lasers.

5.6 Nd - YAG LASER

Nd - YAG laser is **Neodymium based laser**. Nd stands for Neodymium (rare earth element) and YAG for **Y**ttrium **A**luminium **G**arnet ($Y_3 Al_5O_{12}$).

It is a four - level solid state laser.

Principle

The active medium Nd-YAG rod is optically pumped by krypton flash tube. The neodymium ions (Nd^{3+}) are raised to excited energy levels. During transition from metastable state to ground state, a laser beam of wavelength 1.064 μm is emitted.

Construction

The construction of Nd-YAG laser is shown in fig. 5.17.

A small amount of yttrium ions (Y^{3+}) is replaced with neodymium ions (Nd^{3+}) in the active medium of Nd-YAG rod.

The active medium Nd-YAG crystal is cut into a cylindrical rod. The ends of this rod are highly polished and optically flat and parallel. The cylindrical rod (laser rod) and a pumping source (krypton flash tube) are kept in an elliptical reflector cavity in order to focus most of the light into Nd YAG rod.

The optical resonator is formed by using two external reflecting mirrors. One mirror M_1 is fully reflecting while the other mirror M_2 is partially reflecting.

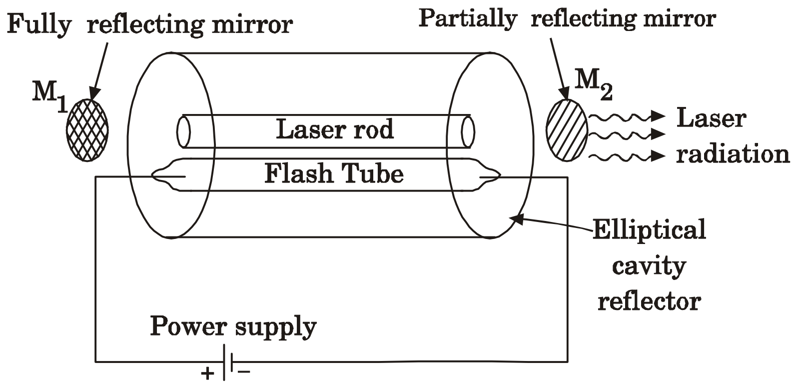


Fig. 5.17 Nd - YAG Laser

Working

The energy level of Nd^{3+} ion in Nd - YAG laser is shown in fig 5.18.

- When krypton flash tube is switched on, the neodymium ions are excited from ground state E_o to upper energy levels E_3 and E_4 (pump bands) due to absorption of light radiation of wavelengths 0.73 μm and 0.80 μm.
- The neodymium ions from these excited energy levels make a transition to energy level E_2 by non-radiative transition. E_2 is a metastable state.
- Now, the neodymium ions are collected in this energy level E_2. Thus, the population inversion is achieved between E_2 and E_1.
- A neodymium ion makes a spontaneous transition from E_2 to E_1 by emitting a photon of energy $h\nu$. This emitted photon triggers a chain of stimulated photons between E_2 and E_1.

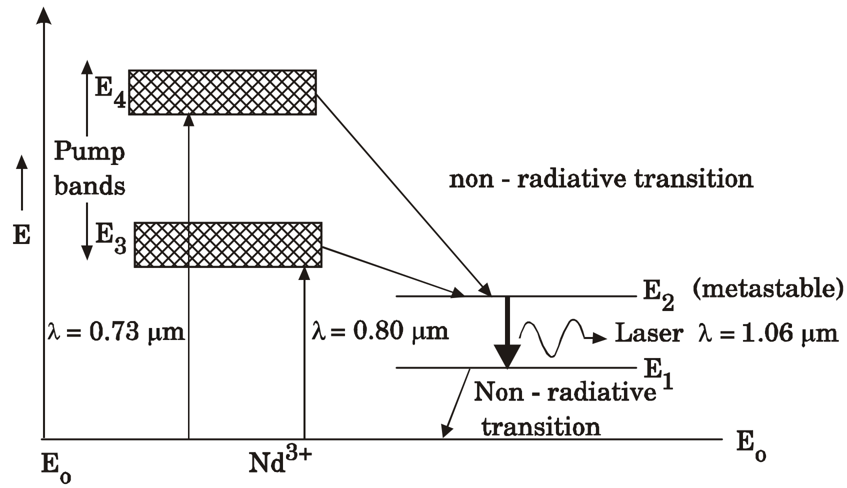


Fig. 5.18 Energy level of Nd^{3+} in Nd - YAG laser

- The photons thus generated travel back and forth between two mirrors and grow in strength due to induced emission. Hence, the photon number multiplies more rapidly.
- After enough strength is attained, an intense laser light of wavelength 1.06 μm (10600 Å) is emitted and flow through the partially reflecting mirror M_2. This corresponds to transition of ions from $\mathbf{E_2}$ **to** $\mathbf{E_1}$.

Characteristics

- **Type:** It is a four-level solid state laser. (E_1, E_2, E_3 and E_4)
- **Active medium:** Nd-YAG rod.
- **Pumping method:** Optical pumping.
- **Pumping source:** Krypton flash tube.
- **Optical resonator:** Two ends of Nd-YAG rod polished with silver (one end is fully silvered and other is partially silvered).
- **Power output:** 20 kW.

- **Nature of output:** Pulsed or continuous beam of light.
- **Wavelength of output:** 1.06 μm (infra-red).

Advantages

- This laser has high energy output.
- It is much easier to achieve population inversion.

Disadvantages

The electron energy level structure of Nd^{3+} in Nd - YAG is complicated.

Applications

- Nd-YAG laser is used in range finders and illuminators.
- It is widely used in resistor trimming, scribing, micro-machining operations such as welding, drilling etc.
- It finds many medical applications such as endoscopy, urology, neurosurgery.

Molecular Gas Laser

In a molecular gas laser, laser action takes place by transitions between vibrational and rotational energy levels of gas molecules.

5.7 CARBON DIOXIDE (CO_2) LASER

The first molecular CO_2 gas laser was developed by **Indian born American Scientist Prof. C.K.N. Patel**.

It is a four-level molecular gas laser. In this laser, transition takes place between vibrational energy states of carbon dioxide molecules. It is a very efficient laser.

Energy states of CO_2 molecules

A carbon dioxide molecule has a carbon atom at the centre with two oxygen atoms attached, one at each sides. Such a molecule vibrates in **three independent modes**. They are

(a) Symmetric stretching mode

(b) Bending mode

(c) Asymmetric stretching mode

(a) Symmetric stretching

In this mode of vibration, carbon atom is at rest. Both oxygen atoms vibrate such that they are moving away or approaching the fixed carbon simultaneously along the axis of the molecule (Fig. 5.19).

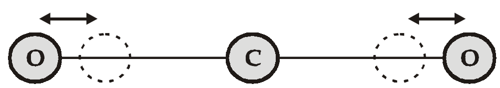


Fig. 5.19 Symmetric stretching mode

(b) Bending

In this mode of vibration, both oxygen atoms and carbon atom vibrate perpendicular to molecular axis (Fig. 5.20).

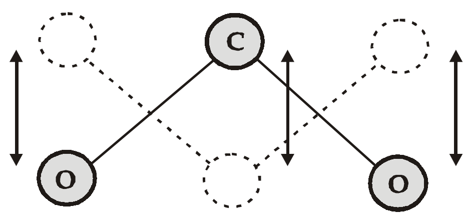


Fig 5.20 Bending mode

(c) Asymmetric stretching

In this mode of vibration, both oxygen atoms and carbon atom vibrate asymmetrically, i.e., oxygen atoms move in one direction while carbon atom moves in the opposite direction (Fig. 5.21).

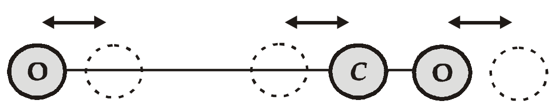


Fig. 5.21 Asymmetric stretching mode

Principle

The laser transition takes place between the vibrational energy states of CO_2 molecules.

Construction

It consists of a quartz discharge tube 5 m long and 2.5 cm in diameter (Fig. 5.22). This discharge tube is filled with the gas mixture of CO_2, nitrogen and helium with suitable partial pressures.

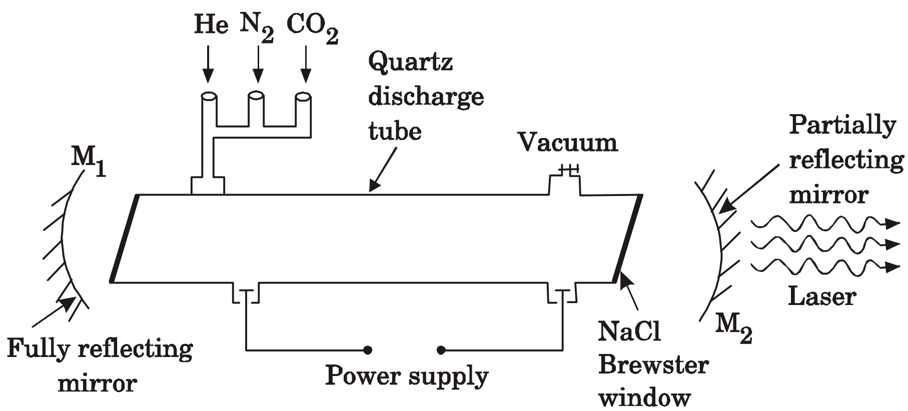


Fig. 5.22 CO_2 laser

The terminals of the discharge tube are connected to D.C. power supply. The ends of this tube are fitted with NaCl Brewster windows so that the laser light generated is plane polarised.

The optical resonator is formed with two concave mirrors one fully reflecting M_1 with the other partially reflecting M_2.

Working

The energy level diagram of nitrogen and carbon dioxide molecules is shown in fig 5.23.

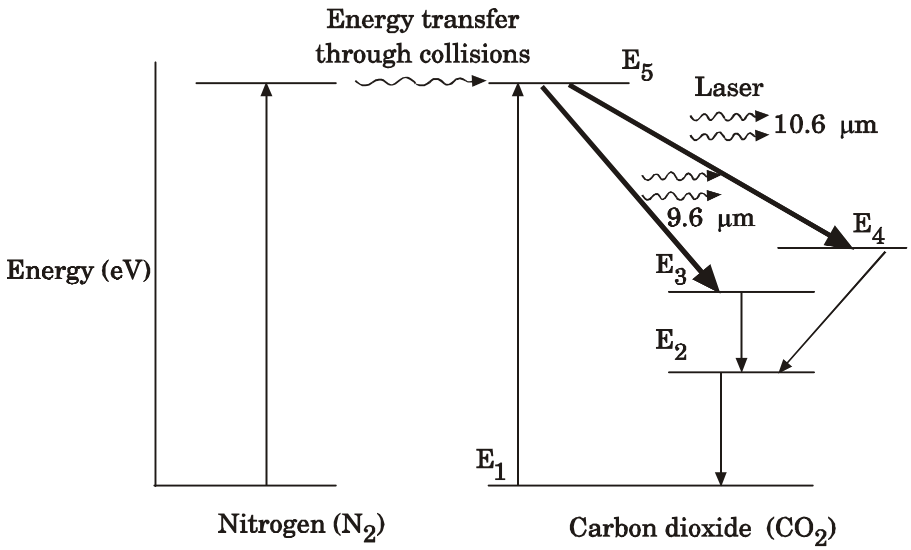


Fig. 5.23. Energy levels of nitrogen and carbon dioxide molecules and transitions between these energy levels

When the electrical discharge occurs in gas mixture, the electrons collide with nitrogen molecules and they are raised to excited energy states.

This process is represented by the equation

$$N_2 + e^* \rightarrow N_2^* + e$$

$N_2 \rightarrow$ Nitrogen molecule in ground state

$e^* \rightarrow$ Electron with high energy

$N_2^* \rightarrow$ Nitrogen molecule in excited state

$e \rightarrow$ Same electron with lesser energy

Since excited energy level of nitrogen is very close to E_5 energy level of CO_2 molecules, CO_2 molecules are excited by energy transfer and population inversion is achieved.

This process is represented by the equation

$$\boxed{N_2^* + CO_2 \rightarrow CO_2^* + N_2}$$

$N_2^* \rightarrow$ Nitrogen molecule in excited state

$CO_2 \rightarrow$ Carbon dioxide molecule in ground state

$CO_2^* \rightarrow$ Carbon dioxide molecule in excited state

$N_2 \rightarrow$ Nitrogen molecule in ground state

Any of the spontaneously emitted photon triggers laser action in the tube.

There are two possible types of laser transition.

(i) Transition $E_5 - E_4$

This transition produces a laser beam of wavelength 10.6 μm.

(ii) Transition $E_5 - E_3$

This transition produces a laser beam of wavelength 9.6 μm.

Normally 10.6 μm transition is more intense than 9.6 μm transition. The power output from this laser is 10 kW.

Note: The helium gas is used to conduct heat generated in the central region of the discharge tube to the walls of the discharge tube.

Characteristics

- **Type :** Molecular gas and four level laser.
- **Active medium :** A gas mixture of CO_2, N_2 and helium.
- **Pumping method :** Electrical discharge method.
- **Optical resonator :** It is formed with two concave mirrors.
- **Power output :** 10 kW.
- **Nature of output :** Continuous wave or pulsed wave.
- **Wavelength of output :** 9.6 μm and 10.6 μm (96000 Å and 106000 Å infra red).

Advantages

- The construction of CO_2 laser is simple.
- The output from this laser is continuous.
- It has high efficiency.
- It has very high output power.
- The output power can be increased by increasing the length of discharge tube.

Disadvantages

- The contamination of oxygen by carbon monoxide has some effect on laser action.
- The operating temperature plays an important role in determining the output power of the laser.
- The corrosion may occur at the surfaces of the discharge tube.
- Due to high power laser light, accidental exposure may damage eyes, since it is invisible (infra red region) to our eyes.

Applications

- High-power CO_2 laser finds application in material processing, welding, drilling, cutting, soldering, etc.
- The low atmospheric attenuation (10.6 μm) makes CO_2 laser suitable for open air communication.
- It is used in remote sensing.
- It is used in the treatment of liver and lung diseases.
- It is mostly used in neurosurgery and general surgery.
- It is used to perform microsurgery and bloodless operations.

Solid-state Diode lasers (Semiconductor diode laser)

Laser light can also be produced in semiconductors. The most compact of all lasers is **semiconductor diode laser**. It is also called **injection laser**.

Types of semiconductor diode lasers

Broadly there are two types of semiconductor diode lasers. They are

1. ***Homojunction semiconductor diode lasers***
2. ***Heterojunction semiconductor diode lasers***.

Homojunction Semiconductor Laser

A diode laser which makes use of the same type of semiconductor material on both sides of the junction is known as a ***homojunction diode laser***.

Example: Gallium Arsenide (GaAs) laser

Heterojunction Semiconductor Laser

A diode laser which makes use of different semiconductor materials on the two sides of the junction is known as ***hetrojunction diode laser***.

Example: A junction laser having GaAs on one side and GaAlAs on the other side.

Direct band gap semiconductor

In this type of semiconductor during the recombination of hole and electron, a photon of light is released.

Example: Ga As

It is used in the fabrication of light emitting diodes (LEDs) and lasers.

Indirect band gap semiconductor.

During the recombination of hole and electron, heat energy is released.

Example: Germanium and silicon.

5.8 SEMICONDUCTOR LASER (Homo - junction)

Definition

It is a specially fabricated p - n junction diode.

This diode emits laser light when it is forward - biased.

Principle

When the p-n junction diode is forward-biased (Fig. 5.24(a)), the electrons from n-region and holes from p-region cross the junction and recombine with each other.

During the recombination process, the light radiation (photons) is released from a certain specified direct band gap semiconductors like Ga-As. This light radiation is known as **recombination radiation** (Fig. 5.24(b)).

The photon emitted during recombination stimulates other electrons and holes to recombine. As a result, stimulated emission takes place and laser light is produced.

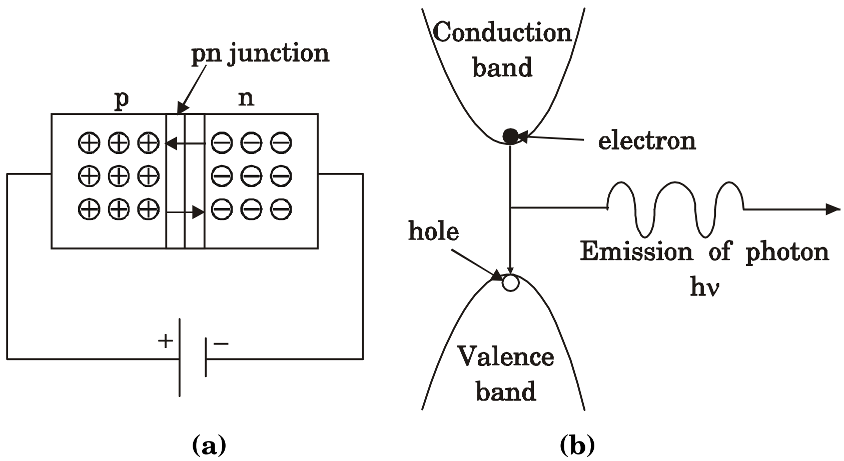


Fig. 5.24 Semiconductor diode laser principle

Construction

The construction of homo-junction semiconductor laser is shown in fig 5.25. The active medium is a p - n junction diode made from a single crystal of gallium arsenide. This crystal is cut in the form of a platelet having a thickness of 0.5 mm.

This platelet consists of two regions *n*− type and *p*−type.

The metal electrodes are connected to both upper (p-region) and lower (n-region) surfaces of the semiconductor diode. The forward bias voltage is applied through metal electrodes.

Now the photon emission is stimulated in a very thin layer of *pn* junction.

The end faces of the *pn* junction are well polished and parallel to each other. They act as an optical resonator through which the emitted light comes out.

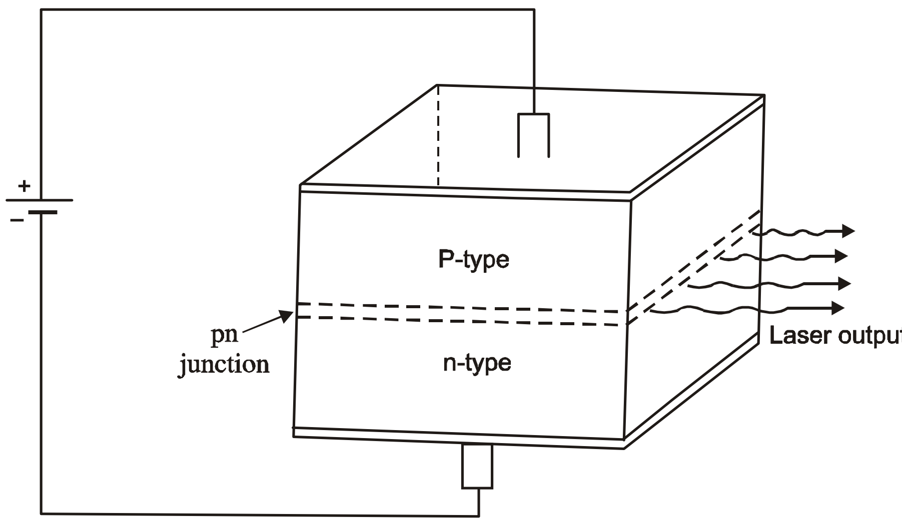


Fig. 5.25 Semiconductor diode laser (Homojunction)

Working

The energy level diagram of homojunction semiconductor laser is shown in fig 5.26.

When the *pn* junction is forward-biased, the electrons and holes are injected into junction region.

The region around junction contains a large number of electrons in the conduction band and holes in the valance band.

Now the electrons and holes recombine with each other. During recombination, light photons are produced.

When the forward - biased voltage is increased, more light photons are emitted. These photons trigger a chain of stimulated recombinations resulting in the emission of more light photons in phase.

These photons moving at the plane of the junction travel back and forth by reflection between two polished surfaces of the junction. Thus, the light photons grow in strength

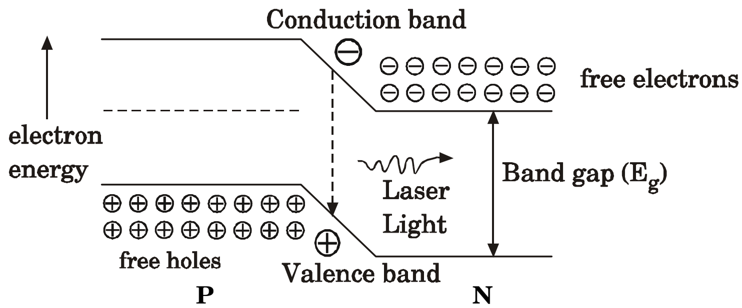


Fig. 5.26 Energy level diagram of a semiconductor laser

After gaining enough strength, laser beam of wavelength 8400 Å is emitted from the junction.

The wavelength of laser light is given by

$$E_g = h\nu = \frac{hc}{\lambda}$$

$$\boxed{\lambda = \frac{hc}{E_g}} \qquad \left(\because \nu = \frac{c}{\lambda}\right)$$

where $E_g \rightarrow$ band gap energy in joule

Characteristics

- **Type :** Solid state semiconductor laser.
- **Active medium :** A *pn* junction diode made from a single crystal of gallium arsenide.
- **Pumping method :** Direct conversion method.
- **Power output :** a few mW.
- **Nature of output :** Continuous wave or pulsed output.
- **Wavelength of output :** 8300 Å to 8500 Å.

Advantages

- This laser is very small in size and compact.
- It has high efficiency.
- The laser output can be easily increased by increasing the junction current.

- It is operated with less power than ruby and CO_2 lasers.
- It requires very little additional equipment.
- It emits a continuous wave output or pulsed output.

Disadvantages

- Laser output beam has large divergence.
- The purity and monochromacity are poor.
- It has poor coherence and stability.

Application

- This laser is widely used in fibre optic communication.
- It is used in laser printers and CD players.
- It is used to heal the wounds by infrared radiation.
- It is also used as a pain killer.

Semiconductor Laser (Hetero Junction)

A diode laser with a *pn* junction made up of different semiconductor materials in two regions ie., *n* - type and *p* - type is known as heterojunction semiconductor laser.

Principle

When the *pn* junction diode is forward biased, the electrons from ***n***-region and the holes from ***p***-region recombine with each other at the junction. During recombination process, light photon is released.

Example

Diode laser having a junction of GaAs and GaAlAs

Construction

Generally, this laser consists of five layers as shown in fig. 5.27. A layer of GaAs *p*-type (3rd layer) acts as active region. This layer is kept between two layers having wider band gap Ga Al As - *p* - type (2nd layer) and Ga Al As *n* - type (4th layer).

The bias voltage is applied through the metal electrodes fixed on top and bottom layers of hetero junction semiconductor laser.

The end faces of the junctions of 3rd and 4th layers are well polished and parallel to each other. They act as an optical resonator.

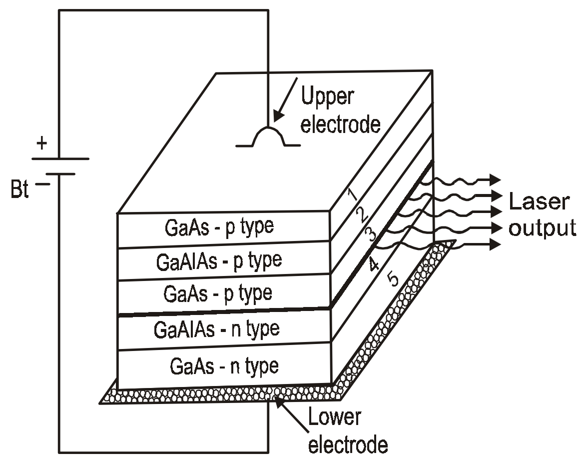


Fig. 5.27 Heterojunction Semiconductor laser

Table 5.4
Comparison table of Different Types of Lasers

S. No.	Characteristics	Nd-YAG laser	CO_2 laser	Semiconductor laser
1.	Type	Solid state laser	Molecular gas laser	Semiconductor laser
2.	Active medium	Yitrium Aluminium Garnet (Y_3 Al_5 O_{12})	Gas mixture of CO_2, N_2 and He	*pn* junction
3.	Pumping method	Optical pumping	Electrical discharge method	Direct conversion
4.	Optical resonator	Ends of the polished rods in silver	Metallic concave mirror of gold or silicon coated with aluminium	End faces of the junction diode

S. No.	Characteristics	Nd-YAG laser	CO_2 laser	Semiconductor laser
5.	Power output	2×10^4 W	10 kW	1 mW
6.	Nature of output	Pulsed	Continuous or pulsed	Pulsed or continuous wave form
7.	Wavelength	1.06 μm	9.6 μm and 10.6 μm	8300 - 8500 Å

5.9 BASIC APPLICATIONS OF LASER IN INDUSTRY

Material processing

Material processing involves cutting, welding, drilling and surface treatment.

When the material is exposed to laser light, then light energy is converted into heat energy. Due to heating effect, the material is heated then melted and vapourised.

Laser instrumentation for material processing

A laser setup used for material process, such as surface treatment, welding, cutting and drilling is shown in fig 5.28.

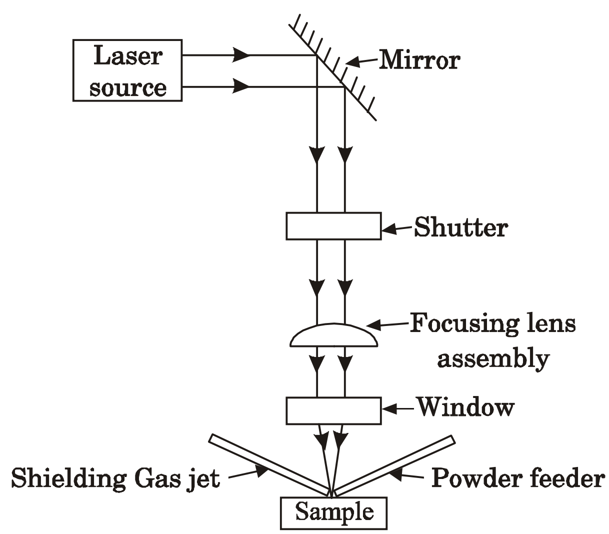


Fig. 5.28 Laser instrumentation for material processing

The light output from the laser source is incident on a plane mirror. After reflection, the laser light passes through a shutter to control its intensity.

There is a focusing lens assembly to get a fine beam. Further, there are shielding gas jet and powder feeder.

The shielding gas is used (i) to remove the molten material and help in vaporisation (ii) to provide cooling effect.

For different materials, different gases are used.

The powder feeder is used to spray the metal powder on the substrate for alloying or cladding.

The different types of laser heat treatment processes are shown in fig. 5.29.

Laser Annealing

In annealing, there is no separate heat affected zone and melting takes place over few picometer thickness.

Laser hardening

In hardening process, there is a heat affected zone in the form of hemisphere.

Laser surface alloying

Laser alloying involves the controlled melting of a work piece surface to a desired depth using laser. The powdered alloying element is added simultaneously with heating in small time intervals (0.1 to 10 seconds).

Laser cladding

In this process, a laser beam melts a very thin layer of work piece. This thin layer mixes with the liquid cladding alloy and form metallurgical bonding between the cladding and subtrate.

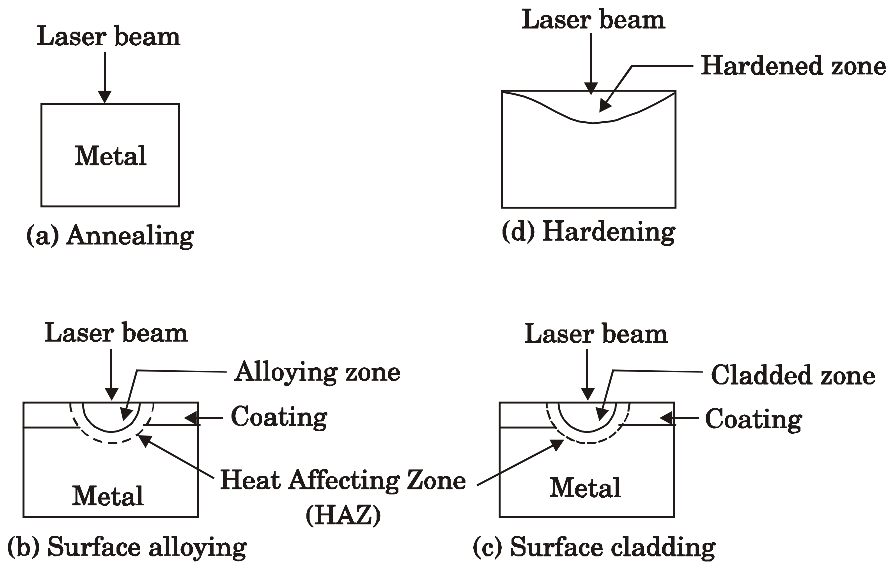


Fig. 5.29 Laser Heat Treatment Processes

Advantages

- Heat treatment of metals using laser radiation is very fast.
- Compared to other methods of heating, lasers are able to localise thermal treatment even to spots inaccessible by other methods.

Laser welding

Welding is joining of two or more metal pieces into a single unit.

For welding of two metal plates, the metal plates are held in contact at their edges and laser beam is allowed to move along the line of contact of the plates.

The laser beam heats the edges of the two plates to their melting points. Metals fuse together where they are in contact. (Fig. 5.30)

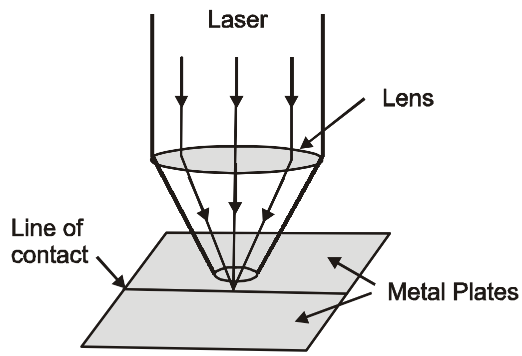


Fig. 5.30 Laser Welding

Advantages

- It is a contact - less process and hence, there is no possibility of impurities into joint.
- The heat affected zone is relatively small because of the small spot size of laser beam.
- Laser welding can be done even with very small pieces without any difficultly.
- The welding is done at very high rates. (faster)
- Any dissimilar metals can be welded.

Laser cutting (or) Drilling

The principle of laser cutting is the vaporization of the material at point of focus of the laser beam.

The laser cutting setup is shown in fig. 5.31. When laser beam is incident on the material, due to heating effect the material is melted and vapourised at the point of incidence. The vaporized material is removed with the help of a gas jet.

The gas jet is also used to cool the adjacent edges of the cut metal.

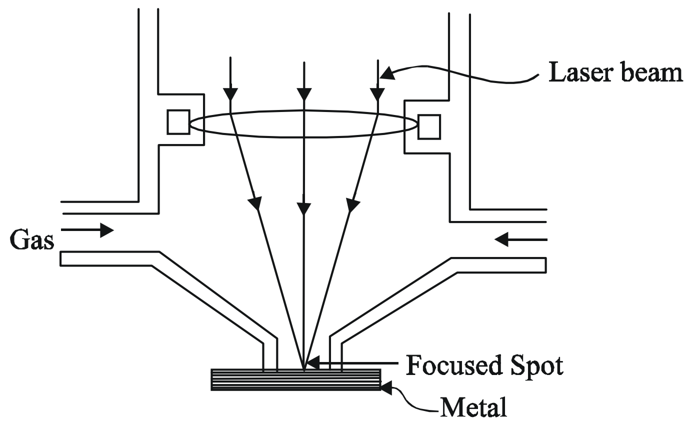


Fig. 5.31 Laser Cutting or Drilling

Advantages

- Laser cutting can be done at room temperature and pressure without preheating and vacuum condition.
- The microstructure of surrounding layers are not affected since heat affected zone is very narrow.
- Higher cutting speed can be achieved.
- The laser cutting has improved edge quality. The surface quality is maintained in the original condition since there is no bead formation due to scattering of molten material.

Soldering

It is a process in which two or more metals are joined together by melting and putting a **filler metal (solder)** into the joint, the filler metal having a lower **melting point** than the adjoining metal.

- Laser soldering, the newest soldering method.

Laser Soldering

It is a process in which selectively heats solder by means of laser irradiation to form a bond between two parts.

Principle

Laser soldering is a technique where a precisely focused laser beam provides controlled heating of the solder alloy leading to a fast and non-destructive of an electrical joint.

The process uses a controlled laser beam to transfer energy to a soldering location. The absorbed energy heats the solder until it reaches its melting temperature leading to the soldering of the contact and this completely eliminates any mechanical contact.

Working

Laser soldering is a technique where a 30 - 50 W laser is used to melt and solder an electrical connection joint. Diode laser systems based on semiconductor junctions are used for this purpose.

The wavelengths are typically 808 nm through 980 nm. The beam is delivered via an optical fiber to the workpiece, with fiber diameters 800 μ m and smaller.

Since the beam out of the end of the fiber diverges rapidly, **lenses** are used to create a suitable spot size on the workpiece at a suitable working distance. A wire feeder is used to supply solder.

Both lead - tin and silver - tin material can be soldered.

Laser Soldering Process

1. The laser illuminates the soldering point. (Fig. 5.32)
2. The illuminated area emits heat (surface heat emission).
3. The heat transfers into the surrounding area and is raised to the melting temperature.
4. Solder is supplied.

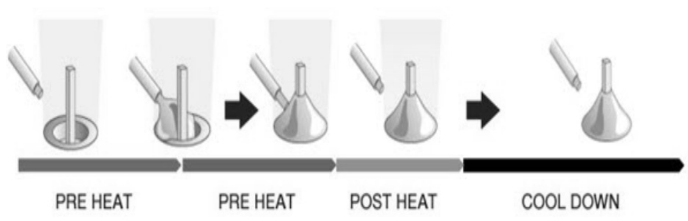


Fig. 5.32 Laser soldering

Types of lasers used in soldering

Three main types of lasers are found suitable for soldering process. They are

- Carbon dioxide laser (gas laser)
- Nd:YAG laser (a solid state laser)
- Semiconductor Laser (diode laser).

Advantages of laser soldering:

In contrast to other conventional soldering techniques, laser soldering offers a lot of advantages. They include

- Localized / selective heat input and Ideal for heat sensitive components.
- High precision – spot sizes in the order of 100s of microns
- Fast control of heat input (laser on / off)
- It reduces intermetallic compound formation and produce high quality joint.
- It also has low maintenance

Applications

The primary application for laser soldering is laser soldering of circuit boards in the electronics industry.

Surface Defect Detection

High intensity laser beam is used to study the surface defects in materials such as ICs, aircrafts, etc. The laser beam reflected from the surface of the material under investigation. The laser light is also directly obtained from the source as a reference are used to produce interference between the two laser beams. Information about the material is obtained by forming the interference pattern.

Fig. 5.33 shows the experimental arrangement to study surface defects in materials.

A high intensity laser beam from the source S falls on the converging lens L_1. The lens L_1 focuses the laser beam on the object ***O.*** The optical diffraction pattern of the image is focused on the photographic plate (P) or photodiode (D).

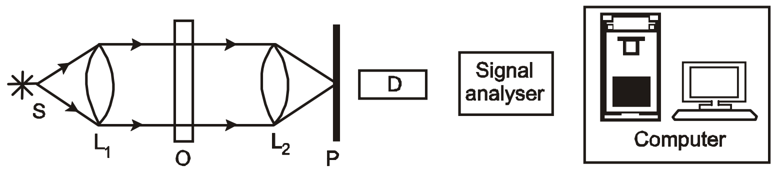


Fig. 5.33 Surface Defect Detection

The photodiode senses the light and converts the light energy into electrical signals. Using the signal analyser along with the necessary software, the image of the defect is obtained. Thus surface defects in materials can be studied.

Types of laser for material processing

In most of the material processing applications, the following lasers are used

(a) High power CO_2 - laser with continuous waves or pulsed waves.

(b) Nd - YAG laser.

Laser Application in Holography

One of the most important applications of the laser in the production of three dimensional images of an object in a process called holography.

In conventional photography a negative is made first and using it a positive print is produced later. The positive print is only a two dimensional record of light intensity received from a three dimensional object.

It contains information about the square of the amplitude of the light wave that produce the image but information about the phase of the light wave is not recorded and is lost.

In 1947 Dennis Gabor the English physicist developed a new technique of photographing objects.

He called this technique **wavefront construction**.

According to this technique both the phase and intensity components of the wave are recorded and when viewed the photograph shows a three dimensional image of the object. This technique is named **holography**.

Fig. 5.34 explains the principle of holography. A weak but broad beam of laser light is split into two beams namely a **reference beam** and **object beam**. The reference beam is allowed to reach the photographic plate directly while the object beam illuminates the object.

The part of the light scattered by the object travels towards the photographic plate and interferes with the reference beam and produces an interference pattern on the photo graphic plate.

The photographic plate carrying the interference pattern is called a **hologram**. Holos means complete in Greek and "gramma" means writing. Thus a hologram means complete recording. Like any ordinary photographic plate, a hologram is developed, fixed and stored.

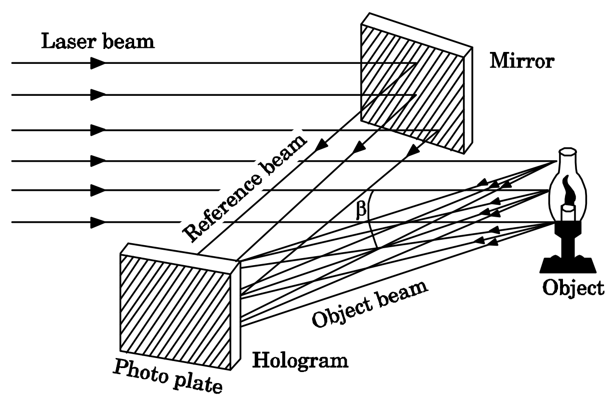


Fig. 5.34 Generation of a hologram

Applications of Laser in Communication

1. Since laser beam has enormous bandwidth and it permits 10 million telephone conversation or 8000 TV programmes simultaneously.

Note: Gabor was awarded in 1971 the Nobel prize in physics for this invention.

2. Narrow angular spread and directionality of laser beam makes it a very useful tool for communications with satellites and rockets to the moon and other planes i.e., inter planetary communications.
3. Since the laser light is not absorbed by water, it is possible to establish underwater communication between sub-marines.
4. **Fibre Guides:** The optical fibers are increasingly used in communications, where these are termed as **fibre guide**. (Fig. 5.35)

   Here laser light is used as a source.

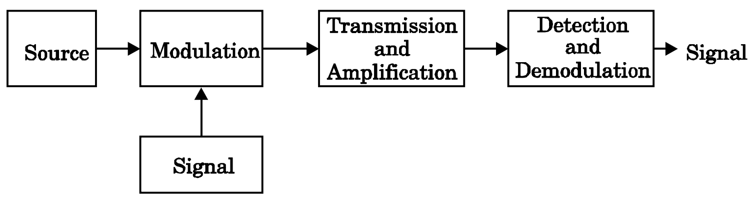


Fig. 5.35 Laser communication system block diagram

Applications of Laser in Engineering

Since large quantity of coherent energy can be concentrated at a point with laser it finds applications in defence.

1. **Death Ray:** Since laser beam is highly energetic, a powerful laser beam is used to destroy very big objects like aircrafts, missiles etc. in a few seconds by directing the laser beam on the objects. For this reason it is called as **death ray of war weapon**. (Fig. 5.36)

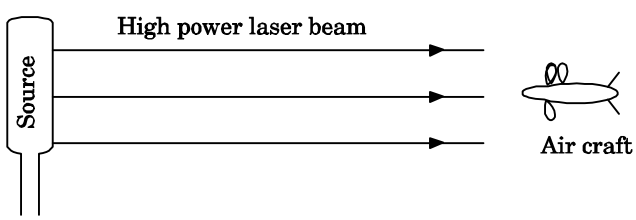


Fig. 5.36 Laser as a Death Ray

2. **Laser Gun:** In a laser gun a highly convergent beam is focussed on the enemy at a short range, which can

vapourise the governing part of the weapon system or sensitive part of enemy body.

3. **LIDAR:** Laser beam can exactly determine the size, form, distance, velocity and direction of any distant objects (enemy plane, miscible, etc.) by receiving the reflected laser beam on a cathode screen as in RADARS (Radio Detection And Ranging).

   In the case of RADAR, radio waves are used. But it laser is used, it is called LIDAR (Laser RADAR) i.e., Light (laser) Detection And Ranging (Fig. 5.37).

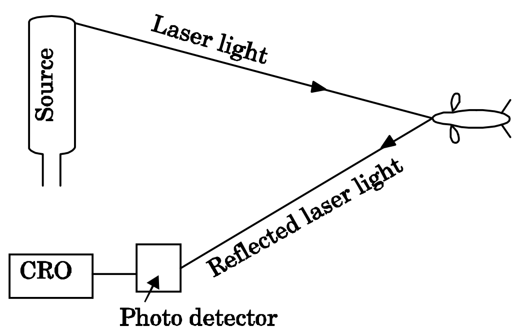


Fig. 5.37 LIDAR

4. Laser can be used for automatically guiding rocket and satellites.
5. Laser can be used for forecasting earthquakes.
6. Computer print outs are done with laser printers.

Chemical applications

Laser can be used to

- accelerate some chemical reactions.
- create new chemical compounds by destroying atomic bonds between molecule.

Biological applications

- drill minute holes in cell walls without damaging the cell itself.
- carry out minute microsurgical operations within cells.

ANNA UNIVERSITY SOLVED PROBLEMS

Problem 5.1

Calculate the relative population of sodium atoms in sodium lamp in the first excited state and the ground state at a temperature of 250°C. [λ = 590 nm]

[A.U. Jan 2018]

Given data:

Planck's constant (h) $= 6.625 \times 10^{-34}$ Js

Velocity of light (c) $= 3 \times 10^{8}$ ms^{-1}

Boltzmann's constant (k) $= 1.38 \times 10^{-23}$ JK^{-1}

Temperature $T = 250°C = 250 + 273 = 523$ K

Wavelength $\lambda = 590 \text{ nm} = 590 \times 10^{-9}$ m

Solution:

Let N_2 be the population of the first excited state and N_1 be the population of the ground state.

We know that

$$\frac{N_2}{N_1} = \frac{e^{-E_2/kT}}{e^{-E_1/kT}} = e^{-(E_2 - E_1)/kT} = e^{-h\nu/kT} = e^{-hc/\lambda kT}$$

Substituting the given values,

$$\frac{N_2}{N_1} = e^{\left[\frac{-6.625 \times 10^{-34} \times 3 \times 10^{8}}{5.9 \times 10^{-7} \times 1.38 \times 10^{-23} \times 523}\right]}$$

$$= e^{[-46.674]} = 5.37 \times 10^{-21}$$

$$\boxed{\frac{N_2}{N_1} = 5.367 \times 10^{-21}}$$

The ratio between the atoms in the first excited state and the ground state is 5.367×10^{-21}.

Problem 5.2

Laser action occurs by transition from an excited state (E_2) to the ground state ($E_1 = 0$). If the transition produces a light of wavelength 6930 Å, find the energy level of the excited state. ***(A.U. Dec 2017)***

Given data

$$\lambda = 6930 \text{ Å} = 6930 \times 10^{-10} \text{ m}$$

$$h = 6.625 \times 10^{-34} \text{ Js}$$

$$c = 3 \times 10^{8} \text{ ms}^{-1}$$

$$E_1 = 0. \quad \text{(ground state))}$$

Solution

$$E_2 - E_1 = h\nu = \frac{hc}{\lambda}.$$

Substituting the given values, we have

$$= \frac{6.625 \times 10^{-34} \times 3 \times 10^{8}}{6930 \times 10^{-10}}$$

Energy of the excited state

$$\boxed{E_2 = 2.868 \times 10^{-19} \text{ J} = 1.79 \text{ eV.}}$$

Problem 5.3

Find the relative population of the two states in a Nd-YAG laser that produces a light beam of wavelength 6943 Å at 300 K. ***(A.U. May 2019)***

Given data

$\lambda = 6943\ \text{Å} = 6943 \times 10^{-10}\ \text{m}$

$T = 300\ \text{K}$

Solution

The population ratio is given by

$$\frac{N_2}{N_1} = \exp\left[-(E_2 - E_1)/kT\right]$$

$$E_2 - E_1 = h\nu = \frac{hc}{\lambda}$$

$$= \frac{6.625 \times 10^{-34} \times 3 \times 10^{8}}{6943 \times 10^{-10}\ \text{m}}$$

$$= 2.863 \times 10^{-19}\ \text{J}$$

$$\frac{N_2}{N_1} = \exp\left[\frac{-2.863 \times 10^{-19}}{1.38 \times 10^{-23} \times 300}\right] = \exp(-69.2)$$

$$\boxed{\frac{N_2}{N_1} = 8 \times 10^{-31}}$$

Problem 5.4

In a laser action the energy of the stimulated photon is 39.62×10^{-20} J. What is the wavelength of the stimulated photon? ***(A.U. Jan 2016)***

Given data

Energy of the stimulated photon $E = 39.62 \times 10^{-20}$ J

Planck's constant $h = 6.625 \times 10^{-34}$ Js

Solution

We know that $E = h\nu$

$$\nu = \frac{E}{h}$$

$$\nu = \frac{39.62 \times 10^{-20}}{6.625 \times 10^{-34}} = 5.98 \times 10^{14}$$

We know that $c = \nu\lambda$

$$\lambda = \frac{c}{\nu}$$

$$\lambda = \frac{3 \times 10^{8}}{5.98 \times 10^{14}}$$

$$= 0.50 \times 10^{-6}$$

$$\boxed{\lambda = 5 \times 10^{-7} \text{ m}}$$

Problem 5.5

Determine the wavelength of radiation given out by a laser with an energy of 3 eV, given that $h = 6.63 \times 10^{-34}$ Js and $c = 3 \times 10^{8}$ ms^{-1}. ***(A.U. May 2020)***

Given data

$$\text{Energy } (E) = 3 \text{ eV} = 3 \times 1.6 \times 10^{-19} \text{ J}$$

$$h = 6.63 \times 10^{-34} \text{ Js}$$

$$c = 3 \times 10^{8} \text{ ms}^{-1}$$

Solution

We know that $E = h\nu$ and

$$c = \nu\lambda \text{ or } \nu = \frac{c}{\lambda}$$

$$\therefore \quad E = h\nu = \frac{hc}{\lambda}$$

$$\lambda = \frac{hc}{E}$$

Substituting the given values, we have

$$\lambda = \frac{hc}{E} = \frac{6.63 \times 10^{-34} \times 3 \times 10^{8}}{3 \times 1.6 \times 10^{-19}}$$

$$\lambda = 4.14 \times 10^{-7} \text{ m}$$

$$\lambda = 414 \times 10^{-9} \text{ m} = 414 \text{ nm}$$

$$\boxed{\lambda = 414 \text{ nm}}$$

Problem 5.6

Calculate the wavelength of light emission from GaAs whose band gap is 1.44 eV ***(A.U. Jan. 2018)***

Given data

$$\text{Band gap } E_g = 1.44 \text{ eV} = 1.44 \times 1.6 \times 10^{-19} \text{ J}$$

$$h = 6.625 \times 10^{-34}$$

$$c = 3 \times 10^{8} \text{ ms}^{-1}$$

Solution

$$\text{Wavelength } \lambda = \frac{hc}{E_g}$$

Substituting given values,

$$\lambda = \frac{6.625 \times 10^{-34} \times 3 \times 10^{8}}{1.44 \times 1.6 \times 10^{-19}}$$

$$\lambda = 8.626 \times 10^{-7} \text{ m}$$

$$= 8626 \times 10^{-10} \text{ m}$$

$$\boxed{\text{Wavelength } \lambda = \mathbf{8626\,Å}}$$

ANNA UNIVERSITY PART - A '2' Marks Q & A

1. What is stimulated emission? ***(A.U. Jan. 2017)***

The process of induced emissions of photons caused by the incident photons is called stimulated emission. This process is a key factor for the operation of a laser.

2. What are the conditions necessary for stimulated emission of radiation? ***(A.U. April 2018)***

(i) The atoms must be in the excited state.

(ii) The photon of light radiation must strike the atoms in the excited state.

3. Write the differences between spontaneous emission and stimulated emission ***(A.U. Dec. 2020)***

	Spontaneous emission	**Stimulated emission**
1.	Emission of light radiation is not triggered by external influence	Induced emissions of light radiations caused by incident photons
2.	Emitted photon travels in random direction	Emitted photon travels in particular direction
3.	Emitted photons cannot be controlled	Emitted photons can be controlled.
4.	This process is a key factor for ordinary light.	This process is a key factor for laser operation

4. What is meant by population inversion and how is it achieved? ***(A.U. Jan. 2019)***

The establishment of a situation in which the number of atoms in higher energy level is more than that in lower energy level is called population inversion. It is an essential requirement for producing a laser beam. It is achieved by pumping action.

5. Explain the need for population inversion in the production of laser? ***(A.U. Dec. 2020)***

Stimulated emission process is a key factor for the production of laser. For stimulated emission, more number of atoms must be in the excited state.

Establishing a situation in which number of atoms in higher energy state is more than that in lower energy state is called population inversion. Hence, population inversion is needed in the production of laser.

6. What is pumping action? ***(A.U. May 2019)***

The process of creating a population inversion in the atomic states is known as pumping action. It is essential requirement for producing a laser beam.

7. What are the methods commonly used for pumping action? ***(A.U. Dec 2018)***

(i) Optical pumping (excitation by photons)

(ii) Electrical discharge method (excitation by electrons)

(iii) Direct conversion

(iv) Inelastic collision between atoms.

8. What is optical pumping? ***(A.U. Jan 2019)***

When the atoms are exposed to light radiations (of energy $h\nu$), atoms in the lower energy state absorb these light radiations and go to excited state. This method of pumping is called optical pumping.

It is used in solid state lasers like ruby laser and Nd - YAG laser.

9. What is meant by active material in laser? ***(A.U. Jan 2021)***

A material in which population inversion can be achieved is called as active material.

10. What are the characteristics of the laser? (or) What are the properties of the laser beam?

(A.U. Jan 2018, May 2019, Jan 2020)

- Laser light is highly coherent
- It is highly powerful and intense.
- It is directional and monochromatic.
- It is capable of travelling over long distance without any energy loss.
- It is extremely bright.
- Laser beam is not easily absorbed by the water.

11. Under which conditions a set of laser beams is said to be coherent? ***(A.U. Dec. 2019)***

A set of laser beams is said to be coherent if they have same frequency and constant phase difference among them with respect to space and time.

12. Compare the characteristics of laser with ordinary light. ***(A.U. Jan. 2018, May 2019, Jan 2020)***

	Ordinary light Source	**Laser Source**
1.	Light emitted is not monochromatic.	Light emitted is highly monochromatic.
2.	Light emitted does not have high degree of coherence	It has high degree of coherence.
3.	Emits light in all directions (not directional)	Emits light only in one direction (directional)
4.	Light is less intense and less bright	Laser light is much intense and bright.

13. What is optical resonant cavity? ***(A.U. Jan. 2021)***

It is a pair of mirrors with active material in between them. One of the mirrors of the resonant cavity is made partially reflecting to serve as an output element passing the light (laser) out of the resonator. The other mirror is a highly reflecting one.

14. What is the function of resonator cavity in laser? ***(A.U. May 2019)***

Resonator cavity is made of a pair of fully reflecting plate and a partially reflecting plate. Both of them are optically plane and accurately parallel. The active medium is placed between these mirrors.

The photons emitted along the axial direction during stimulated emission travel back and forth across the active medium and grow in strength. After enough strength is attained, laser beam emerges out from the partial reflector.

15. What is the principle of laser action? ***(A.U. May 2019)***

Stimulated emission process is a key factor for the laser action. This can be multiplied through chain reaction. This multiplication of photons through stimulated emission leads to coherent, powerful, monochromatic, collimated beam of light-emission.

16. What are the three important components of any laser device? ***(A.U. Jan 2016)***

(i) Active medium

(ii) Pumping source

(iii) Optical resonator

17. What are the conditions required for laser action? ***(A.U. Dec 2019)***

- Population inversion should be achieved.
- Stimulated emission should be predominant over spontaneous emission.

18. What are Einstein's coefficients? ***(A.U. Jan 2019)***

In Einstein's theory of spontaneous and stimulated emission, if N_1 and N_2 are the number of atoms in the lower energy state (E_1) and higher energy state (E_2), then the number of stimulated absorption transition is given by

$$\boxed{N_{ab} = B_{12} N_1 Q}$$

The number of spontaneous emission transition is given by

$$\boxed{N_{sp} = A_{21} N_2}$$

The number of stimulated emission transition is given by

$$\boxed{N_{st} = B_{21} N_2 Q}$$

Also $B_{12} = B_{21}$ and $\dfrac{A_{21}}{B_{21}} = \dfrac{8\pi h}{\lambda^3}$

where Q is the energy density of the incident radiation. A_{21}, B_{12} and B_{21} in the above three equations are called Einstein's Coefficients.

19. How lasers are classified? or Mention the various types of lasers. ***(A.U. May 2019)***

(i) Solid state lasers

(ii) Gas lasers

(iii) Liquid lasers

(iv) Dye lasers

(v) Semiconductor lasers.

20. What is Nd - YAG laser? ***(A.U. May 2020)***

Nd - YAG is a neodymium based laser.

Nd – Neodymium (rare earth element Nd^{3+}).

YAG – Yittrium Aluminium Garnet ($Y_3 Al_5 O_{12}$).

It is a four level solid state laser.

21. What are the applications of Nd-YAG laser ?

(A.U. Jan. 2017)

(i) It finds many applications in range finders and illuminators

(ii) It finds applications in resistor trimming, scribing, micro machining operations such as welding, drilling etc.

(iii) It finds applications in medical field like endoscopy, urology, neurosurgery, ENT, gynaecology, dermatology, dental surgery and general surgery.

22. What is CO_2 laser ? ***(A.U. Jan. 2016)***

It is a four level molecular gas laser. The active medium of this laser is CO_2 gas. Laser transition takes place between the vibrational energy states of the CO_2 molecules. It is a very useful and efficient laser.

23. What is the active medium in CO_2 laser ?

(A.U. May 2016)

A gas mixture consisting of CO_2, nitrogen and helium is the active medium.

24. What are the applications of CO_2 laser ?

(A.U. April 2017)

(i) High power CO_2 lasers find applications in materials processing, welding, drilling, cutting, soldering etc., because of their very high output power.

(ii) It is used in laser remote sensing.

(iii) It has found wide applications in neurosurgery and general surgery.

(iv) It is used to perform micro-surgery and bloodless operations.

25. What is semiconductor laser ? ***(A.U. Jan. 2019)***

Semiconductor diode laser is a specially fabricated *pn* junction device. It emits laser light when it is forward biased.

26. What is homo-junction laser ? ***(A.U Jan. 2020)***

Homo-junction means that a p-n junction is formed from a single crystalline material.

Example: Gallium Arsenide (GaAs).

27. What are the drawbacks of homojunction laser diodes ? ***(A.U. April 2019)***

- The output beam has large divergence.
- Coherence and stability are poor.
- Optical confinement is very poor.

28. What are the applications of semiconductor laser ? ***(A.U. May 2018)***

(i) It is mostly used in optical fiber communications.

(ii) It is used to heal the wounds by means of infrared radiation.

(iii) It is used in computer laser printers and for writing and reading CD's.

29. What is laser material processing ? ***(A.U. May 2019)***

Material processing involves cutting, welding, drilling and surface treatment using laser beams.

When the material is exposed to laser light, then light energy is converted into heat energy. Due to heating effect, the material is heated then melted and vapourised. Also a fine beam of laser acts like a machine tool to do cutting, welding etc.

30. Mention the applications of lasers in industry. ***(A.U. Jan. 2015, May 2016)***

Nd : YAG and CO_2 lasers are very much used in industries for the following processes :

(i) Welding

(ii) Cutting

(iii) Drilling

(iv) Heat treatment of metallic and non metallic (plastic, ceramic, glass) materials.

(v) Non - Destructive Testing (NDT): Testing the materials for flaws or defects without damaging them.

31. What is laser welding ? ***(A.U. Dec 2018)***

In this technique, a focussed laser beam is incident on spot where the two parts are to be welded. The spot-contact points get welded due to heating affect of fine laser beams.

32. What are the advantages of laser welding ? ***(A.U May 2020)***

(i) Laser welding is contactless, therefore there is no possibility of introduction of harmful impurities.

(ii) Laser welding can be performed faster in atmospheric pressure unlike electron beam welding where vacuum is a must.

(iii) Dissimilar materials can be welded

33. What is heat treatment of laser ? ***(A.U. Jan 2019)***

A powerful laser beam is incident on a metal surface. That portion at which laser light is incident gets heated. As the beam is moved away to other areas, the heated spot cools down rapidly. This procedure is used for heat treatment of metal surfaces which enhances the strength of the metal.

34. What are advantages laser cutting ? ***(A.U. Jan 2018)***

(i) Laser cutting is used for wide range of processed materials (paper, cloth, plywood, glass, ceramics, sheet metal);

(ii) This laser cutting introduces minimum mechanical distortion and minimum thermal damage in the material being cut

(iii) This cutting process has high chemical purity.

ADDITIONAL Q&A

1. What is laser?

Laser stands for Light Amplification by Stimulated Emission of Radiation.

Laser is a light source. It produces a powerful, monochromatic, collimated beam of light in which the light waves are coherent.

2. What is stimulated absorption?

An atom in the ground state with energy E_1 absorbs a photon of energy $h\nu$ and go to an excited state (higher state) with energy E_2 provided that the photon energy $h\nu$ is equal to the energy difference $(E_2 - E_1)$. This process is called **stimulated absorption** or **simply absorption**.

3. What is spontaneous emission?

The atom in the excited state E_2 (higher energy state) returns to the ground state E_1 (lower energy state) by emitting a photon of energy $h\nu$ without the action of an external agency. Such an emission of radiation which is not triggered by an external influence is called ***spontaneous emission***

4. What is gas laser?

Gas laser is a type of laser, in which gases such as CO_2, Nitrogen and He-Ne are used as active medium for laser operations. CO_2, Nitrogen and He-Ne are the important gas lasers.

Anna University Part - B '16' Marks Questions

1. For atomic transitions, derive Einstein relations and hence deduce the expressions for the ratio of spontaneous emission rate to be stimulated emission rate.

[A.U. Jan. 2019, Jan 2020]

2. Explain the construction and working of Nd - YAG laser with neat diagram. ***[A.U. May 2015, Jan. 2019, Jan. 2021]***

3. Explain the modes of vibrations of CO_2 molecule. Describe the construction and working of CO_2 laser with necessary diagrams. ***[A.U. Jan 2019, Jan 2020, Jan 2011}***

4. Explain the lasing schemes and working of a Nd:YAG laser. ***[A.U. Jan 2020]***

5. Describe the construction and working of CO_2 laser ***[A.U. Dec 2018, 2019, May 2020]***

6. (a) Explain the modes of vibrations of CO_2 molecule. Describe the construction and working of CO_2 laser with necessary diagrams.
   (b) Classify lasers based on active medium with one example for each ***[A.U. Dec 2019]***

7. Describe the construction and working of CO_2 laser. What are the engineering applications of lasers? ***[A.U. Dec 2019, Jan 2020]***

8. (a) What is molecular gas laser? With neat sketch, explain the construction and working of CO_2 laser using energy level diagram.
   (b) List out the applications of laser beam in industries and in medical field. ***[A.U. Jan 2020]***

9. (i) With a neat diagram, explain the construction and working of a CO_2 laser.
   (ii) Outline the use of lasers in heat treatment. ***[A.U. Jan 2021]***

10. Describe the construction and working of semiconductor laser. ***[A.U. Jan 2019]***

11. With suitable diagram explain how laser action is achieved in homojunction Ga-As laser. ***[A.U. Jan 2019, Jan. 2020, Jan 2021]***

12. Explain the principle, construction and working of a semiconductor diode laser with necessary diagrams.

[A.U. Jan 2010]

13. What are different pumping mechanisms used in lasers? Give an example for each. ***[A.U. Jan 2017]***

14. Discuss the applications of Lasers in industry.

PROBLEM FOR PRACTICE

1. Find the wavelength of emitted photons from a GaAs laser diode, which has a bandgap of 1.44 eV. ***(Ans:* 8626 Å*)***

2. A CO_2 laser source emits light at a wavelength of 9.6 μm and has an output power of 10 kW. How many photons are emitted in each hour by this laser? ***(Ans:* 1.739×10^{27}*)***

3. A laser source emits lights of wavelength 0.621 μ m and has an output of 35 mW. Calculate how many photons are emitted per minute by this laser source.

(Ans:* 6.562×10^{18} *photons / minute)

4. Transition occurs between a metastable state E_3 and an energy state E_2, just above the ground state. If emission is at 1.1 μm and $E_2 = 0.4 \times 10^{-19}$ J, find the energy of the E_3 state.

***(Ans:* 2.2068×10^{-19} J)**

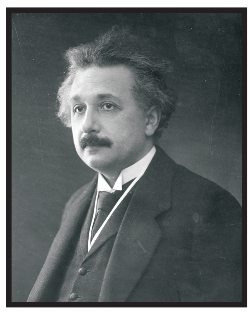
5. If laser action occurs by the transition from an excited state to the ground state $E_1 = 0$ and if it produces light of 650 nm wavelength, what is the energy level of the excited state.

(Ans:* 3.0576×10^{-19} J*)

6. Given E_g = 1.43 eV. Find the wavelength of the light emitted by the diode for the given energy gap. Also find the colour of light emitted. ***(Ans:* λ = 8671.32 Å, IR*)***

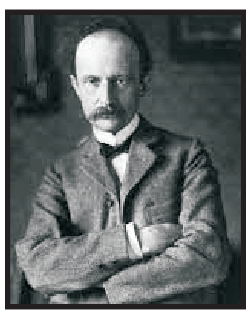
Unit - IV

Louis de Broglie
1892 - 1987

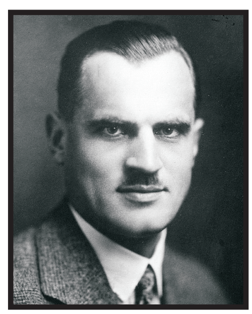


Albert Einstein
1879 - 1955

Erwin Schrödinger
1887 - 1961



Max Planck
1858 - 1947



Compton Arthur Holly
1892 – 1962

Basic Quantum Mechanics

6. Basic Quantum Mechanics

Photons and light waves - Electrons and matter waves - Compton effect - The Schrodinger equation (Time dependent and time independent forms) - meaning of wave function - Normalization - Free particle - particle in a infinite potential well: 1D, 2D and 3D Boxes - Normalization, probabilities and the correspondence principle

Introduction

- The most outstanding development in modern science is the conception of quantum mechanics. The quantum mechanics is better than Newtonian classical mechanics in explaining the fundamental physics.

- The fundamental concepts were not different from those of everyday experience, such as particle, position, speed, mass, force, energy and even field. These concepts are referred as ***'classical'***.

- The world of atoms cannot be described and understood with these concepts. For atoms and molecules, the ideas and concepts used in dealing with optics in day to day life is not sufficient. Thus, it needed new concepts to understand the properties of atoms.

- A group of scientists **Neils Bohr W. Heisenberg, E Schrodinger, P.A.M. Dirac, W. Pauli, and M. Born** conceived and formulated these new ideas in the beginning of 20^{th} century. This new formulation, a branch of physics, was named as ***quantum mechanics***.

Limitations of Classical Mechanics

- The classical physics is complete and beautiful in explaining daily experiences where big bodies are involved. But it breaks down severely at subatomic level and failed to explain some of the phenomenon totally.

- The phenomenas which classical physics failed to explain are black body radiation, photoelectric effect, emission of X-rays, etc.
- In classical physics, a body which is very small in comparison with other body is termed as ***'particle'*** whereas in quantum mechanics, the body which cannot be divided further is termed as ***'particle'***.
- The other main difference is the quantized energy state. In classical physics, an oscillating body can assume any possible energy. On the contrary, quantum mechanics says that it can have only discrete non-zero energy.

Need of Quantum Mechanics

- Classical mechanics successfully explained the motions of object which are observable directly or by instruments like microscope.

  But when classical mechanics is applied to the particles of atomic levels, it fails to explain actual behaviour. Therefore, the classical mechanics cannot used to explain in atomic level, e.g. motion of an electron in an atom.
- The phenomena of black body radiation, photoelectric effect, emission of X-rays, etc. were explained by Max Planck in 1900 by introducing of the formula

$$\boxed{E = nh\nu} \quad \text{... (1)}$$

where, $n = 0, 1, 2, \ldots$

h = Planck's constant = 6.63×10^{-34} J/s

- This is known as ***'quantum hypothesis'*** and marked the beginning of modern physics. The whole microscopic world obeys the above formula.

6.1 PHOTONS AND LIGHT WAVES - (Duality of Radiation and Matter)

The wave and particle duality of radiation is easily understood by knowing a difference between a wave and and a particle.

Wave

- A **wave** originates due to oscillations and it is spread out over a large region of space. A wave cannot be located at a particular place and mass cannot be carred by a wave.
- Actually, a wave is a spread out disturbance specified by its amplitude A, frequency γ, wavelength λ, phase δ and intensity I.
- The phenomena of interference, diffraction and polarisation require the presence of two or more waves at the same time and at the same position.
  It is very clear that two or more particles cannot occupy the same position at the same time. So one has to conclude that radiation behaves like waves.

Particle

- A ***particle*** is located at some definite point and it has mass. It can move from one place to another. A particle gains energy when it is accelerated and it loses energy when it is slowed down.
- A particle is characterized by mass m, velocity v, momentum p and energy E.
- Spectra of black body radiation, Compton effect, photoelectric effect, etc. could not be explained on wave nature of radiation.
  These phenomena established that radiant energy interacts with matter in the form of **photons or quanta.** Therefore, Planck's quantum theory came to conclude that radiation behaves like particles.
- Thus, radiation sometimes behaves as a wave and at some other times as a particle. Now, wave - particle duality of radiation is universally accepted.

Scattering of X - rays

Like ordinary light waves, X - rays are scattered by matter in two different ways. They are

(a) **Coherent scattering** or **classical scattering or Thomson scattering.**

(b) **Incoherent scattering** or **Compton scattering**.

(a) Coherent scattering

In coherent scattering, X - rays are scattered by electrons without any change in their wavelengths. This type of scattering was explained by Thomson on the basis of classical electromagnetic theory.

(b) Incoherent scattering (Compton scattering)

In Compton scattering, the scattered X - ray consists of two components, one is having the same wavelength as that of the incident X - rays and the other has a slightly longer wavelength. This type of scattering was studied by Compton on the basis of particle nature of light radiation.

6.2 COMPTON EFFECT

Compton effect refers to the change in the wavelength of scattered X - rays by a material.

Statement

When a beam of X - rays is scattered by a substance of low atomic number, the scattered X - ray radiation consists of two components.

One component has the same wavelength λ as the incident ray and the other component has a slightly longer wavelength λ'.

The change in the wavelength of scattered X - rays is known as Compton shift. The phenomenon is called Compton effect.

The radiations of unchanged wavelength in the scattered radiations are called **unmodified radiations**. The radiations of longer wavelength are known as **modified radiations**.

The change of wavelength is due to loss of energy of the incident X - rays.

Explanation

The compton effect was explained on the basis of quantum theory of radiation. The X - radiation consists of quanta or photons each having an energy of $h\nu$. These photons move with velocity of light (c). They obey the laws of conservation of energy and momentum when they undergo collision.

The whole process is treated as a particle - particle collision between X - ray photon and a loosely bound electron of the atom in the scattering substance.

When a photon of energy $h\nu$ collides with a free electron of the scattering substance, which is assumed to be at rest initially, the photon transfers some of its energy to the electron (Fig. 6.1).

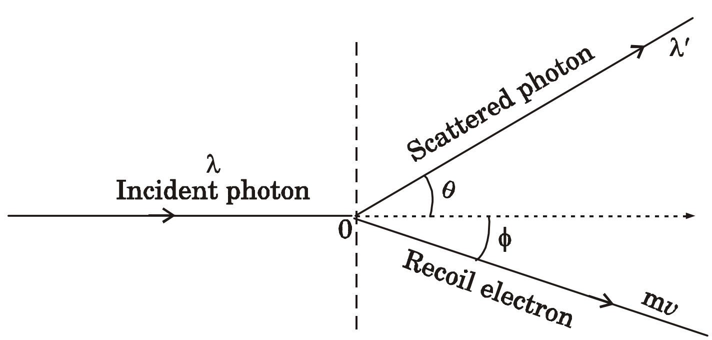
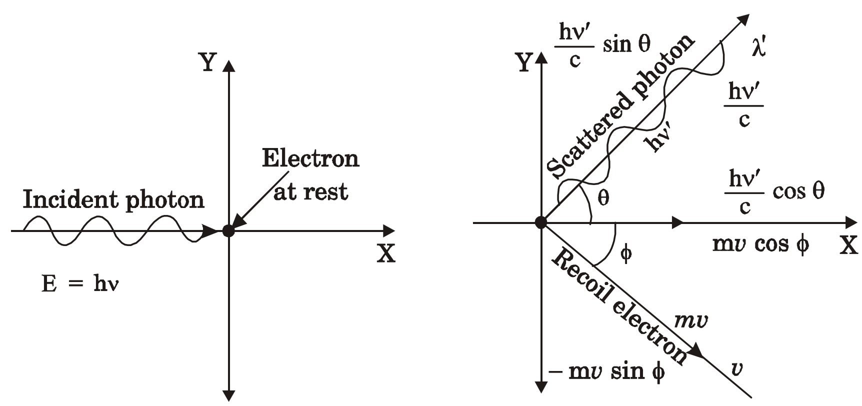


Fig. 6.1 Compton effect

The electron gains kinetic energy and it recoils with velocity v. Therefore, the scattered photon has lower energy, (lower frequency and longer wavelength) than that of the incident one.

Theory of Compton effect (Derivation)

Consider an X - ray photon striking an electron at rest (Fig 6.2 a). This X - ray photon is scattered through an angle θ to X-axis from its initial direction of motion (Fig. 6.2 b).



(a) Before collision **(b) After collision**

Fig. 6.2 Compton effect

Let the frequency of scattered photon be ν′ and its energy $h\nu'$. During the collision, X - ray photon gives a fraction of its energy to the free electron. This free electron of mass m gains energy and it moves with a velocity v at an angle ϕ to X-axis.

Total energy before collision

Energy of incident photon $= h\nu$

Energy of electron at rest $= m_o c^2$

where m_o – rest mass of the electron,

c – velocity of light.

Total energy before collision $= h\nu + m_o c^2$

Total energy after collision

Energy of scattered photon $= h\nu'$

Energy of scattered electron $= mc^2$

where m is the mass of electron.

when it moves with velocity v.

Total energy after collision $= h\nu' + mc^2$

Applying the law of conservation of energy, i.e.,

Total energy before collision = Total energy after collision

$$h\nu + m_o c^2 = h\nu' + mc^2$$

$$mc^2 = h\nu - h\nu' + m_o c^2$$

$$\boxed{\boldsymbol{mc^2 = h(\nu - \nu') + m_o c^2}} \quad ...(1)$$

Total momentum along X-axis

Before collision

Momentum of photon along X-axis $= \dfrac{h\nu}{c}$

Momentum of electron along X-axis = 0

Total momentum along X-axis $= \dfrac{h\nu}{c}$

Note: From de-Broglie's concept,

momentum of the particle $p = \dfrac{h}{\lambda}$

But, $\lambda = \dfrac{c}{\nu}$ for photon $\quad \therefore \; p = \dfrac{h}{c/\nu} = \dfrac{h\nu}{c}$

After collision

Since the momentum is vector quantity, it is resolved along X-axis and Y-axis, then

Momentum of photon along X-axis $= \dfrac{h\nu'}{c}\cos\theta$

Momentum of electron along X-axis = $mv \cos\phi$

Total momentum along X-axis after collision

$$= \frac{h\nu'}{c}\cos\theta + mv\cos\phi$$

Applying the law of conservation of momentum ie.,

Total momentum before collision = Total momentum after collision

$$\frac{h\nu}{c} = \frac{h\nu'}{c}\cos\theta + mv\cos\phi \quad \text{....(2)}$$

$$\frac{h\nu}{c} - \frac{h\nu'}{c}\cos\theta = mv\cos\phi$$

$$\frac{h}{c}(\nu - \nu'\cos\theta) = mv\cos\phi$$

$$h(\nu - \nu'\cos\theta) = mvc\cos\phi$$

$$mvc\cos\phi = h(\nu - \nu'\cos\theta) \quad \text{....(3)}$$

Total momentum along Y-axis

Before collision

Momentum of photon along Y-axis = 0

Momentum of electron along Y-axis = 0

Total momentum along Y-axis = 0

After collision

Momentum of photon along Y - axis $= \frac{h\nu'}{c}\sin\theta$

Momentum of electron along Y - axis = $-mv\sin\phi$

[negative sign indicates negative Y - direction]

Total momentum along Y - axis $= \frac{h\nu'}{c}\sin\theta - mv\sin\phi$

Applying the law of conservation of momentum ie.,

Total momentum before collision = Total momentum after collision

$$0 = \frac{h\nu'}{c} \sin\theta - mv \sin\phi$$

$$mv \sin\phi = \frac{h\nu'}{c} \sin\theta \qquad ...(4)$$

$$mvc \sin\phi = h\nu' \sin\theta \qquad ....(5)$$

Squaring eqn (3) and eqn (5) and then adding, we get

$$(mvc \cos\phi)^2 + (mvc \sin\phi)^2 = h^2(\nu - \nu' \cos\theta)^2 + (h\nu' \sin\theta)^2 \qquad ... (6)$$

L.H.S. of eqn (6)

$$= m^2v^2c^2\cos^2\phi + m^2v^2c^2 \sin^2\phi$$

$$= m^2v^2c^2 (\sin^2\phi + \cos^2\phi)$$

$$= m^2v^2c^2 \qquad [\because \sin^2\phi + \cos^2\phi = 1]$$

R.H.S. of eqn (6)

$$= h^2(\nu^2 - 2\nu\nu' \cos\theta + \nu'^2 \cos^2\theta) + h^2\nu'^2 \sin^2\theta$$

$$= h^2 [\nu^2 - 2\nu\nu' \cos\theta + \nu'^2 \cos^2\theta + \nu'^2 \sin^2\theta]$$

$$= h^2 [\nu^2 - 2\nu\nu' \cos\theta + \nu'^2 (\sin^2\theta + \cos^2\theta)]$$

$$= h^2(\nu^2 - 2\nu\nu' \cos\theta + \nu'^2) \qquad [\because \sin^2\theta + \cos^2\theta = 1]$$

L.H.S = R.H.S of eqn. 6

$$\boxed{m^2v^2c^2 = h^2(\nu^2 - 2\nu\nu' \cos\theta + \nu'^2)} \qquad ...(7)$$

Squaring eqn (1). on both sides, we get

$$(mc^2)^2 = (h(\nu - \nu') + m_oc^2)^2 \qquad ...(8)$$

$$m^2c^4 = h^2(\nu - \nu')^2 + m_o^2 c^4 + 2h\,(\nu - \nu')\, m_o\, c^2$$

$$\boxed{m^2c^4 = h^2(\nu^2 - 2\nu\,\nu' + \nu'^2) + 2h(\nu - \nu')\, m_o\, c^2 + m_o^2\, c^4} \quad ....(9)$$

Subtracting eqn (7) from eqn (9), we get

$$m^2c^4 - m^2v^2c^2 = h^2(\nu^2 - 2\nu\nu' + \nu'^2) + 2h(\nu - \nu')\, m_o\, c^2$$
$$+ m_o^2\, c^4 - h^2(\nu^2 - 2\nu\nu' \cos\theta + \nu'^2)$$

$$m^2\, c^2\, (c^2 - v^2) = h^2\nu^2 - 2h^2\, \nu\nu' + h^2\, \nu^{2\prime} + 2h\,(\nu - \nu')\, m_o\, c^2$$

$$+ m_o^2\, c^4 - h^2\nu^2 + 2h^2\nu\nu'\cos\theta - h^2\nu'^2$$

$$m^2c^2\,(c^2 - v^2) = -2h^2\nu\nu' + 2h(\nu - \nu')\, m_o c^2 + 2h^2\nu\nu'\cos\theta + m_o^2\, c^4$$

$$\boxed{m^2c^2\,(c^2 - v^2) = -2h^2\,\nu\nu'\,(1 - \cos\theta) + 2h\,(\nu - \nu')\, m_o\, c^2 + m_o^2\, c^4} \quad ...(10)$$

From the theory of relativity, the variation of mass with velocity is given by

$$m = \frac{m_o}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}} \quad ...\ (11)$$

Squaring the eqn (11) on both sides, we have

$$m^2 = \frac{m_o^2}{1 - \frac{v^2}{c^2}} = \frac{m_o^2}{\frac{c^2 - v^2}{c^2}} = \frac{m_o^2\, c^2}{c^2 - v^2}$$

$$m^2\,(c^2 - v^2) = m_o^2\, c^2$$

Multiplying c^2 on both sides, we have

$$m^2 c^2 (c^2 - v^2) = m_o^2 c^2 c^2$$

$$\boxed{m^2 c^2 (c^2 - v^2) = m_o^2 c^4} \quad ...(12)$$

Substituting eqn (12) in eqn (10), we get

$$m_o^2 c^4 = -2h^2 \nu\nu' (1 - \cos\theta) + 2h(\nu - \nu') m_o c^2 + m_o^2 c^4$$

$$2h(\nu - \nu') m_o c^2 = 2h^2 \nu \nu' (1 - \cos\theta)$$

or $$\frac{\nu - \nu'}{\nu\nu'} = \frac{h}{m_o c^2}(1 - \cos\theta)$$

$$\frac{\nu}{\nu\nu'} - \frac{\nu'}{\nu\nu'} = \frac{h}{m_o c^2}(1 - \cos\theta)$$

$$\boxed{\frac{1}{\nu'} - \frac{1}{\nu} = \frac{h}{m_o c^2}(1 - \cos\theta)} \quad ...(13)$$

Multiplying c on both sides of eqn (13), we have

$$\frac{c}{\nu'} - \frac{c}{\nu} = \frac{hc}{m_o c^2}(1 - \cos\theta)$$

$$\frac{c}{\nu'} - \frac{c}{\nu} = \frac{h}{m_o c}(1 - \cos\theta) \qquad \left[\because \frac{c}{\nu} = \lambda, \quad \frac{c}{\nu'} = \lambda'\right]$$

$$\lambda' - \lambda = \frac{h}{m_o c}(1 - \cos\theta)$$

Therefore, the change in wavelength is given by

$$\boxed{d\lambda = \frac{h}{m_o c}(1 - \cos\theta)} \quad ...(14)$$

It is found that the change in wavelength ($d\lambda$) does not depend on the wavelength of the incident radiation

and the nature of the scattering substance. But it depends only on the angle of scattering (θ).

Case - 1 When $\theta = 0$, then

$$d\lambda = \frac{h}{m_o c}(1 - \cos 0)$$

$$d\lambda = \frac{h}{m_o c}(1 - 1) \qquad [\because \cos 0 = 1]$$

$$= \frac{h}{m_o c} \times 0$$

$$\boxed{d\lambda = 0}$$

i.e. Along the incident direction, there is no change in wavelength

Case - 2 When $\theta = 90°$, then

$$d\lambda = \frac{h}{m_o c}(1 - \cos 90°)$$

$$d\lambda = \frac{h}{m_o c}(1 - 0) \qquad [\because \cos 90° = 0]$$

$$d\lambda = \frac{h}{m_o c}$$

Substituting for h, m_o and c, we have

$$d\lambda = \frac{6.625 \times 10^{-34}}{9.11 \times 10^{-31} \times 3 \times 10^{8}}$$

$$\boxed{d\lambda = 0.0243 \text{ Å}}$$

This difference in wavelength is known as ***Compton wavelength*** of electron.

Case - 3 When $\theta = 180°$, then

$$d\lambda = \frac{h}{m_o c}(1 - \cos 180°)$$

$$d\lambda = \frac{h}{m_o c}(1 - (-1)) \qquad [\because \cos 180° = -1]$$

$$d\lambda = \frac{h}{m_o c}(1 + 1) = \frac{2h}{m_o c}$$

$$d\lambda = \frac{2h}{m_o c}$$

$$d\lambda = 2 \times 0.0243\,\text{Å} \left[\because \frac{h}{m_o c} = 0.0243\,\text{Å}\right]$$

$$\boxed{d\lambda = 0.0486\,\text{Å}}$$

Thus, the change in wavelength is maximum at $\theta = 180°$.

Experimental verification of Compton effect

A beam of monochromatic X-rays of wavelength λ is made to incident on a scattering substance (Fig. 6.3). The scattered X-rays are received by Bragg spectrometer.

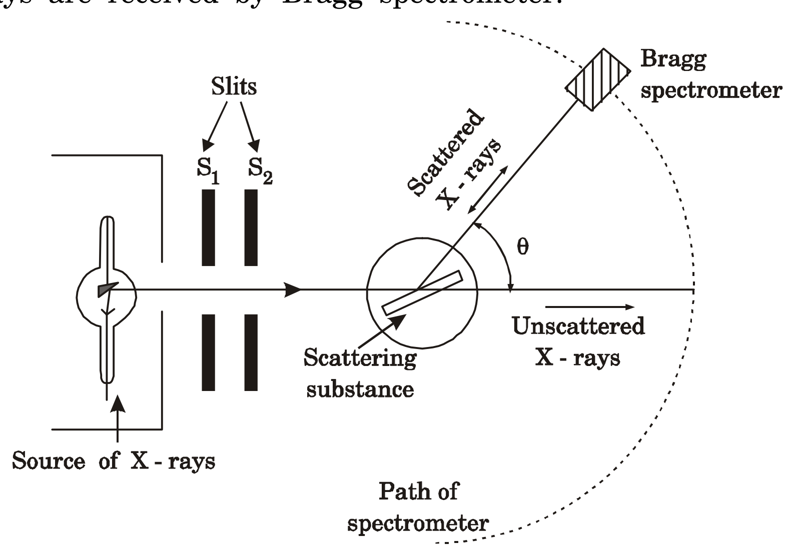


Fig. 6.3 Experimental verification of Compton effect

The intensity of scattered X-rays is measured for various scattering angles. The graph is plotted (intensity Vs wavelength) as shown in fig. 6.4.

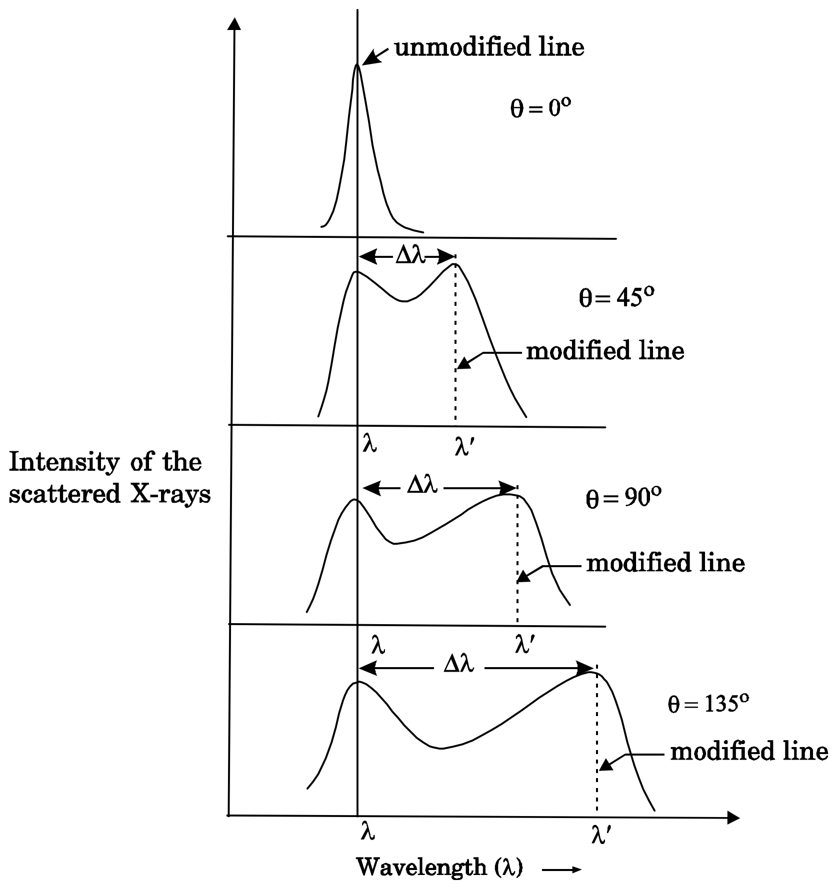


Fig. 6.4 Graph between intensity of the scattered X-rays and wavelength (λ)

It is found that the curves have two peaks, one corresponding to unmodified radiation and other corresponding to modified radiation.

The difference between two peaks on the wavelength axis gives **Compton shift**.

The curves show that the greater the scattering angle, the greater is Compton shift in accordance with the expression.

$$\lambda' - \lambda = d\lambda = \frac{h}{m_o c}(1 - \cos\theta)$$

The change in wavelength $d\lambda = 0.0243$ Å at $\theta = 90°$ is found to be in good agreement with the theoretical value 0.0243 Å. **Thus, Compton effect is experimentally verified**.

The experimental observations were in perfect agreement with the prediction of the above equation, thus holding the Einstein's concept of photon as completely valid.

Physical significance of compton effect

In the compton effect, the compton shift is explained on the basis that the X-ray photons collide with electrons. In the collision, the energy exchange between the two occurs as it is a kind of particle-particle collision.

Thus, it demonstrates the particle nature of X-rays which we know are electromagnetic waves, or in other words, it signifies the particle nature of waves in general.

6.3 ELECTRONS (Particles) AND MATTER WAVES - (Concept of Matter Waves)

- Particle nature of matter is very well established. Now it is known that matter is composed of atoms, electrons, protons and neutrons. They are the building blocks of all types of atoms.

- Electromagnetic theory and quantum theory established wave-particle duality of radiation. But inspite of the success of this duality of radiation, the postulates of Bohr's theory of atomic structure remained unexplained for a long time.

- On this background, in 1924 De Broglie extended the idea of dual nature of radiation to matter and proposed that matter possesses particle as well as wave characteristics.

- ***He believed that motion of electron within an atom is guided by a peculiar kind of waves called 'Pilot waves'***.
- While introducing the concept of matter waves, De Broglie was guided by wave - particle duality of radiation and the way in which nature manifests herself.

The concept of wave nature of matter is developed from the dual character of radiation which sometimes behaves as a wave and at other times as a particle.

de - Broglie's Hypothesis

Louis de - Broglie proposed a very bold and novel suggestion that **like light radiation, matter or material particle also posseses dual (two) characteristics i.e., particle – like and wave – like**.

The moving particles of matter such as electrons, protons, neutrons, atoms or molecules exhibit the wave nature in addition to particle nature.

According to de - Broglie hypothesis, a moving particle is always associated with waves. (Fig. 6.5)

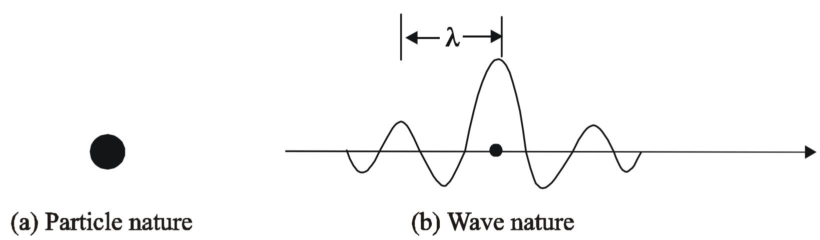


Fig. 6.5 particle and wave nature of moving particle

- Waves and particles are the only two modes through which energy can propagate in nature.
- Our universe is fully composed of light radiation and matter.
- Since **nature loves symmetry**, **matter and waves must be symmetric**.

- If electromagnetic radiation like light, X - rays can act like a wave and a particle, then material particles (electrons, protons etc) should also act like a particle and a wave.
- Every moving particle is always associated with a wave.

de - Broglie waves and its wavelength

The waves associated with the matter particles are called matter waves or de - Broglie waves.

From Planck's theory, the energy of a photon (particle nature) of frequency ν is given by

$$E = h\nu \quad \text{... (1)}$$

According to Einstein's mass - energy relation

$$E = mc^2 \quad \text{... (2)}$$

where m - mass of the photon

c - velocity of the photon.

Equating (1) and (2), we get

$$h\nu = mc^2 \quad \text{...(3)}$$

$$\frac{hc}{\lambda} = mc^2 \qquad \left(\therefore \ \nu = \frac{c}{\lambda} \right)$$

$$\lambda = \frac{hc}{mc^2}$$

$$\lambda = \frac{h}{mc} \text{ (for electromagnetic radiation)}$$

Since $mc = p$ momentum of a photon,

$$\text{then } \lambda = \frac{h}{p} \quad \text{...(4)}$$

The wavelength of de - Broglie wave associated with any moving particle of mass m with velocity v (momentum $p = mv$) is given by

$$\boxed{\lambda = \frac{h}{p} = \frac{h}{mv}} \quad ...(5)$$

This equation (5) is known as **de - Broglie's wave equation**.

de - Broglie wavelength in terms of energy

We know that the kinetic energy $E = \frac{1}{2} mv^2$

Multiplying by m on both sides we get,

$$mE = \frac{1}{2} m^2 v^2 \quad ...(6)$$

$$2mE = m^2 v^2$$

or $$m^2 v^2 = 2mE$$

Taking square root on both sides,

$$\sqrt{m^2 v^2} = \sqrt{2\,mE}$$

$$mv = \sqrt{2\,mE}$$

We know that $\lambda = \frac{h}{mv}$ ... (7)

substituting for mv in eqn (7), we have

de - Broglie wavelength $\boxed{\lambda = \frac{h}{\sqrt{2mE}}}$

de - Broglie's wavelength in terms of accelerating potential associated with electrons

When an electron of charge e is accelerated by a potential difference of V volts, then the electron gains a velocity v and hence,

Workdone on the electron $= eV$ ... (1)

This workdone is converted into the kinetic energy of the electron as $\frac{1}{2}mv^2$

$$\text{Workdone} = \text{kinetic energy}$$

$$e\text{V} = \frac{1}{2}mv^2 \quad \text{... (2)}$$

$$2\,eV = mv^2$$

$$mv^2 = 2\,e\text{V}$$

Multiply by m on both sides, we have

$$m^2v^2 = 2\,me\text{V}$$

Taking square root on both sides, we get

$$\sqrt{m^2v^2} = \sqrt{2me\text{V}}$$

$$mv = \sqrt{2me\text{V}} \quad \text{... (3)}$$

From the de - Broglie's concept, the wavelength associated with any moving particle is given by

$$\lambda = \frac{h}{mv} \quad \text{... (4)}$$

Substituting eqn (3) in eqn (4), we have

$$\boxed{\lambda = \frac{h}{\sqrt{2me\text{V}}}} \quad \text{... (5)}$$

Substituting the given values, we have

$$h = 6.625 \times 10^{-34}\,\text{Js}, \quad e = 1.6 \times 10^{-19}\,\text{C}$$

$$m = 9.1 \times 10^{-31}\,\text{kg}$$

$$\lambda = \frac{6.625 \times 10^{-34}}{\sqrt{2 \times 9.1 \times 10^{-31} \times 1.6 \times 10^{-19} \times V}}$$

$$\lambda = \frac{12.25 \times 10^{-10}}{\sqrt{V}} \text{ metre} \quad ...(6)$$

$$\lambda = \frac{12.25}{\sqrt{V}} \text{ Å} \quad ...(7)$$

Properties of Matter Waves

1. If the mass of the particle is smaller, then the wavelength associated with that particle is longer.
2. If the velocity of the particle is small, then the wavelength associated with that particle is longer.
3. If $v = 0$, then $\lambda = \infty$, i.e., the wave becomes indeterminate and if $v = \infty$, then $\lambda = 0$. This indicates that de - Broglie waves are generated by the motion of particles.
4. These waves do not depend on the charge of the particles. This shows that these waves are not electromagnetic waves.
5. The velocity of de - Broglie's waves is not constant since it depends on the velocity of the material particle.

ANNA UNIVERSITY SOLVED PROBLEMS

Problem 6.1

An electron is accelerated by a potential difference of 150 V. What is the wavelength of that electron wave?

(A.U. Jan 2017)

Given data

Accelerating voltage applied to the electron $V = 150$ V

Solution

We know that the de - Broglie wavelength

$$\lambda = \frac{12.25 \times 10^{-10}}{\sqrt{V}} \text{ metre}$$

Substituting the given values, we have

$$\lambda = \frac{12.25 \times 10^{-10}}{\sqrt{150}} \text{ metre}$$

$$\lambda = \frac{12.25 \times 10^{-10}}{12.24} \text{ metre}$$

$$\lambda = 1.001 \times 10^{-10} \text{ metre}$$

$$\boxed{\lambda = 1.001 \text{ Å}}$$

Problem 6.2

Calculate de – Broglie wavelength of an electron of energy 100 eV ***(A.U. Jan 2011)***

Given data

Energy of electron E = 100 eV

$$E = 100 \times 1.6 \times 10^{-19} \text{ joule}$$

$$(\because\ 1 \text{ eV} = 1.6 \times 10^{-19} \text{ J})$$

Mass of the electron $m = 9.1 \times 10^{-31}$ kg

Planck's constant $h = 6.625 \times 10^{-34}$ Js

Solution

We know that $\lambda = \frac{h}{mv} = \frac{h}{\sqrt{2mE}}$

Substituting the given values, we have

$$\lambda = \frac{6.625 \times 10^{-34}}{\sqrt{2 \times 9.1 \times 10^{-31} \times 100 \times 1.6 \times 10^{-19}}}$$

de - Broglie's wavelength $\lambda = 1.235 \times 10^{-10}$ metre

$$\boxed{\lambda = \mathbf{1.235\ \text{Å}}}$$

Note: At the time de - Broglie proposed his hypothesis, there was no experimental evidence to show that the particles of matter like electrons, protons, neutrons etc., have wave properties.

However, the wave nature of atomic particles was experimentally verified by the American scientists Davisson and Germer in 1927 and by G.P. Thomson in 1928 and thus the truth of de Broglie concept was established.

Concept of Wave Function

In quantum mechanics it is postulated that there exists a function determined by the physical situation. The function is called **wave function**.

It is also postulated to contain all possible information about the system. Hence, it is also called as **state function**.

The wave function may vary with respect to both the position coordinates of the physical system and the time. It is then called the **total wave function**, and is represented by the capital form of the greek letter Ψ (pronounced as psi).

If the wave function has variation only with position (i.e., space) and not with time, it is represented by the lower case greek letter ψ.

The total wave function can be represented by the equation,

$$\Psi = Ae^{i\,(kx - \omega t)} \quad ...(1)$$

where, A is a constant, and ω is the angular frequency of the wave.

Separating the space and time dependent parts, Ψ can be expressed as,

$$\Psi = Ae^{ikx}\,e^{-i\,\omega t} \quad ...(2)$$

In the above equation, separating out the time dependent part, we can represent the rest as a space dependent wave function.

$$\psi = Ae^{ikx} \quad ...(3)$$

ψ is time independent wave function. The total wave function is now written as,

$$\Psi = \psi\, e^{-i\,\omega t}$$

The wave function in quantum mechanics accounts for the wave-like properties of a particle.

It is obtained by solving a fundamental equation called **Schrondinger equation**. To solve the schrodinger equation it requires the knowledge of,

(i) **potential energy of the particle**,

(ii) **initial conditions, and**

(iii) **boundary conditions**.

6.4 SCHRODINGER WAVE EQUATION

Schrodinger wave equation describes the wave nature of a particle in mathematical form. It is the basic equation of motion for matter waves.

If the particle has wave properties, then there should be some sort of wave equation to describe the behaviour of that particle.

Schrodinger connected the expression of de-Broglie's wavelength with the classical wave equation for a moving particle. He obtained a new wave equation. This wave equation is known as **Schrodinger wave equation**.

Forms of Schrodinger wave equations

There are two forms of Schrodinger wave equations.

They are:

(a) Time independent wave equation

(b) Time dependent wave equation

One, which is general and takes care of both the position and the time variations of the wave function, is called **time-dependent Schrodinger equation**.

It involves the imaginary quantity '*i*'.

The other one is applicable only to steady state conditions in which case, the wave function can have variation only with position but not with time. It is called **time-independent Schrodinger's equation** and is simpler than the other one. It doesn't involve '*i*'.

Accordingly we get the corresponding wave functions by solving the respective types of schrodinger equation. The wave functions obtained as solutions of time-dependent schrodinger equation will always be complex.

But the wave functions obtained as solutions of time-independent schrodinger equation are not necessarily complex functions, though they could be complex under certain conditions.

6.5 SCHRODINGER TIME INDEPENDENT WAVE EQUATION (Derivation)

Consider a wave associated with a moving particle.

Let x, y, z be the coordinates of the particle and ψ wave function for de - Broglie's waves at any given instant of time t. (Fig 6.6)

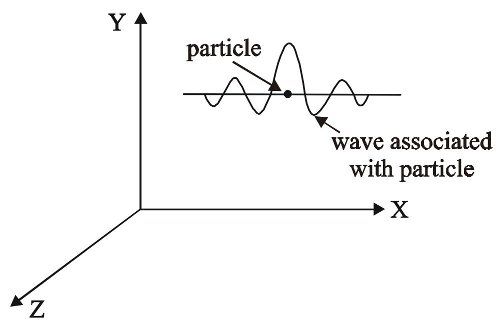


Fig 6.6

The classical differential equation for wave motion is given by

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2} \quad ...(1)$$

Here, v is wave velocity.

The eqn (1) is written as

$$\nabla^2\psi = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2} \quad ...(2)$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the Laplacian's operator.

The solution of eqn (2) gives ψ as a periodic variations in terms of time t,

$$\psi(x, y, z, t) = \psi_o(x, y, z)\, e^{-i\omega t}$$

$$\psi = \psi_o\, e^{-i\omega t} \quad ...(3)$$

Here, $\psi_o(x, y, z)$ is a function of x, y, z only which is the amplitude at the point considered. ω is angular velocity of the wave.

Differentiating the eqn (3) with respect to t, we get

$$\frac{\partial \psi}{\partial t} = -i\,\omega\,\psi_o\,e^{-i\omega t}$$

Again differentiating with respect to t, we have

$$\frac{\partial^2 \psi}{\partial t^2} = (-i\omega)(-i\omega)\,\psi_o\,e^{-i\omega t}$$

$$\frac{\partial^2 \psi}{dt^2} = i^2\,\omega^2 \psi_o\,e^{-i\omega t}$$

$$\frac{\partial^2 \psi}{\partial t^2} = -\omega^2 \psi \qquad \ldots (4)$$

$$\left[\because\ i^2 = -1 \quad \psi = \psi_o\,e^{-i\omega t}\right]$$

Substituting eqn (4) in eqn (2), we have

$$\nabla^2 \psi = -\frac{\omega^2}{v^2}\psi$$

$$\nabla^2 \psi + \frac{\omega^2}{v^2}\psi = 0 \qquad \ldots (5)$$

We know that angular frequency $\omega = 2\pi\nu = 2\pi\left(\frac{v}{\lambda}\right)$

Here, ν is the frequency $\left(\because\ \nu = \frac{v}{\lambda}\right)$

$$\frac{\omega}{v} = \frac{2\pi}{\lambda} \qquad \ldots (6)$$

Squaring the eqn (6) on both sides, we get

$$\frac{\omega^2}{v^2} = \frac{2^2\pi^2}{\lambda^2} = \frac{4\pi^2}{\lambda^2} \quad \text{... (7)}$$

Substituting eqn (7) in eqn (5), we have

$$\nabla^2\psi + \frac{4\pi^2}{\lambda^2}\psi = 0 \quad \text{... (8)}$$

on substituting, $\lambda = \frac{h}{mv}$ in eqn (8), We get

$$\nabla^2\psi + \frac{4\pi^2}{\frac{h^2}{m^2v^2}}\psi = 0$$

$$\nabla^2\psi + \frac{4\pi^2 m^2 v^2}{h^2}\psi = 0 \quad \text{... (9)}$$

If E is total energy of the particle, V is potential energy and $\frac{1}{2}mv^2$ is kinetic energy, then

Total energy = Potential energy + Kinetic energy

$$E = V + \frac{1}{2}mv^2$$

$$\text{or} \quad E - V = \frac{1}{2}mv^2$$

$$2(E - V) = mv^2$$

$$mv^2 = 2(E - V)$$

Multiplying by m on both sides, we have

$$m^2v^2 = 2m(E - V) \quad \text{... (10)}$$

Substituting eqn (10) in eqn (9), we get

$$\nabla^2\psi + \frac{4\pi^2}{h^2} \times 2m(E - V)\,\psi = 0$$

$$\boxed{\nabla^2\psi + \frac{8\pi^2 m}{h^2}(E - V)\,\psi = 0} \quad \ldots (11)$$

The eqn (11) is known as Schrodinger time independent wave equation for three dimensions.

Let us now introduce $\hbar = \frac{h}{2\pi}$ in eqn (11),

$$\hbar^2 = \frac{h^2}{2^2\pi^2} = \frac{h^2}{4\pi^2} \quad \ldots (12)$$

where $\hbar$ is a reduced Planck's constant

The eqn (11) is modified by substituting $\hbar$,

$$\nabla^2\psi + \frac{m}{\frac{h^2}{8\pi^2}}(E - V)\,\psi = 0$$

$$\nabla^2\psi + \frac{m}{\frac{h^2}{2 \times 2^2\pi^2}}(E - V)\,\psi = 0$$

$$\nabla^2\psi + \frac{2m}{\frac{h^2}{4\pi^2}}(E - V)\,\psi = 0 \quad \ldots (13)$$

on substituting eqn (12) in eqn (13), Schrodinger time-independent wave equation is written as

$$\boxed{\nabla^2\psi + \frac{2m}{\hbar^2}(E - V)\,\psi = 0} \quad \ldots (14)$$

(or)

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = E\psi \quad \ldots (15)$$

Note: In eqn (14), there is no term representing time. That is why it is called as time independent equation.

Special case

If we consider one-dimensional motion ie., particle moving along only X - direction, then Schrodinger time independent equation (14) reduces to

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}(E - V)\psi = 0 \quad \ldots (16)$$

6.6 SCHRODINGER TIME DEPENDENT WAVE EQUATION

Schrodinger time dependent wave equation is derived from Schrodinger time independent wave equation.

The solution of classical differential equation of wave motion is given by

$$\psi(x, y, z, t) = \psi_0(x, y, z)\, e^{-i\omega t} \quad \ldots (1)$$

Differentiating eqn (1) with respect to time t, we get

$$\frac{\partial\psi}{\partial t} = -i\,\omega\,\psi_0\, e^{-i\omega t} \quad \ldots (2)$$

$$\frac{\partial\psi}{\partial t} = -i\,(2\pi\nu)\,\psi_0\, e^{-i\omega t} \quad (\because \omega = 2\pi\nu)$$

$$\frac{\partial\psi}{\partial t} = -2\pi i\nu\psi \quad \ldots (3)$$

$$(\because \psi = \psi_o\, e^{-i\omega t})$$

$$\frac{\partial \psi}{\partial t} = -2\pi i \frac{E}{h} \psi \qquad \left(\because E = h\nu \text{ or } \nu = \frac{E}{h} \right)$$

$$\frac{\partial \psi}{\partial t} = -i \frac{E}{\frac{h}{2\pi}} \psi = -i \frac{E}{\hbar} \psi \qquad \left[\because \hbar = \frac{h}{2\pi} \right]$$

$$\frac{\partial \psi}{\partial t} = -i \frac{E}{\hbar} \psi \qquad \text{... (4)}$$

Multiplying i on both sides in eqn (4), we have

$$i \frac{\partial \psi}{\partial t} = -i \times i \left(\frac{E}{\hbar} \right) \psi = -i^2 \left(\frac{E}{\hbar} \right) \psi$$

$$i \frac{\partial \psi}{\partial t} = \frac{E}{\hbar} \psi \qquad [\because i \times i = i^2 = -1]$$

$$i \hbar \frac{\partial \psi}{\partial t} = E\psi \qquad \text{... (5)}$$

Schroedinger time independent wave equation is

$$\boxed{-\frac{\hbar^2}{2m} \nabla^2 \psi + V\psi = E\psi}$$

Substituting for Eψ from eqn (5)

$$-\frac{\hbar^2}{2m} \nabla^2 \psi + V \psi = i \hbar \frac{\partial \psi}{\partial t}$$

$$\Rightarrow \left(-\frac{\hbar^2}{2m} \nabla^2 + V \right) \psi = i \hbar \frac{\partial \psi}{\partial t} \qquad \text{... (7)}$$

$$\text{or} \quad \boxed{H\psi = E\psi} \qquad \text{... (8)}$$

where $H = \left(-\frac{\hbar^2}{2m} \nabla^2 + V \right)$ is Hamiltonian operator

$E = i\hbar \frac{\partial}{\partial t}$ is energy operator.

The eqn (7) is known as **Schrodinger time dependent wave equation**.

> ***Note:*** In eqn (7), there is term $\frac{\partial \psi}{\partial t}$ representing time. That is why it is called **time dependent wave equation**.

6.7 MEANING OR PHYSICAL SIGNIFICANCE OF WAVE FUNCTION ψ

1. The variable quantity which describes de-Broglie wave is called **wave function ψ.**
2. It connects the particle nature and its associated wave nature statistically.
3. The wave function associated with a moving particle at a particular instant of time and at a particular point in space is related to the probability of finding the particle at that instant and at that point.
4. The probability 0 corresponds to the certainty of not finding the particle and probability 1 corresponds to certainty of finding the particle.

   i.e., $\iiint \psi^* \psi \, d\tau = 1$, if particle is present.

   $= 0$, if particle is not present.

   where ψ^* – complex conjugate of ψ
5. The probability of finding a particle at a particular region must be real and positive, but the wave function ψ is in general a complex quantity.

Applications of Schrodinger's time independent wave equation

6.8 MOTION OF A FREE PARTICLE

Let us consider electrons propagating freely in space in the positive x-direction and not acted upon by any force.

As the electrons are not acted upon by any force, their potential energy V is zero. Schrodinger equation

$$\nabla^2 \psi + \frac{8\pi^2 m}{\hbar^2}(E - V)\psi = 0 \text{ reduces to}$$

$$\boxed{\frac{d^2\psi}{dx^2} + \frac{8\pi^2 m}{h^2} E\psi = 0} \qquad \ldots (1)$$

Taking $\frac{8\pi^2 mE}{\hbar^2} = k^2$ in the above equation, we get

$$\boxed{\frac{d^2\psi}{dx^2} + k^2\psi = 0}$$

The general solution of the above equation is

$$\boxed{\psi(x) = Ae^{ikx} + Be^{-ikx}}$$

where A and B are constants. As it is assumed that the waves propagate only in the positive x-direction, we can write

$$\boxed{\psi(x, t) = Ae^{ikx} e^{-i\omega t}}$$

There are no boundary conditions to be considered and hence there are no restrictions on k. All values of the energy are allowed. The allowed energy values form a continuum and are given by

$$\boxed{E = \frac{h^2 k^2}{8\pi^2 m}} \qquad \ldots (2)$$

A freely moving electron therefore possess a continuous energy spectrum as shown in fig. 6.7.

It is noted from equation (2) that

$$k = \sqrt{\frac{2mE}{\hbar^2}} \qquad (\because\ p = \sqrt{2\,m\,E})$$

$$k = \frac{\sqrt{2\,m\,E}}{\hbar} = \frac{p}{\hbar} \qquad \left(\frac{p}{\hbar} = \frac{1}{\lambda}\right)$$

$$= \frac{p}{\frac{\hbar}{2\pi}} = \frac{2\pi p}{\hbar} = \frac{2\pi}{\lambda} \qquad \text{(de Broglie equation)}$$

The k known as wave vector describes the wave properties of the electrons. Further, it is seen from the relation (2) that

$$\boxed{E \propto k^2}$$

The plot of E as a function of k gives a parabola, as explained in fig. 6.8.

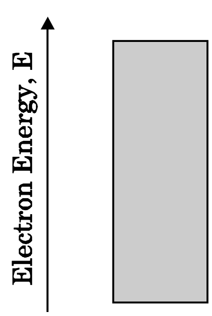


Fig. 6.7: The energy continuum of free electron

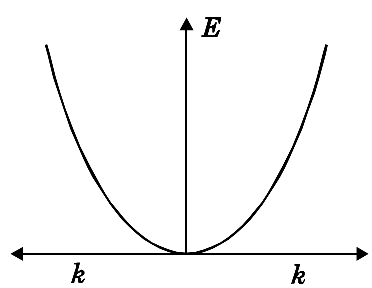


Fig. 6.8: The parabolic relationship between energy and *E* and wave vector *k* in case of a free electron.

The momentum is well defined in this case. Therefore, according to uncertainty principle it is difficult to assign a position to the electron. The uncertainty in position will be infinity which means that the electron position is indeterminate.

6.9 PARTICLE IN A INFINITE POTENTIAL (One - Dimensional Box)

Consider a particle of mass m moving between two rigid walls of a box or infinite deep potential well at $x = 0$ and $x = a$ along x-axis.

This particle is bouncing back and forth between the walls of the infinite well. The potential energy (V) of the particle inside the box is constant. It is taken as zero for simplicity (Fig. 6.9).

The walls are infinitely high. The potential energy V of the particle is infinite outside the walls.

Thus, the potential function is given by

$$V(x) = 0 \text{ for } 0 < x < a$$

$$V(x) = \infty \text{ for } 0 \geq x \geq a$$

This potential function is known as **square well potential**. (Fig. 6.9)

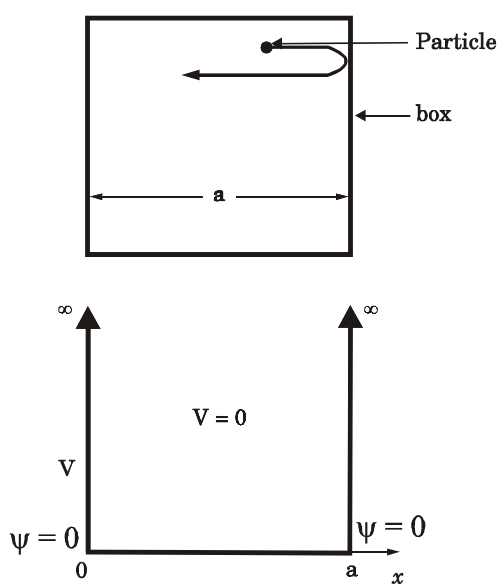


Fig. 6.9 Particle in a one-dimensional rigid box (infinite deep potential well)

The particle cannot come out of the box well. Also, it can not exist on the walls of the box. So, its wave function ψ is 0 for $x \leq 0$ and $x \geq a$. **Now, task is to find the value ψ within the box i.e., between $x = 0$ and $x = a$.**

Schroedinger's wave equation in one-dimension is given by

$$\boxed{\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}(E - V)\psi = 0} \quad \text{... (1)}$$

Since V = 0 between the walls, the eqn (1) reduces to

$$\frac{d^2\psi}{dx^2} + \frac{2mE}{\hbar^2}\psi = 0 \quad \text{... (2)}$$

Substituting $\frac{2mE}{\hbar^2} = k^2$ in eqn (2), we get

$$\boxed{\frac{d^2\psi}{dx^2} + k^2\psi = 0} \quad \text{... (3)}$$

The general solution of eqn (3) is given by

$$\boxed{\psi(x) = A \sin kx + B \cos kx} \quad \text{... (4)}$$

Here, A and B are two unknown constants.

The values of the constants A and B are determined by applying the boundary conditions.

Boundary condition (i)

$$\boldsymbol{\psi = 0 \text{ at } x = 0}$$

Applying this condition to eqn (4), we have

$$0 = A \sin 0 + B \cos 0 \quad \left[\because \begin{array}{l} \sin 0 = 0 \\ \cos 0 = 1 \end{array}\right]$$

$$0 = 0 + B \times 1$$

Hence, $\quad B = 0$

Boundary condition (ii)

$$\psi = 0 \text{ at } x = a$$

Applying this condition to eqn (4), we have

$$0 = A \sin ka + 0 \qquad [\because B = 0]$$

$$A \sin ka = 0$$

It is found that either $A = 0$ or $\sin ka = 0$

A cannot be '0' since already one of the constants B is '0'. If A is also 0, then the wave function is zero even in between walls of the box. Hence, A should not be zero.

$$\therefore \quad \sin ka = 0$$

$\sin ka$ is '0' only when ka takes the value of $n\pi$

$$\text{ie.,} \quad ka = n\pi$$

where n is positive integer 1, 2, 3 ...

$$k = \frac{n\pi}{a} \quad \text{... (5)}$$

On squaring eqn (5), we have

$$k^2 = \frac{n^2\pi^2}{a^2} \quad \text{... (6)}$$

We know that $k^2 = \dfrac{2mE}{\hbar^2} = \dfrac{2mE}{\dfrac{h^2}{4\pi^2}}$ $\qquad \left[\because \hbar = \dfrac{h}{2\pi}\right]$

$$k^2 = \frac{(2m \times 4\pi^2)E}{h^2}$$

$$k^2 = \frac{8\pi^2 mE}{h^2} \quad \text{... (7)}$$

Equating eqn (6) and eqn (7), we have

$$\frac{n^2\pi^2}{a^2} = \frac{8\pi^2 mE}{h^2}$$

Energy of the particle

$$\boxed{E_n = \frac{n^2h^2}{8ma^2}} \quad \text{... (8)}$$

substituting eqn (5) in eqn (4), we have

$$\boxed{\psi_n(x) = A\sin\frac{n\pi x}{a}} \quad \text{... (9)}$$

Here $n = 1, 2, 3 \ldots$

For each value of n, there is an **energy level**.

The particle in a box cannot possess any arbitrary amount of energy. It can only have discrete energy values specified by eqn (8).

In otherwords, its energy is quantised.

Each value of $\mathbf{E_n}$ is known as eigen value and the corresponding ψ_n is called as eigen function.

6.10 NORMALISATION OF WAVE FUNCTION

The constant A is determined by normalisation of wave function as follows.

Probability density is given by $\psi^* \psi$

We know that $\psi_n(x) = A\sin\frac{n\pi x}{a}$

$$\psi^* \psi = A\sin\frac{n\pi x}{a} \times A\sin\frac{n\pi x}{a}$$

[$\because \psi = \psi^*$ the wave function is real (not complex)]

$$\psi^* \psi = A^2 \sin^2\left[\frac{n\pi x}{a}\right] \quad \text{... (10)}$$

It is certain that the particle is some where inside the box. Thus, the probability of finding the particle inside the box of length a is given by

$$\int_0^a \psi^* \psi \, dx = 1 \qquad \text{... (11)}$$

Substituting $\psi^*\psi$ from eqn (10) in eqn (11), we have

$$\int_0^a A^2 \sin^2\left(\frac{n\pi x}{a}\right) dx = 1$$

$$A^2 \int_0^a \frac{\left(1 - \cos\left(\frac{2n\pi x}{a}\right)\right)}{2} dx = 1 \quad \left[\because \sin^2\theta = \frac{1-\cos 2\theta}{2}\right]$$

$$\frac{A^2}{2}\left[\int_0^a dx - \int_0^a \cos\left(\frac{2n\pi x}{a}\right) dx\right] = 1$$

$$\frac{A^2}{2}\left[[x]_0^a - \left[\frac{\sin\left(\frac{2n\pi x}{a}\right)}{\frac{2n\pi}{a}}\right]\right]_0^a = 1$$

The second term of the integral becomes zero at both limits

$$\frac{A^2}{2}[x]_0^a = 1$$

Thus, $\dfrac{A^2 a}{2} = 1$ or $A^2 = \dfrac{2}{a}$

$$\boxed{A = \sqrt{\frac{2}{a}}} \qquad \text{... (12)}$$

On substituting eqn (12) in eqn (9), we have

The eigen function (ψ_n) belongs to eigen energy values E_n and it is expressed as

$$\boxed{\psi_n = \sqrt{\frac{2}{a}} \ \sin \frac{n \pi x}{a}} \qquad \text{... (13)}$$

This expression (13) is known as **normalised eigen function.** The energy E_n and normalised wave functions ψ_n are shown in fig. 6.10.

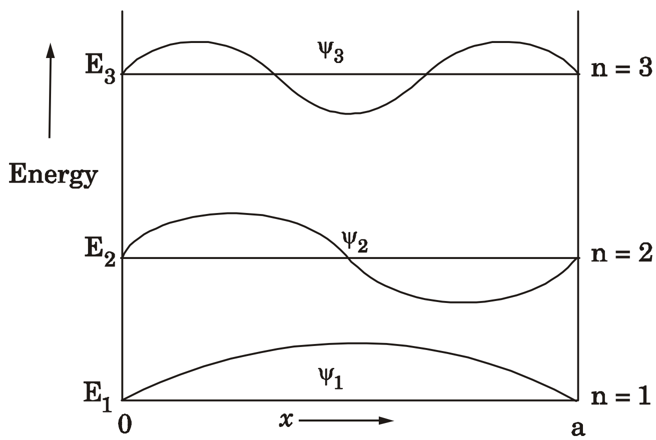


Fig. 6.10 Energy levels and wave functions

Special cases

From eqns (8) and (13), the following cases can be taken and they explain the motion of electron in one dimensional box.

Case (i) : For $n = 1$

$$E_1 = \frac{h^2}{8ma^2}$$

$$\psi_1(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{\pi x}{a}\right)$$

Hence, $\psi_1(x)$ is maximum at exactly middle of the box as shown in fig. 6.10.

Case (ii) : For $n = 2$

$$E_2 = \frac{4h^2}{8ma^2} = 4E_1$$

$$\psi_2(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{2\pi x}{a}\right)$$

Hence, $\psi_2(x)$ is maximum at quarter distance from either sides of the box as shown in 6.10.

Case (iii): For $n = 3$

$$E_3 = \frac{9h^2}{8ma^2} = 9E_1$$

$$\psi_3(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{3\pi x}{a}\right)$$

Hence, $\psi_3(x)$ is maximum at exactly middle and one-sixth distance from either sides of the box as shown in fig. 6.10.

ANNA UNIVERSITY SOLVED PROBLEM

Problem 6.3

Find the energy of an electron moving in one - dimension in an infinitely high potential box of width 0.1 nm.

[A.U. Jan 2019]

Given data

$$a = 0.1 \text{ nm} = 0.1 \times 10^{-9} \text{ m}$$

Solution

We know that energy of an electron

$$E_n = \frac{n^2 h^2}{8ma^2}$$

where $n = 1, 2, 3, \ldots$

The lowest energy of the particle is obtained for $n = 1$.

$$E = \frac{1^2 h^2}{8ma^2} = \frac{h^2}{8ma^2}$$

Substituting the given values, we have

$$E = \frac{(6.625 \times 10^{-34})^2}{8 \times 9.11 \times 10^{-31} \times (0.1 \times 10^{-9})^2}$$

$$E = 6.022 \times 10^{-18} \text{ J}$$

$$E = \frac{6.022 \times 10^{-18}}{1.6 \times 10^{-19}}$$

$$\boxed{E = 37.69 \text{ eV}}$$

6.11 EXTENSION TO TWO DIMENSIONS (2D Boxes)

The solution of one-dimensional potential well is extended for a two-dimensional potential well.

In a two-dimensional potential well, the particle (electron) can freely move in two directions (say x and y) direction. Therefore, instead of one quantum number n, we have to use two quantum numbers, n_x and n_y corresponding to the two coordinate axes namely x and y respectively.

If a and b are the lengths of the well as shown in fig. 6.11 along x and y axes, then

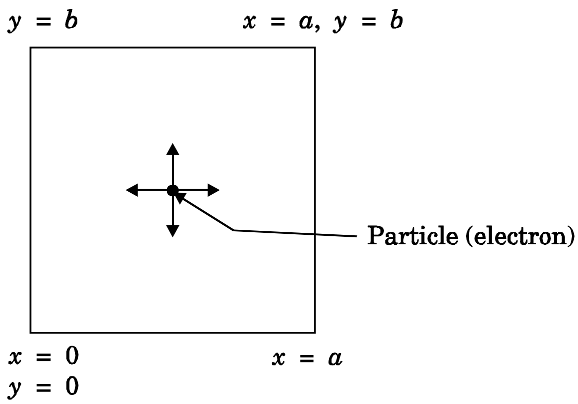


Fig. 6.11 Particle in 2-Dimensional box

Energy of the particle $E = E_{n_x} + E_{n_y}$

$$\text{i.e., } E_{n_x n_y} = \frac{n_x^2 h^2}{8ma} + \frac{n_y^2 h^2}{8ma^2}$$

If $a = b$

$$\boxed{E_{n_x n_y} = \frac{h^2}{8m}\left[\frac{n_x^2}{a^2} + \frac{n_y^2}{a^2}\right]}$$

$$E_{n_x n_y} = \frac{h^2}{8ma^2}[n_x^2 + n_y^2]$$

The corresponding normalised wave function of the particle in the two dimensional well is written as

$$\psi_{n_x n_y} = \sqrt{\frac{2}{a}} \sin\left(\frac{n_x \pi x}{a}\right) \times \sqrt{\frac{2}{b}} \sin\left(\frac{n_y \pi y}{b}\right)$$

$$\psi_{n_x n_y} = \sqrt{\frac{2}{a}} \times \sqrt{\frac{2}{b}} \sin\left(\frac{n_x \pi x}{a}\right) \sin\left(\frac{n_y \pi y}{b}\right)$$

$$\therefore \quad \boxed{\psi_{n_x n_y} = \sqrt{\frac{4}{ab}} \left(\sin\frac{n_x \pi x}{a}\right) \sin\left(\frac{n_y \pi y}{b}\right)} \quad \text{...(2)}$$

From equations (1) and (2), we understand that several combinations of the two quantum numbers (n_x and n_y) lead to different energy eigen values and eigen functions.

Example

Suppose a state has quantum numbers

$$n_x = 1, \quad n_y = 2$$

$$\text{Then,} \quad n_x^2 + n_y^2 = 1^2 + 2^2 = 1 + 4 = 5$$

Similarly, for a combination $n_x = 2$, $n_y = 1$

we have $n_x^2 + n_y^2 = 2^2 + 1^2 = 4 + 1 = 5$

$$\therefore \quad \boxed{E_{12} = E_{21} = \frac{5h^2}{8ma^2}} \quad \text{...(3)}$$

The corresponding wave functions is written as

$$\boxed{\psi_{12} = \sqrt{\frac{4}{ab}} \sin\frac{\pi x}{a} \sin\frac{2\pi y}{b}}$$

$$\boxed{\psi_{21} = \sqrt{\frac{4}{ab}} \sin\left(\frac{2\pi x}{a}\right) \sin\left(\frac{\pi y}{b}\right)} \quad \text{... (4)}$$

Note: Example for particle in two dimensional infinite well is quantum well.

6.12 EXTENSION TO INFINITE WELL THREE DIMENSIONS (3D Box)

The solution of one-dimensional potential well is extended for a three-dimensional (3D) potential box.

In a three-dimensional potential box, the particle (electron) can move in any direction in space. Therefore, instead of one quantum number n, we have to use three quantum numbers, n_x, n_y and n_z, corresponding to the three coordinate axes namely x, y and z respectively.

If a, b, c are the lengths of the box as shown in fig. 6.12 along x, y and z axes, then

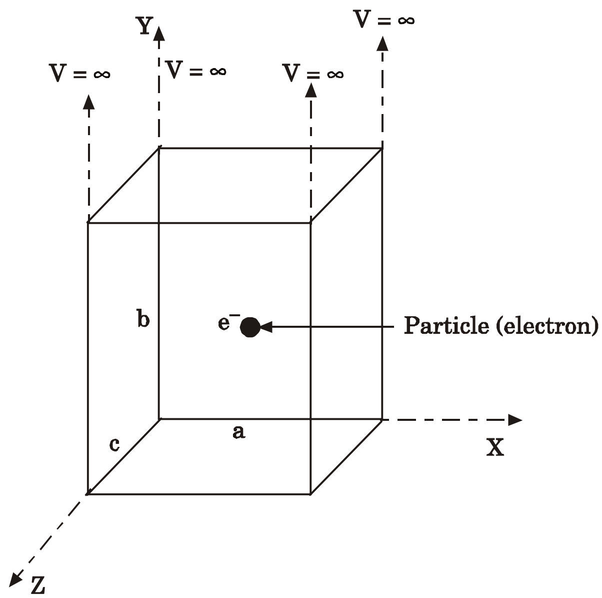


Fig. 6.12 Particle in a three-dimensional box

Energy of the particle $= E_x + E_y + E_z$

$$\text{i.e., } E_{n_x n_y n_z} = \frac{n_x^2 h^2}{8ma^2} + \frac{n_y^2 h^2}{8mb^2} + \frac{n_z^2 h^2}{8mc^2}$$

If $a = b = c$ as for a ***cubical box***, then

$$E_{n_x n_y n_z} = \frac{h^2}{8m}\left[\frac{n_x^2}{a^2} + \frac{n_y^2}{a^2} + \frac{n_z^2}{a^2}\right]$$

$$\boldsymbol{E_{n_x n_y n_z} = \frac{h^2}{8ma^2}[n_x^2 + n_y^2 + n_z^2]} \quad ...(1)$$

The corresponding normalised wave function of the particle in the three dimension well is written as

$$\psi_{n_x n_y n_z} = \sqrt{\frac{2}{a}}\sin\left(\frac{n_x \pi x}{a}\right) \cdot \sqrt{\frac{2}{b}}\sin\left(\frac{n_y \pi y}{b}\right) \cdot \sqrt{\frac{2}{c}}\sin\left(\frac{n_z \pi z}{c}\right)$$

$$\psi_{n_x n_y n_z} = \sqrt{\frac{2}{a} \times \frac{2}{b} \times \frac{2}{c}}\sin\left(\frac{n_x \pi x}{a}\right)\sin\left(\frac{n_y \pi y}{b}\right)\sin\left(\frac{n_z \pi z}{c}\right)$$

$$\therefore\ \boldsymbol{\psi_{n_x n_y n_z} = \sqrt{\frac{8}{abc}}\sin\left(\frac{n_x \pi x}{a}\right)\sin\left(\frac{n_y \pi y}{b}\right)\sin\left(\frac{n_z \pi z}{c}\right)} \quad ...(2)$$

From equations (1) and (2), we understand that several combinations of the three quantum numbers (n_x, n_y and n_z) lead to different energy eigen values and eigen functions.

Note: Three dimensional infinite potential well is an example of quantion dot.

Example

Suppose a state has quantum numbers

$$n_x = 1, \quad n_y = 1, \quad n_z = 2$$

Then, $\quad n_x^2 + n_y^2 + n_z^2 = 1^2 + 1^2 + 2^2 = 1 + 1 + 4 = 6$

Similarly, for a combination $n_x = 1$, $n_y = 2$, $n_z = 1$ and for a combination $n_x = 2$, $n_y = 1$, $n_z = 1$

we have $n_x^2 + n_y^2 + n_z^2 = 1^2 + 1^2 + 2^2 = 1 + 1 + 4 = 6$

$$\therefore \quad \boldsymbol{E_{112} = E_{121} = E_{211} = \frac{6h^2}{8ma^2}} \quad ...(3)$$

The corresponding wave functions is written as

$$\psi_{112} = \sqrt{\frac{8}{a^3}} \sin\frac{\pi x}{a} \sin\frac{\pi y}{b} \sin\frac{2\pi z}{c}$$

$$\psi_{121} = \sqrt{\frac{8}{a^3}} \sin\frac{\pi x}{a} \sin\frac{2\pi y}{b} \sin\frac{\pi z}{c} \quad ... (4)$$

$$\psi_{211} = \sqrt{\frac{8}{a^3}} \sin\frac{2\pi x}{a} \sin\frac{\pi y}{b} \sin\frac{\pi z}{c}$$

Degeneracy

It is noted seen from equations (3) and (4) that, for several combinations of quantum numbers, we have the same energy eigen value but different eigen functions. Such a state of energy levels is called ***degenerate state.***

The three combinations of quantum numbers, (112), (121) and (211), which give the same eigen value but different eigen functions are called 3-fold degenerate state.

Non-degenerate state:

When only one wave function corresponds to the energy eigen vlave, such a state is called non-degenerate state.

Suppose

$n_x = 2, \quad n_y = 2, \quad n_z = 2.$

Then, $\quad E_{222} = \dfrac{12\,h^2}{8ma^2}$

and $\psi_{222} = \sqrt{\dfrac{8}{a^3}} \sin\dfrac{2\pi x}{a} \sin\dfrac{2\pi y}{a} \sin\dfrac{2\pi z}{a}$

6.13 PROBABILITY DENSITY

Probability of finding the particle between positions x and $x + dx$

$$P(x) = |\psi_n|^2 dx = \frac{2}{a}\sin^2\left(\frac{n\pi x}{a}\right)dx$$

$$\therefore \text{ Probability density, } P(x) = \frac{2}{a}\sin^2\left(\frac{n\pi x}{a}\right)$$

Probability density is maximum when

$$\frac{n\pi x}{a} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots\ldots$$

or
$$x = \frac{a}{2n}, \frac{3a}{2n}, \frac{5a}{2n}, \ldots\ldots$$

- For $n = 1$ (lowest energy state), $x = \frac{a}{2}$, *i.e.,* the particle is most likely to be in the middle of the box (because $|\psi_1|^2$ is maximum there).
- For $n = 2$ (next energy state), $x = \frac{a}{4}$ and $\frac{3a}{4}$ *i.e.,* the particle is most likely to be at $\frac{a}{4}$ and $\frac{3a}{4}$ and never found in the middle because $|\psi_2|^2$ is zero there.
- For $n = 3$, the most likely positions of particle are $x = \frac{a}{6}, \frac{3a}{6}, \frac{5a}{6}$

The variation of probability densities $|\psi_1|^2$, $|\psi_2|^2$ and $|\psi_3|^2$ (for $n = 1, 2, 3$) with x is shown in fig. 6.13.

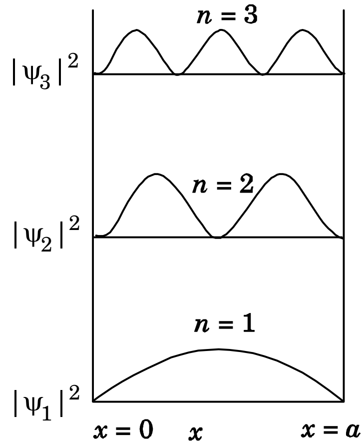


Fig. 6.13

It is noted that quantum mechanical results as vary drastically from the classical results.

While classical mechanics predicts the same probability for the particle being anywhere in the box, quantum mechanics predicts that the probability is different at different points and there are points (nodes) where the particle is never found.

6.14 PARTICLE IN A RECTANGULAR THREE - DIMENSIONAL INFINITE WELL

Let a particle of mass m be in motion in a rectangular potential deep potential (Fig. 6.14) with sides of lengths a, b, c, parallel to the x, y and z-axes respectively.

If there is no force acting on the particle inside the box, so that in the region.

$$0 < x < a$$

$$0 < y < b$$

$$0 < z < c$$

the potential energy $V(x, y, z) = 0$

and outside the box $V(x, y, z) = \infty$

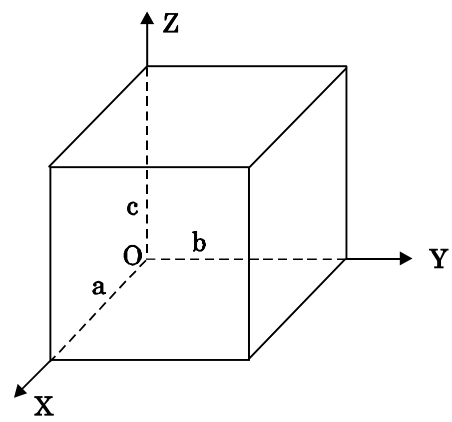


Fig. 6.14

Wave Equation of the Particle

For the motion of the particle inside the box, the Schrodinger time - independent wave equation is:

$$-\frac{\hbar^2}{2m}\nabla^2\psi = E\psi \quad \text{... (1)}$$

or
$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} + \frac{2m}{\hbar^2}E\psi = 0 \quad \text{... (2)}$$

It is assumed that the wave-function $\psi(x, y, z)$ is equal to the product of three functions X, Y, and Z each of which is a function of one variable only.

Thus we have

$$\psi(x, y, z) = X(x)\,Y(y)\,Z(z) \quad \text{... (3)}$$

Substituting this equation in eqn. (2), we have,

$$YZ\frac{d^2X}{dx^2} + ZX\frac{d^2Y}{dy^2} + XY\frac{d^2Z}{dz^2} + \frac{2mE}{\hbar^2}XYZ = 0 \quad \text{... (4)}$$

Note: We used ordinary derivatives instead of partial derivatives because each of the functions X, Y and Z is a function of one variable only.

Dividing eqn. (4) by *XYZ*, we get

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} + \frac{2mE}{\hbar^2} = 0 \qquad \text{... (5)}$$

In this equation $\frac{2mE}{\hbar^2}$ is a constant for a particular value of the kinetic energy.

Since the velocity of the particle, being a vector quantity, can be resolved into three components along the coordinate axes, the kinetic energy E is expressed as the sum of the corresponding terms E_x, E_y and E_z.

Hence,

$$E = E_x + E_y + E_z \qquad \text{... (6)}$$

Therefore, from equations (5) and (6), we get

$$\left[\frac{1}{X}\frac{d^2X}{dx^2} + \frac{2mE_x}{\hbar^2}\right] + \left[\frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{2mE_y}{\hbar^2}\right] + \left[\frac{1}{Z}\frac{d^2Z}{dz^2} + \frac{2mE_z}{\hbar}\right] = 0$$

This equation gives three independent equations:

$$\frac{d^2X}{dx^2} + \frac{2mE_x}{\hbar^2}X = 0 \qquad \text{... (7)}$$

$$\frac{d^2Y}{dy^2} + \frac{2mE_y}{\hbar^2}Y = 0 \qquad \text{... (8)}$$

$$\frac{d^2Z}{dz^2} + \frac{2mE_z}{\hbar^2}Z = 0 \qquad \text{... (9)}$$

The eqn. (7) is the equation for the one-dimensional case. The boundary condition applicable to the solution is:

$$X(0) = X(a) = 0$$

So the eigen values of E_x are given by

$$\boxed{E_x = \frac{\pi^2 \hbar^2}{2ma^2} n_x^2} \quad \text{... (10)}$$

where $n_x = 1, 2, 3, \ldots\ldots$

and the corresponding normalized eigen functions are given by:

$$\boxed{X(x) = \sqrt{\frac{2}{a}} \sin \frac{n_x \pi x}{a}} \quad \text{... (11)}$$

The solution for Y and Z are of the same form, therefore, we have:

$$E_y = \frac{\pi^2 \hbar^2}{2mb^2} n_y^2 \quad \text{... (12)}$$

$$Y(y) = \sqrt{\frac{2}{b}} \sin \frac{n_y \pi y}{b} \quad \text{... (13)}$$

and

$$E_z = \frac{\pi^2 \hbar^2}{2mc^2} n_z^2 \quad \text{... (14)}$$

$$Z(z) = \sqrt{\frac{2}{c}} \sin \frac{n_z \pi z}{c} \quad \text{... (15)}$$

Eigen Values of Energy

Substituting the expressions for E_x, E_y and E_z in eqn. (6), we have:

$$\boxed{E_{n_x, n_y, n_z} = \frac{\pi^2 \hbar^2}{2m}\left[\frac{n_x^2}{a^2} + \frac{n_y^2}{b^2} + \frac{n_c^2}{c^2}\right]} \quad \text{... (16)}$$

where $n_x = 1, 2, 3, \ldots\ldots$

$n_y = 1, 2, 3, \ldots\ldots$

$n_z = 1, 2, 3, \ldots\ldots$

This equation gives the eigen values of the energy of the particle. **These values are called the energy-levels of the particle.**

Wave function

The total normalized wave-function inside the box for the stationary states is given by:

$$\psi_{n_x, n_y, n_z}(x, y, z) = X(x)\, Y(y)\, Z(z) = \sqrt{\frac{8}{abc}} \sin\frac{n_x \pi x}{a} \cdot \sin\frac{n_y \pi y}{b} \cdot \sin\frac{n_z \pi z}{c} \quad \ldots (17)$$

where n_x, n_y and n_z are integers.

The wave-function is zero outside the box. It is easily to proved that the wave-function is normalized, because:

$$\frac{8}{abc} \int_0^a \left(\sin\frac{n_x \pi x}{a}\right)^2 dx \cdot \int_0^b \left(\sin\frac{n_y \pi y}{b}\right)^2 dy \cdot \int_0^c \left(\sin\frac{n_z \pi z}{c}\right)^2 dz = 1$$

From Eqs. (16) and (17) we get the following conclusions:

1. Three integers n_x, n_y and n_z, which are called *quantum numbers*, are required to describe each stationary state.

   If we change the sign of the quantum numbers, there is no change in the energy and in the wave-function except that the minus will appear on the right hand side of Eq. (17).

   Therefore, all the stationary states are given by the positive integral values of n_x, n_y and n_z.

No quantum number can be zero, because if any one of them is taken zero, then $\psi(x, y, z) = 0$, which would mean that the particle does not exist in the box.

2. The lowest possible energy, *i.e.,* the energy in the ground state, occurs when $n_x = n_y = n_z = 1$ and it depends on the values of a, b and c.

3. If the particle is confined in a cubical box in which $a = b = c = a$, the eigen-values of energy are given by:

$$E_{n_x, n_y, n_z} = \frac{\pi^2 \hbar^2}{2ma^2}(n_x^2 + n_y^2 + n_z^2) \quad \text{... (18)}$$

In this case energy of the particle in the ground state is given by:

$$E_{111} = \frac{3\pi^2 \hbar^2}{2ma^2} \quad \text{... (19)}$$

No other state will have this energy, and this state has only one wave-function. Therefore, the ground state and the energy-level are said to be ***non-degenerate***.

4. In a cubical box, the energy depends on the sum of the squares of the quantum numbers. Consequently the particle having the same energy in an excited state will have several different stationary states, or different wave-functions. Such states and energy-levels are said to be ***degenerate***.

6.15 CORRESPONDENCE PRINCIPLE

In 1932 Niels Bohr proposed a correspondence principle.

Classical mechanics deals with the Laws of Physics governing macroscopic bodies. The Laws of Quantum Physics are applicable to the behaviour of microscope particles *viz.*, atoms, nuclei etc.

Bohr's correspondence principle bridges the gap between the classical mechanics and quantum mechanics. It removes the apparent discontinuity between the two.

According to the correspondence principle, a system in a state of higher quantum number (higher excitation) is governed by the laws of classical mechanics. It does not mean that the laws of classical mechanics are applicable to all large systems.

For systems at very low temperatures (near zero degree Kelvin) or in very low state of quantum numbers (low excitation), laws of quantum mechanics are applicable. The example are He^3, He II, supper conductors etc.

Statement

The principle states that for large quantum numbers, quantum physics gives the same results as those of classical physics.

Proof

According to classical electro-magnetic theory, an electron revolving in a circular orbit radiates electro-magnetic waves having a frequency equal to the frequency of revolution, including harmonics which are integral multiples of that frequency.

The velocity of an electron revolving round the nucleus in an orbit of radius r is given by

$$v^2 = \frac{ke^2}{mr} \quad ...(1)$$

Taking root on both sides, we have

$$\sqrt{v^2} = e\sqrt{\frac{ke^2}{mr}} = \frac{\sqrt{k}\sqrt{e^2}}{\sqrt{mr}}$$

$$v = e\sqrt{\frac{k}{mr}} = \frac{e}{\sqrt{4\pi\,\varepsilon_0\,mr}} \quad ...(2)$$

$$r = \frac{n^2 h^2}{4\pi^2 m\,kZ\,e^2}$$

and $$r = \frac{n^2 h^2}{\frac{4\pi^2 m \times 1}{4\pi\varepsilon_o} \times 1 \times e^2} = \frac{n^2 h^2 \varepsilon_0}{\pi m e^2} \qquad ...(3)$$

$$\left[\text{Putting } Z = 1 \text{ for hydrogen and } k = \frac{1}{4\pi\varepsilon_0}\right]$$

The frequency of revolution

$$\nu = \frac{v}{2\pi r} \qquad ...(4)$$

Substituting for ν we have

$$\nu = \frac{1}{2\pi}\frac{e}{\sqrt{(4\pi\varepsilon_o mr)}\, r} = \frac{1}{2\pi}\frac{e}{(4\pi\varepsilon_0 mr)^{1/2} r^1} = \frac{1}{2\pi}\frac{e}{(4\pi\varepsilon_o m)^{½} r^{½} e^1}$$

$$= \frac{1}{2\pi}\frac{e}{(4\pi\varepsilon_o m)^{½} r^{3/2}}$$

Substituting for r, we have

$$= \frac{1}{2\pi}\cdot\frac{e}{(4\pi\varepsilon_0 m)^{1/2}\left(\frac{n^2 h^2 \varepsilon_0}{\pi m e^2}\right)^{3/2}}$$

$$\boxed{\nu = \frac{me^4}{4\varepsilon_0^2 h^3}\cdot\frac{1}{n^3}} \qquad ...(5)$$

According to Bohr's theory of the hydrogen atom

$$\frac{1}{\lambda} = R\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right)$$

$$= \frac{me^4}{8\varepsilon_o^2 h^3}\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right)$$

$$\nu = \frac{c}{\lambda} = \frac{me^4}{8\varepsilon_o^2 h^3}\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right) \qquad ...(6)$$

When the quantum number involved are large *i.e.* $n_1 = n$ and $n_2 = n + 1$ where $n >> 1$

$$\nu = \frac{me^4}{8\varepsilon_o^2 h^3}\left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right)$$

$$= \frac{me^4}{8\varepsilon_o^2 h^3}\left[\frac{(n+1)^2 - n^2}{n^2 (n+1)^2}\right]$$

$$= \frac{me^4}{8\,\varepsilon_o^2 h^2}\left[\frac{n^2 + 1^2 + 2n - n^2}{n^2 (n+1)^2}\right]$$

$$= \frac{me^4}{8\varepsilon_o^2 h^3}\left[\frac{2n + 1}{n^2 (n+1)^2}\right]$$

As $n >> 1$, so neglecting 1 as compared to n and $2n$, we get

$$\nu = \frac{me^4}{8\varepsilon_o^2 h^3} \cdot \frac{2}{n^3} \qquad ...(7)$$

$$\boxed{\nu = \frac{me^4}{4\varepsilon_o^2 h^3}} \qquad ...(8)$$

Comparing equations (4) and (5) we find that the classical orbital frequency and frequency of radiation emitted as calculated on the basis of quantum theory have the same value. Hence both classical and quantum theories of the hydrogen atom make identical predictions in the case of very large quantum numbers.

In fact ***'the greater the quantum number, the closer quantum physics approaches classical physics'.***

Significance of correspondence principle

The correspondence principle has proved to be of great use in the computation of the intensity, polarisation and coherence of spectral radiation. It has also been helpful in the formulation of '*selection rules*'.

ANNA UNIVERSITY SOLVED PROBLEM

Problem 6.4

In a Compton scattering experiment, the incident photons have a wavelength of 3×10^{-10} m. Calculate the wavelength of scattered photons if they are viewed at an angle of 60° to the direction of incidence.

[A.U April 2017]

Given data

$$\text{Wavelength of incident X-rays } \lambda = 3 \times 10^{-10} \text{ m,}$$

$$\text{Angle of scattering } \theta = 60°$$

$$h = 6.625 \times 10^{-34} \text{ Js}$$

$$m_o = 9.1 \times 10^{-31}$$

$$c = 3 \times 10^{8} \text{ ms}^{-1}$$

Solution:

We know that

$$\Delta\lambda = \lambda' - \lambda = \frac{h}{m_o c}(1 - \cos\theta)$$

$$\text{or } \lambda' = \lambda + \frac{h}{m_o c}(1 - \cos\theta)$$

Substituting the given values, we have

$$\lambda' = 3 \times 10^{-10} + \frac{6.625 \times 10^{-34}}{9.1 \times 10^{-31} \times 3 \times 10^{8}}(1 - \cos 60°)$$

$$= 3 \times 10^{-10} + \frac{6.625 \times 10^{-34}}{2.730 \times 10^{-22}}(1 - 0.5)$$

$$= 3 \times 10^{-10} + 2.427 \times 10^{-12} \times 0.5$$

$$= 3 \times 10^{-10} + 1.2132 \times 10^{-12}$$

$$\lambda' = 3.012 \times 10^{-10} \text{ m}$$

$$\boxed{\lambda' = 3.012 \text{ Å}}$$

Problem 6.5

X-rays of 1.0 Å are scattered from a carbon block. Find the wavelength of the scattered beam in a direction making 90° with the incident beam. How much kinetic energy is imparted to the recoiling electron?
[A.U May 2018]

Given data

Wavelength of incident X - rays $\lambda = 1 \text{ Å} = 1 \times 10^{-10}$ m

Angle of scattering $\theta = 90°$

$$h = 6.625 \times 10^{-34} \text{ Js.}$$

$$c = 3.0 \times 10^{8} \text{ ms}^{-1}$$

$$1 \text{ eV} = 1.6 \times 10^{-19} \text{ joule.}$$

Solution

The change in wavelength is given by

$$\Delta\lambda = \lambda' - \lambda = \frac{h}{m_o c}(1 - \cos\theta)$$

$$\Delta\lambda = \frac{6.625 \times 10^{-34}}{9.11 \times 10^{-31} \times 3 \times 10^{8}} (1 - \cos 90°)$$

$$\Delta\lambda = 0.242 (1 - 0) \times 10^{-11}$$

$\Delta\lambda = 0.0242 \times 10^{-10} = 0.0242\,\text{Å}$

Now $\lambda' = \lambda + \Delta\lambda = 1.0 + 0.0242 = 1.0242\,\text{Å}$

$= 1.0242 \times 10^{-10}$ m

Energy of incident X-ray photon $= \dfrac{hc}{\lambda}$

Energy of scattered X-ray photon $= \dfrac{hc}{\lambda'}$

$\therefore$ Energy imparted to the recoiling electron

$$= \frac{hc}{\lambda} - \frac{hc}{\lambda'}$$

$$= hc\left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right)$$

$$= \frac{hc\,(\lambda' - \lambda)}{\lambda\,\lambda'} = \frac{hc\Delta\lambda}{\lambda\,\lambda'}$$

$$= \frac{6.625 \times 10^{-34} \times 3 \times 10^{8} \times 0.0242 \times 10^{-10}}{1.0 \times 10^{-10} \times 1.024 \times 10^{-10}}$$

$$= 4.66 \times 10^{-17} \text{ joule}$$

$$= \frac{4.66 \times 10^{-17}}{1.6 \times 10^{-19}} = 291 \text{ eV}$$

Problem 6.6

A neutron of mass 1.675×10^{-27} kg is moving with a kinetic energy 10 keV. Calculate the De-Broglie wavelength associated with it. ***[A.U Jan 2019]***

Given data

Mass of the neutron $= 1.675 \times 10^{-27}$ kg

Kinetic energy = 10 keV = 10×10^{3} eV

$$= 10 \times 10^{3} \times 1.6 \times 10^{-19} \text{ J}$$

Planck's constant $h = 6.625 \times 10^{-34}$ Js

Solution:

$$\text{We know that } \lambda = \frac{h}{\sqrt{2mE}}$$

Substituting the given values, we have

$$= \frac{6.625 \times 10^{-34}}{\sqrt{2 \times 1.675 \times 10^{-27} \times 10 \times 10^{3} \times 1.6 \times 10^{-19}}}$$

$$= \frac{6.625 \times 10^{-34}}{\sqrt{5.36 \times 10^{-42}}}$$

$$\boldsymbol{\lambda = 2.862 \times 10^{-13} \text{ m}}$$

Problem 6.7

An electron at rest is accelerated through a potential of 5000 V. Calculate de - Broglie wavelength of matter wave associated with it. ***[A.U. Jan 2020]***

Given data

Accelerating potential (V) = 5000 V

Solution

$$\text{We know that } \lambda = \frac{h}{\sqrt{2meV}}$$

$$\lambda = \frac{12.26}{\sqrt{V}} \times 10^{-10} \text{ m}$$

Substituting the given values, we have

$$\lambda = \frac{12.26 \times 10^{-10}}{\sqrt{5000}}$$

$$\lambda = \frac{12.26 \times 10^{-10}}{70.71}$$

$$\lambda = 0.173 \times 10^{-10} \text{ m}$$

$$\boxed{\lambda = \mathbf{0.173\ \text{Å}}}$$

Problem 6.8

Calculate de - Broglie wavelength associated with a proton moving with a velocity equal to one-thirtieth of the velocity of light. ***(A.U. Dec. 2019)***

Given data

Velocity of the proton $v = \frac{1}{30} \times$ velocity of light

$$= \frac{1}{30} \times 3 \times 10^{8} \text{ ms}^{-1}$$

$$= 1 \times 10^{7} \text{ ms}^{-1}$$

Mass of the proton $m = 1.67 \times 10^{-27}$ kg

Planck's constant $h = 6.625 \times 10^{-34}$ J s

Solution

Weknowthatde - Brogliewavelength

$$\lambda = \frac{h}{mv}$$

Substituting the given values, we have

$$\lambda = \frac{6.625 \times 10^{-34}}{1.67 \times 10^{-27} \times 1 \times 10^{7}}$$

$$\boxed{\lambda = 3.97 \times 10^{-14}\ \text{m}}$$

Problem 6.9

If the momentum of two particles are in the ratio 1 : 0.25, compare their de - Broglie wave lengths.

(A.U. Jan 2018)

de - Broglie wavelengths associated with two particles of momentum in the ratio 1 : 0.25 are λ_1 and λ_2

$$\lambda = \frac{h}{mv} = \frac{h}{p}$$

$$\lambda_1 = \frac{h}{p_1}, \qquad \lambda_2 = \frac{h}{p_2}$$

$$\lambda_1 : \lambda_2$$

$$\frac{h}{p_1} : \frac{h}{p_2}$$

$$\frac{1}{1} : \frac{1}{0.25}$$

$$1 : 4$$

de - Broglie wavelengths are in the ratio

$$\boxed{1 : 4}$$

Problem 6.10

Calculate the de - Broglie's wave length of an electron having a velocity of 10^6 m/sec. ***(A.U. Dec. 2018)***

Given data

Velocity of the electron $v = 10^6 \text{ ms}^{-1}$

Mass of the electron $m = 9.1 \times 10^{-31}$ kg

Planck's constant $h = 6.625 \times 10^{-34}$ Js

Solution

We know that de - Broglie's wavelength $\lambda = \dfrac{h}{mv}$

Substituting the given values, we have

$$\lambda = \frac{6.625 \times 10^{-34}}{9.1 \times 10^{-31} \times 10^{6}}$$

$$\lambda = 7.28 \times 10^{-10} \text{ m}$$

$$\boxed{\lambda = 7.28 \text{ Å}}$$

Problem 6.11

Calculate the de - Broglie's wavelength associated with an electron which travels with a velocity 500 km s^{-1}. **(A.U. Jan. 2017)**

Given data

Velocity of the electron

$$v = 500 \text{ km / sec} = 500 \times 10^3 \text{ m s}^{-1}$$

Planck's constant $h = 6.625 \times 10^{-34}$ Js

Mass of the electron $m = 9.1 \times 10^{-31}$ kg

Solution

We know that de-Broglie's wavelength associated with electrons

$$\lambda = \frac{h}{mv}$$

Substituting the given values, we have

$$\lambda = \frac{6.625 \times 10^{-34}}{9.1 \times 10^{-31} \times 500 \times 10^{3}}$$

$$\lambda = 0.00145 \times 10^{-6}$$

$$\lambda = 14.5 \times 10^{-10} \text{ m}$$

$$\lambda = 14.5 \text{ Å}$$

Problem 6.12

Calculate the minimum energy an electron can possess in an infinitely deep potential well of width 4 nm.

[A.U. Jan 2015]

Given data

Width of potential well $a = 4 \text{ nm} = 4 \times 10^{-9}$ m

For minimum energy, $n = 1$

Mass of the electron $m = 9.1 \times 10^{-31}$ kg

Planck's constant $h = 6.625 \times 10^{-34}$ Js

Solution:

We know that $E_n = \frac{n^2 h^2}{8ma^2}$

Substituting the given values, we have

$$E_1 = \frac{1^2 \times (6.625 \times 10^{-34})^2}{8 \times 9.11 \times 10^{-31} \times (4 \times 10^{-9})^2}$$

$$E_1 = 3.764 \times 10^{-21} \text{ J}$$

$$E_1 = \frac{3.764 \times 10^{-21}}{1.6 \times 10^{-19}} \text{ eV} \qquad [\because 1 \text{ eV} = 1.6 \times 10^{-19} \text{ J}]$$

$$\boxed{\mathbf{E_1 = 0.024\ eV}}$$

Problem 6.13

An electron is trapped in a one-dimensional box of length 0.1 nm. Calculate the energy required to excite the electron from its ground state to the fifth excited state. **[A.U. April 2016]**

Given data

Length of the one dimensional box

$$a = 0.1 \text{ nm} = 0.1 \times 10^{-9} \text{ m}$$

For ground state $n = 1$

For 5^{th} excited state, $n = 6$

Solution

We know that $E_n = \dfrac{n^2 h^2}{8ma^2}$

$$E_1 = \frac{1^2 \times (6.625 \times 10^{-34})^2}{8 \times 9.11 \times 10^{-31} \times (0.1 \times 10^{-9})^2}$$

$$E_1 = 6.022 \times 10^{-18} \text{ J}$$

For 5^{th} excited state, $n = 6$

$$E_6 = \frac{6^2 \times (6.625 \times 10^{-34})^2}{8 \times 9.11 \times 10^{-31} \times (0.1 \times 10^{-9})^2}$$

$$E_6 = 2.168 \times 10^{-16}\ J$$

The energy required to excite the electron from its ground state to the fifth excited state is $\Delta E = E_6 - E_1$

$$\Delta E = 2.168 \times 10^{-16} - 6.022 \times 10^{-18}$$

$$= 2.168 \times 10^{-16} - 0.06022 \times 10^{-16} = 2.108 \times 10^{-16}\ J$$

$$= \frac{2.108 \times 10^{-16}}{1.6 \times 10^{-19}}\ eV \qquad [\because 1\ eV = 1.6 \times 10^{-19}\ J]$$

$$\boxed{\Delta E = 1317\ eV}$$

ANNA UNIVERSITY PART - A
'2' MARKS Q&A

1. State compton effect. ***(A.U. Jan 2018)***

When a beam of X - rays is scattered by a substance of low atomic number, the scattered radiation consists of two components. One has the same wavelength λ as the incident ray and the other has a slightly longer wavelength λ'. This phenomenon of change in wavelength of scattered X - rays is known as **compton effect**.

2. What is Compton wavelength? ***(A.U. Jan 2019)***

The change in wavelength corresponding to scattering angle of 90° obtained in Compton effect is called Compton wavelength.

Mathematically, $\Delta \lambda = \frac{h}{m_o c}(1 - \cos\theta)$

m_o - rest mass of electron $= 9.11 \times 10^{-31}$ kg

When $\theta = 90°$, $\Delta\lambda = \frac{h}{m_o c}(1 - \cos 90°)$

$$= \frac{h}{m_o c}(1 - 0)$$

$$\frac{h}{m_o c} = 0.0243 \text{ Å}$$

This is known as Compton wavelength of electron.

3. What are matter waves? ***(A.U. Dec. 2018)***

The waves associated with moving particles of matter (e.g., electrons, photons, etc) are known as matter waves or de-Broglie waves.

4. How De-Broglie justified his concept? ***(A.U. May 2020)***

- Our universe is fully composed of light and matter.
- Nature loves symmetry. If radiation like light can act like wave and particle, then material particles (e.g., electron, neutron etc.) should also act as particle and wave.
- Every moving particle has always associated with a wave.

5. Write an expression for the wavelength of matter waves? (or) What is de - Broglie's wave equation? ***(A.U. Jan 2017)***

Wavelength for matter waves is

$$\lambda = \frac{h}{mv} = \frac{h}{p}$$

where $h \rightarrow$ planck's constant

$m \rightarrow$ mass of the particle

$v \rightarrow$ velocity of the particle with which the wave is associated.

$p \rightarrow$ momentum of the particle.

6. Write an expression for the de - Broglie wavelength associated with electrons. ***(A.U. Dec. 2018)***

De-Broglie wave length associated with electrons accelerated by the potential V.

$$\lambda = \frac{h}{\sqrt{2\, m_o\, e\, V}}$$

where $h \rightarrow$ planck's constant
$e \rightarrow$ charge of the electron
$m \rightarrow$ mass of the electron
$V \rightarrow$ accelerating voltage

7. State the properties of the matter waves.

(A.U. Jan 2020)

(i) Lighter is the particle, greater is the wavelength associated with it.

(ii) Smaller is the velocity of the particle, greater is wavelength associated with it.

(iii) These waves are not electromagnetic waves.

(iv) The velocity of deBroglie wave is equal to the velocity of the material particle.

8. Write down Schroedinger time independent and dependent wave equations. ***(A.U. Jan 2021)***

Schroedinger time independent wave equation is

$$\nabla^2 \psi + \frac{2m}{\hbar^2}(E - V)\psi = 0$$

Schroedinger time dependent wave equation

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\psi = i\hbar\frac{\partial \psi}{\partial t}$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is Laplacian operator.

ψ – Wave function

m – Mass of the particle.

E – Total energy of the particle.

V – Potential energy.

and $\hbar = \dfrac{h}{2\pi}$

9. Mention some of the physical significances of the wave function. ***(A.U. Jan. 2017, May 2018, Jan 2020)***

(i) The wave function (ψ) relates the particle and wave nature of matter statistically.

(ii) It is a complex quantity and hence we cannot measure it.

(iii) If the particle is certainly to be found somewhere in a space of dimensions dx, dy, dz, then the probability value is equal to one.

$$\text{i.e., } P = \int\int_V\int |\psi|^2 \, dx\, dy\, dz = 1$$

10. What are eigen values and eigen function?

(A.U. Jan. 2018)

Energy of a particle moving in one dimensional box of width a is given by

$$\boxed{E_n = \frac{n^2 h^2}{8ma^2}}$$

For each value of n, there is an energy level. Each value of E_n is called an eigen value.

For every quantum state (i.e., for different 'n' values), there is a corresponding wave function ψ_n. This corresponding wave function is called eigen function.

Eigen function associated with an electron in a one dimensional box is given by

$$\boxed{\psi_n = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right).}$$

ADDITIONAL PART - A '2' MARKS Q & A

1. What is Schrodinger wave equation?

The equation that describes the wave nature of a particle in mathematical form is known as Schrodinger wave equation.

2. What is a wave function?

A variable quantity which characterises de - Broglie wave is known as wave function and it is denoted by the symbol ψ.

3. Define correspondance principle.

Any new theory in Physics must reduce to well - established corresponding classical theory when the new theory is applied to the special situation in which the less general theory is know to be valid.

Anna University Part - B (16 Marks) Questions

1. (i) Define Compton effect.
   (ii) Derive an expression for the wavelength of the scattered photon (compton shift). ***[A.U. May 2017]***

2. (i) What is Compton effect?
   (ii) Give the theory of Compton effect and show that the Compton shift $\Delta \lambda = \frac{h}{m_o c}(1 - \cos \theta)$ ***[A.U. May 2019, Jan. 2020]***

3. Explain Compton effect and derive an expression for the wavelength of scattered photon. Also briefly explain its experimental verification. ***[A.U. Jan 2020, Jan. 2021]***

4. Arrive at the Schrondinger wave equation and apply the same for a particle in a rectangular box to obtain the energy eigen values and the corresponding eigen functions. ***[A.U. Dec. 2016, April 2017, Jan. 2018, Jan. 2020]***

5. Derive the time independent Schrodinger equation for a one-dimensional case. Use it to prove that a particle enclosed in a one-dimensional box has quantised energy values. ***[A.U. Jan 2016]***

6. Solve Schrodinger wave equation of a particle in box (one dimensional) and obtain the energy eigen values. ***[A.U. May 2018, Jan. 2019, Jan. 2020]***

7. Derive an expression for energy levels of a particle enclosed in one-dimensional potential box of width 'a' and infinite height. ***[A.U. Jan 2019]***

8. Discuss free particle problem starting from schrodinger wave equation eigen.

9. Derive eigen value end eigen energy function for 2D dimensional box.

10. Derive eigen value end eigen function for 3D dimensional box.

11. State Band prove Bohr's correspondance principle.

ADDITIONAL PART B '16' MARKS' QUESTIONS

1. Obtain the eigen values and eigen functions for an electron enclosed in a one dimensional potential box.

ASSIGNMENT PROBLEMS

1. In Compton scattering, the incident photons have a wavelength 0.5 nm. Calculate the wavelength of scattered radiation if they are viewed at angle of 45° to the direction of incidence. ***[Ans:*** $\lambda' = 0.5007$ **nm]**

2. X-rays of 1.0 Å are scattered from a carbon block. Find the wavelength of the scattered beam in a direction making 60° with the incident beam. How much kinetic energy is imparted to the recoiling electron? ***[Ans:*** $\lambda' = 1.0121$ **Å K.E. = 149 eV]**

3. Find the change in the wavelength of an X-ray photon when it is scattered through an angle of 180°. ***[Ans: 0.0484 Å]***

4. Monochromatic X-rays of wavelength 0.7078 Å are scattered by carbon at an angle of 90° with the direction of incident beam. What is the wavelength of scattered X-rays?

[Ans: 0.7320 Å]

5. Estimate the potential difference through which a proton is needed to be accelerated so that its de Broglie wavelength becomes equal to 1 Å.

(Given mass of proton $= 1.673 \times 10^{-27}$ kg.) ***[V = 0.082 V]***

6. Calculate the de Broglie wavelength associated with an electron carrying an energy 2000 eV. **[$\lambda = 2.74 \times 10^{-11}$ m]**

7. Prove that the de Broglie wavelength of an electron accelerated through a potential difference of V volts is

$$\sqrt{\frac{150}{V}}\ \text{Å}.$$

8. Calculate the zero point energy for an electron in a one dimensional box of width 10 Å. ***[Ans: 0.376 eV]***

9. A beam of X-rays are scattered by free electrons. At 45° from the beam direction, the At scattered X-rays have a wavelength of 0.022 Å. What is the wavelength of the incident beam? ***(Ans: 0.0149 Å)***

10. X-rays of wavelength 0.324 Å are scattered by a carbon block. Find the wavelengh of scattered X-rays for a scattering angle of 180°. ***(Ans: 0.1725 Å)***

11. X-rays of wavelength 0.3 Å undergoes Compton scattering at an angle of 60°. Find the wavelength of the scattered photon and energy of the recoil electron.

(Ans: **$\lambda = 0.3121$, $E = 2.571 \times 10^{-16}$ J)**

12. Calculate the de Broglie wavelength of an electron accelerated to a potential of 2 kV. ***(Ans: 0.2744 Å)***

Unit - V

Scanning Tunneling Microscope (STM)

Applied Quantum Mechanics

7. Applied Quantum Mechanics

The harmonic oscillator (qualitative) – Barrier penetration and quantum tunneling (qualitative) – Tunneling microscope – Resonant diode – Finite potential wells (qualitative) – Bloch's theorem for particles in a periodic potential – Basics of Kroning – Penney model and origin of energy bands.

Introduction

- In quantum mechanics, the wave function of a system gives the description of that system. We apply Schrodinger's wave equation to a system and then solve it to find the wave function of the system.

- We shall study how Schrodinger's time independent wave equation can be applied to a system and then solved to find the energy and wave function of the system under given conditions.

- We also aim at learning characteristic properties of solutions of this equation and comparing the predictions of quantum mechanics with those of Newtonian mechanics.

- As simple applications of Schrodinger's time independent wave equation, here we shall discuss the problems of:

  – **Harmonic oscillator**

  – **Barrier penetration and Quantum tunneling**

  – **Finite potential wells**

7.1 HARMONIC OSCILLATOR (Qualitative)

Definition

A particle undergoing simple harmonic motion is called a harmonic oscillator.

In harmonic oscillator, the force applied is directly proportional to the displacement and is always directed towards the mean position.

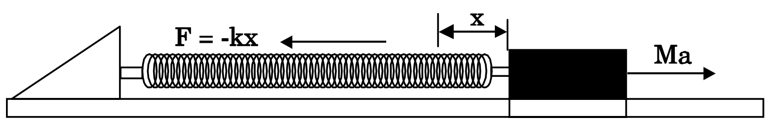


Fig. 7.1 Harmonic Oscillator

Examples. Familiar examples are; a simple pendulum, an object floating in a liquid, a diatomic molecule and an atom in a crystal lattice.

If applied force moves the particle through x, then restoring force F is given by

$$F \propto -x$$

$$F = -kx \quad \text{... (1)}$$

The potential energy of the oscillator is

$$V = -\int F dx$$

$$V = k\int x dx = \frac{1}{2}kx^2$$

$$\boxed{V = \frac{1}{2}kx^2} \quad \text{... (2)}$$

where k is force constant.

In harmonic oscillator, angular frequency is given by

$$\omega = \sqrt{\frac{k}{m}}$$

Squaring on both sides

$$\omega^2 = \left(\sqrt{\frac{k}{m}}\right)^2$$

$$\omega^2 = \frac{k}{m}, \qquad k = m\omega^2$$

where m - mass of the particle

Substituting k in eqn (1), we have

$$\boxed{V = \frac{1}{2} m\,\omega^2 x^2} \qquad \text{... (3)}$$

Wave equations for the oscillator

The time - independent Schrodinger wave equation for linear motion of a particle along the x-axis is:

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}(E - V)\psi = 0 \qquad \text{... (4)}$$

where E – Total energy of the particle,

V – Potential energy and

ψ – Wave-function for the particle which is function of x alone.

Substituting for V in equation (4) we get:

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}\left(E - \frac{1}{2} m\,\omega^2 x^2\right)\psi = 0 \qquad \text{... (5)}$$

$$\frac{d^2\psi}{dx^2} + \frac{2mE}{\hbar^2} - \frac{2m}{\hbar^2} \times \frac{1}{2}\, m\omega^2 x^2\,\psi = 0$$

or

$$\boxed{\frac{d^2\psi}{dx^2} + \left(\frac{2mE}{\hbar^2} - \frac{m^2\,\omega^2}{\hbar^2} x^2\right)\psi = 0} \qquad \text{... (6)}$$

This is Schrodinger wave equation for the oscillator.

Simplification of the wave equation

To simplify eqn. (6), a dimensionless independent variable y is introduced. It is related to x by the equation

$$y = ax \qquad \text{... (7)}$$

$$\therefore \qquad x = \frac{y}{a}, \qquad \text{where } a = \sqrt{\frac{m\,\omega}{\hbar}}$$

Now we have

$$\frac{d\,\psi}{dx} = \frac{d\,\psi}{dy}\frac{dy}{dx} = \frac{d\,\psi}{dy}\,a$$

$$\left(y = ax \quad dy = a\,dx \quad \frac{dy}{dx} = a \right)$$

Differentiating

$$\frac{d^2\,\psi}{dx^2} = \frac{d^2\psi}{dy^2}\,\frac{d^2y}{dx^2}$$

and $$\frac{d^2\,\psi}{dx^2} = \frac{d^2\,\psi}{dy^2}\,a^2$$

$$\frac{d^2\psi}{dx^2} = a^2\,\frac{d^2\,\psi}{dy^2} \qquad \text{... (8)} \qquad \frac{d^2y}{dx^2} = a^2$$

Substituting for $\frac{d^2\,\psi}{dx^2}$ and x^2 in eqn (6), we have

$$a^2\,\frac{d^2\,\psi}{dy^2} + \left(\frac{2mE}{\hbar^2} - a^4\,\frac{y^2}{a^2}\right)\psi = 0$$

$$a^2\,\frac{d^2\psi}{dy^2} + \left(\frac{2mE}{\hbar^2} - a^2y^2\right)\psi = 0$$

$$\left[\because x = \frac{y}{a} \quad a = \sqrt{\frac{m\,\omega}{\hbar^2}} \quad a^2 = \frac{m\,\omega}{\hbar^2} \quad a^2 = \frac{m^2\,\omega^2}{\hbar^2} \quad a^4 = \frac{m^2\,\omega^2}{\hbar^2}\right]$$

Dividing through out by a^2, we have

$$\frac{d^2\,\psi}{dy^2} + \left(\frac{2mE}{a^2\,\hbar^2} - y^2\right)\psi = 0 \qquad \text{...(9)}$$

Substituting for a^2.

$$\frac{d^2\psi}{dy^2} + \left(\frac{2mE}{\frac{m\omega}{\hbar}\cdot\hbar^2} - y^2\right)\psi = 0 \qquad ...(10)$$

or $$\frac{d^2\psi}{dy^2} + \left(\frac{2E}{\hbar\omega} - y^2\right)\psi = 0$$

or $$\boxed{\frac{d^2\psi}{dy^2} + (\lambda - y^2)\psi = 0} \qquad ...(11)$$

where $\lambda = \dfrac{2E}{\hbar\omega}$

Eigen-values of the total energy E_n

The wave equation for the oscillator is satisfied only for discrete values of total energies given by

$$\frac{2E}{\hbar\omega} = (2n + 1)$$

or $$E_n = \frac{1}{2}(2n+1)\hbar\omega$$

$$\boxed{E_n = \left(n + \frac{1}{2}\right)\hbar\omega} \qquad ...(12)$$

Substituting $\hbar = \dfrac{h}{2\pi}$ and $\omega = 2\pi\nu$, this expression has the form:

$$\boxed{E_n = \left(n + \frac{1}{2}\right)h\nu} \qquad ...(13)$$

where, $n = 0, 1, 2, \ldots\ldots$, and ν is the frequency of the classical harmonic oscillator, given by

$$\nu = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \qquad \left(\because\ \omega = \sqrt{\frac{k}{m}}\right)$$

From eqn. (13),we get the *following conclusions:*

1. The lowest energy of the oscillator is obtained by putting $n = 0$ in eqns (12) and (13) it is:

$$\boxed{E_o = \frac{1}{2}\hbar\omega = \frac{1}{2}h\nu} \quad \text{... (14)}$$

   This is called the ground state energy or the ***zero point*** **vibrational energy of the harmonic oscillator.** The zero-point energy is the characteristic result of quantum mechanics. The values of E_n in terms of E_o are given by:

$$\boxed{E_n = (2n+1)E_o} \quad \text{... (15)}$$

   where $n = 0, 1, 2, 3, \ldots$

2. The eigen-values of the total energy depend only on one quantum number n. Therefore all the energy-levels of the oscillator are non-degenerate.

3. The successive energy-levels are equally spaced; the separation between two adjacent energy-levels being $\hbar\omega$ $(h\nu)$. The energy-level diagram for the harmonic oscillator is shown in fig. 7.2.

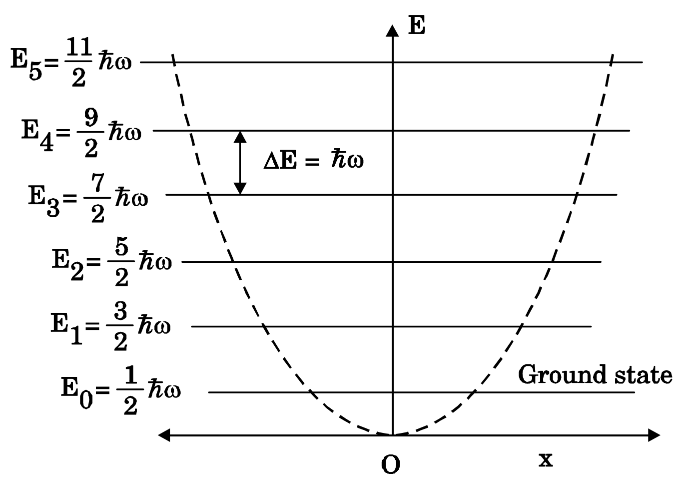


Fig. 7.2 Energy levels allowed for a harmonic oscillator. Note that the oscillator cannot have zero energy

In the figure 7.2 the horizontal lines show the energy levels and the *dashed curve* is parabola representing the potential energy $V = \frac{1}{2} k x^2$

Wave functions of the harmonic oscillator

For each value of the parameter $\lambda = \frac{2E}{\hbar \omega} = 2n + 1$, there is a different wave function ψ_n which consists of:

(i) the normalization constant N_n given by:

$$N_n = \left(\frac{m \omega}{\pi \hbar} \right)^{1/4} (2^n n!)^{-1/2} \quad \text{... (16)}$$

(ii) the exponential factor $e^{-y^2/2}$ and

(iii) a polynomial $H_n(y)$, called **Hermite polynomial** in either odd or even powers of y.

Thus the general formula for the n^{th} wave function is:

$$\psi_n = \left(\frac{m \omega}{\pi \hbar} \right)^{1/4} (2^n n!)^{-1/2} e^{-y^2/2} H_n(y) \quad \text{... (17)}$$

The first six Hermite polynomials are given in the following table:

Table 7.1

n	$\lambda = 2n + 1$	E_n	$H_n(y)$
0	1	$\frac{1}{2} \hbar \omega$	$H_0(y) = 1$
1	3	$\frac{3}{2} \hbar \omega$	$H_1(y) = 2y$
2	5	$\frac{5}{2} \hbar \omega$	$H_2(y) = 4y^2 - 1$

n	$\lambda = 2n + 1$	E_n	$H_n(y)$
3	7	$\frac{7}{2}\hbar\omega$	$H_3(y) = 8y^3 - 12y$
4	9	$\frac{9}{2}\hbar\omega$	$H_4(y) = 16y^4 - 48y^2 + 12$
5	11	$\frac{11}{2}\hbar\omega$	$H_5(y) = 32y^5 - 160y^3 + 120y$

The first six wave functions are shown in fig. 7.3.

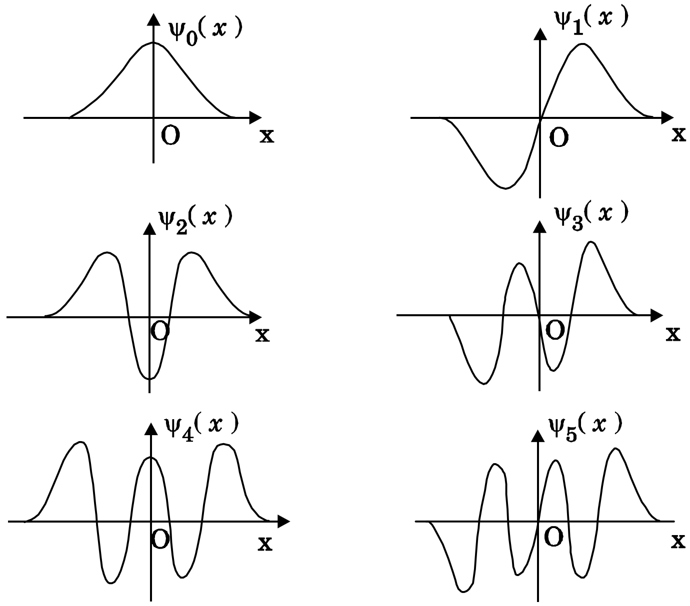


Fig. 7.3 Wave functions for Harmonic Oscillator

Significance of zero point energy

For lowest (ground) state, $n = 0$

$$\boxed{E_0 = \frac{1}{2}h\nu}$$

This is the lowest value of energy, called **zero point energy**. Even it the temperature reduces to absolute zero, the oscillator would still have an amount of energy $\frac{1}{2}h\nu$.

In old quantum mechanics, the energy of n^{th} level.

$$\boxed{E_n = nh\nu}$$

whereas in wave mechanics

$$\boxed{E_n = \left(n + \frac{1}{2}\right)h\nu}$$

A comparison of two results shows that the only difference in old quantum mechanics and wave mechanics is that **all the equally spaced energy levels are shifted upward by an amount equal to half the separation of energy levels**

i.e., $\frac{1}{2}h\nu$ (equal to zero point zero).

The existence of zero point energy is an important feature of wave mechanics and has been found to be experimentally true.

- Experiments in scattering of light by crystals at low temperature show that when temperature is decreased, the intensity of scattered light tends to a finite limit and remains unchanged with further decrease in temperature.
- It indicates that the oscillations of the atom in the crystal do not stop even at absolute zero.

7.2 BARRIER PENETRATION AND QUANTUM TUNNELING (Qualitative)

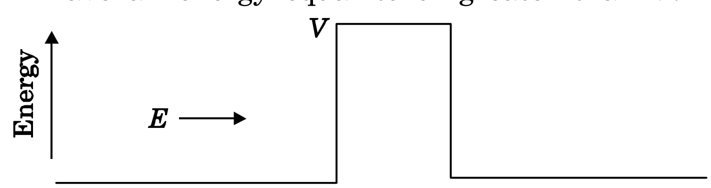
- According to classical ideas, a particle striking a hard wall has no chance of leaking through it. But, the behaviour of a quantum particle is different due to the wave nature associated with it.
- We know that when an electromagnetic wave strikes at the interface of two media, it is partly reflected and partly transmitted through the interface and enters the second medium.

- In a similar way the de Broglie wave also has a possibility of getting partly reflected from the boundary of the potential well and partly penetrating through the barrier.

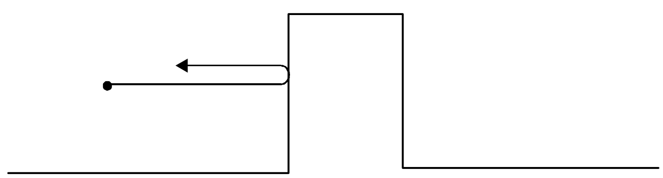
- Fig. 7.4 shows a particle with energy $E < V$ approaching potential barrier of height V.

- An electron of total energy E approaches the barrier from the left. From the view-point of classical physics, the electron would be reflected from the barrier because its energy E is less than V.

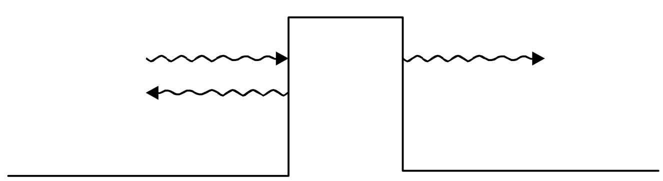
- For the particle to overcome the potential barrier, it must have an energy equal to or greater than V.



(a) A particle with energy E < V approaching a potential barrier.



(b) From classical mechanics, the particle must be reflected by the barrier.



(c) In quantum mechanics, the de Broglie waves that represent the particle are partly flected and partly transmitted, l.e., the particle has finite probability of penetrating the barring

Fig. 7.4

Quantum mechanics leads to an entirely new result. It shows that there is a finite chance for the electron to leak to the other side of the barrier.

It is noted that the electron tunneled through the potential barrier and hence in quantum mechanics, **this phenomenon is called tunneling.**

The transmission of electrons through the barrier is known as barrier penetration.

Expression for Transmission Probability

- Now let us consider the case of a particle of energy $E < V$ approaching a potential barrier of finite height and width as shown in fig. 7.5.
- The particle in region *I* has certain probability of passing through the barrier to reach region II and then emerge out on the other side in region III.
- The particle lacks the energy to go over the top of the barrier, but tunnels through it. Higher the barrier and wider it is, the lesser is the probability of the particle tunneling through it.

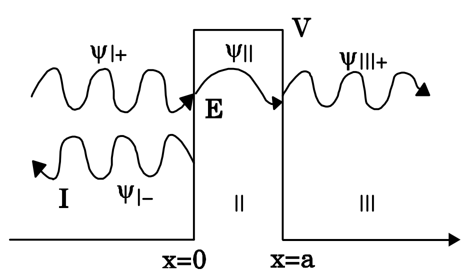


Fig. 7.5 When a particle of energy E < V, approaches a potential barrier, the de Broglie waves that correspond to the particle are particle are partly reflected and partly transmitted. That is the particle has a finite chance of penetrating the barrier

- Let us now consider a beam of identical particles, all having kinetic energy E. The beam is incident on the potential barrier of height V and width a from region I.
- On both sides of the barrier $V = 0$. This means that no forces act on particles in regions I and III.

As shown in fig. 7.5, the wave function ψ_{I-} represents the particle moving towards the barrier from region-I while ψ_{I-} represents the particle reflected moving away from the barrier.

The wave function ψ_{II} represents the particle inside the barrier. Some of the particles end up in region III while the others return to region I.

Quantum mechanics shows that the transmission probability T for a particle to pass through the barrier is given by

$$T = \frac{\text{Number of particles transmitted}}{\text{Number of particles incident}}$$

This probability is approximately given by

$$\boldsymbol{T = T_o e^{-2ka}}$$

where $k = \frac{\sqrt{2m(V-E)}}{h}$ and a is the width of the barrier.

T_o is a constant close to unity. It shows that the probability of particle penetration through a potential barrier depends on the height and width of the barrier.

Significance of the study of barrier penetration problems

1. Tunnelling is a very important physical phenomena which occurs in certain semiconductor diodes. In such diodes electrons pass through potential barriers even though their kinetic energies are smaller than the barrier heights.

2. The tunneling effect also occurs in the case of the alpha particles. The kinetic energy of alpha particle is only a few MeV but it is able to escape from a nucleus whose potential wall is perhaps 25 MeV high.
3. The ability of electrons to tunnel through a potential barrier is used in the Scanning Tunneling Microscope (STM) to study surfaces on an atomic scale of size.

Terminology related to microscope

(a) Microscope

A microscope is an instrument which is used to view the magnified image of a smaller object which cannot be clearly seen with a naked eye.

(b) Optical microscope

It is a microscope which uses light radiation to illuminate the object.

(c) Resolving power

It is the ability of the microscope to show two closer objects as separated ones.

The resolving power is inversely proportional to wavelength of light used. In an electron microscope, beam of electrons are used to illuminate the specimen.

The wave length λ associated with these electrons is about 0.1 Å or less. Hence, its resolving power is very high. The minimum distance that can be resolved in the electron microscope is about 10 Å.

(d) Magnification Power

It is the ability of the microscope to show the image of an object in an enlarged manner.

$$\text{Magnification power} = \frac{\text{size of the image}}{\text{size of the object}}$$

$$= \frac{\Delta}{F} \cdot \frac{D}{f}$$

In an optical microscope,

F $\rightarrow$ Focal length of objective lens in mm

$f \rightarrow$ Focal length of eye piece in mm

Δ – Length of microscope (16 cm)

D – Least distance of distinct vision (25 cm)

Thus, the magnification is about 1000 X (one thousand times).

In the case of electron microscope, Δ is very large (> 1 m) F and f can be reduced to less than a millimetre. So, the magnification power of electron microscope is about 10^5 X.

(e) Depth of focus

It is defined as the ability of the objective of microscope to produce a sharp focussed image when the surface of the object is not truly plane.

The deviation from plane surface occurs when the specimen is severely etched or when certain constituents of the structure are depressed or elevated from the etched surface.

Electron Microscope

Definition

It is a microscope which uses electron beam to illuminate a specimen and it produces an enlarged image of the specimen.

It has very high magnification power and resolving power when compared to optical microscope.

Principle

Like an optical microscope, its purpose is to magnify extremely minute objects. The resolving power of microscope is inversely proportional to the wavelength of the radiation used for illuminating the object under study.

Higher magnification as well as resolving power can be obtained by utilizing waves of shorter wavelength (λ).

Electron microscope uses electron waves whose wavelength is given by the formula $\lambda = \frac{12.25}{\sqrt{V}}$.

For $V = 10{,}000$ V, $\lambda = 0.1225$ Å which is extremely short. Electron microscopes giving magnification more than 2,00,000 X are common in Science & Technology Medical Research Laboratories.

An electron microscope consists of the following essential parts:

(i) **Electron Gun.** Its function is to provide a narrow beam of electrons of uniform velocity.

(ii) **Electrostatic and magnetic lenses.** Their function is to refract and properly focus the electron beam.

(iii) **Fluorescent screen or photographic plate.** They are used to receive the highly magnified image of the extremely small object being studied.

Types of Electron Microscopes

There are four types of electron microscopes. They are

1. **Transmission Electron Microscope (TEM)**

2. **Scanning Electron Microscope (SEM)**

3. **Scanning Transmission Electron Microscope (STEM)**

4. **Scanning Tunneling Microscope (STM).**

Fig. 7.6 gives comparison of an optical and electron microscope.

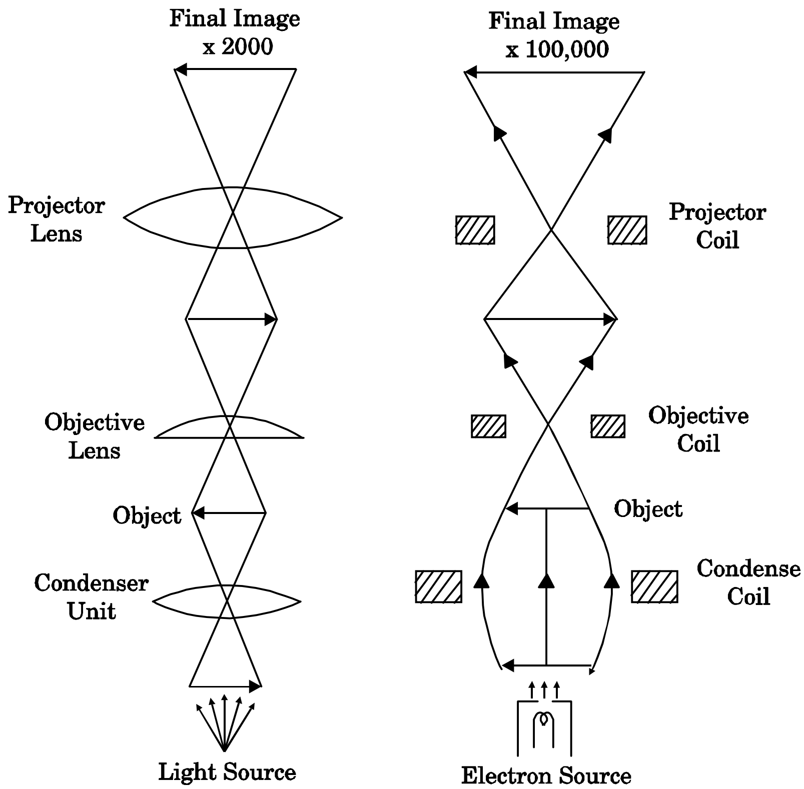


Fig. 7.6

7.3 SCANNING TUNNELING MICROSCOPE (STM)

A **scanning tunneling microscope**, or STM, is a type of electron microscope. It is commonly used in fundamental and industrial research.

- It is an instrument used for imaging surfaces at the atomic level.
- Due to its high resolution, individual atoms within materials are routinely imaged and manipulated.

Note:

Invented in the year 1981 by **Gerd Binning and Heinrich Rohrer** from IBM's Zurich Research Center in Switzerland. It helped them to win Nobel Prize in Physics in the year 1986.

Principle

It is based on the concept of quantum mechanical tunneling of electrons.

- In this technique, a sharp narrow conducting needle (probe) or tip is brought very near to the surface to be examined (Fig.7.7)
- A small voltage difference about 1V is applied between the tip and the surface of the material.
- This allows electrons to tunnel through the vacuum between them and results in **tunneling current.**
- Information about surface morphology is obtained by monitoring the tunneling current. The tip's position scans across the surface and it is usually displayed in image form

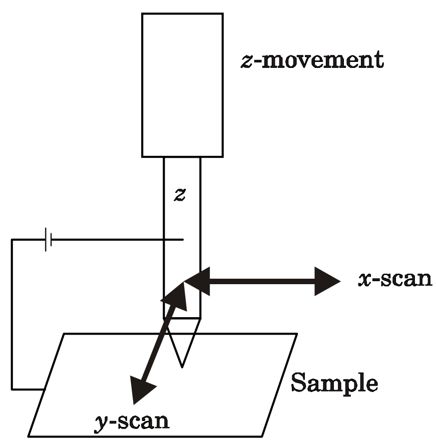


Fig. 7.7 Sketch of STM

Construction

The components of STM include

1. **Scanning needle tip**
2. **Piezoelectric controlled height and surface (*x, y*) scanner**.

3. **Coarse sample to tip control**
4. **Vibration isolation system** and
5. **Computer**

- Needle tip for scanning the sample surface. It is often made of tungsten.
- Piezoelectric tube is provided with tip and electrodes. It is capable of moving X, Y, Z directions. It is used to maintain the tip position with respect to the sample and scanning the sample.
- Coarse sample to tip control is used to bring the tip close to the sample.
- Vibration isolation system: It presents any vibration or sound in the system
- The computer is used to acquire the data and it may also used for enhancing the image with the help of image processing as well as performing quantitative measurement.

Working

The sharp metal needle is brought close to the surface to be imaged. The distance is of the order of a few angstroms.

- A bias voltage is applied between the sample and the tip. When the needle is at a positive potential with respect to the surface, the electrons can tunnel through the gap and set up a small "**tunneling current**" in the needle. This feeble tunneling current is amplified and measured.
- With the help of the tunneling current, the feedback electronics keeps the distance between tip and sample constant.
- Once tunneling is established, the tip's bias and position with respect to the sample can be varied and data are obtained from the resulting changes in current.

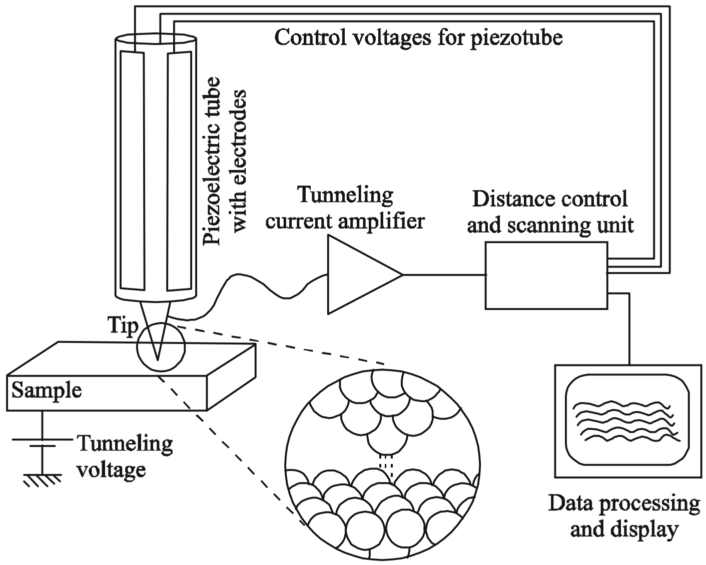


Fig. 7.8 Scanning Tunneling Microscope

Scanning

- If the tip is moved across the sample in the $x - y$ plane, the changes in surface height and density of states causes changes in tip current. These changes are mapped in images to present the surface morphology.

- This change in current with respect to position can be measured itself, or the height Z of the tip corresponding to a constant current can be measured. These two modes are called constant height mode and constant current mode, respectively.

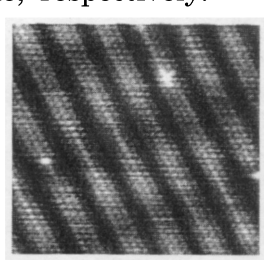


Fig. 7.9 Image of reconstruction on a clean Gold surface

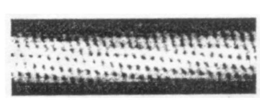


Fig. 7.10 An image of single-walled carbon nonotube

A macroscale image of an etched tungsten STM tip is shown in figure. 7.11.

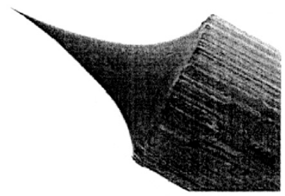


Fig. 7.11 Tungsten tip of STM

The figure 7.12 shows a tube of Piezoelectric Tube (PZT), coated with metal both inside and outside.

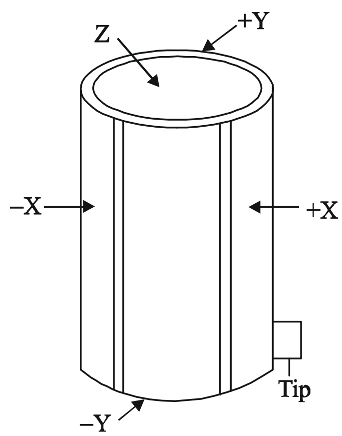


Fig. 7.12 A piezoelectric tube used in STM tip

Advantages of STM

STMs are helpful because they can give researchers a three dimensional profile of a surface. It allows researchers to examine a multitude of characteristics, including roughness, surface defects and determining things about the molecules such as size and conformation.

- For an STM, good resolution is 0.1 nm lateral resolution and 0.01 nm depth resolution.
- The high resolution of STMs enable researchers to examine surfaces at an atomic level.

- Capable of capturing much more detail than other microscopes. This helps researchers better understand the subject of their research on a molecular level.
- STMs are also versatile. They can be used in ultra high vacuum, air, water and other liquids and gasses.
- They will operate in temperatures as low as zero Kelvin up to a few hundred degrees Celsius.

Disadvantages of STM

There are very few disadvantages to using a scanning tunneling microscope.

- STMs can be difficult to use effectively. There is a very specific technique that requires a lot of skill and precisions.
- STMs require very stable and clean surfaces, excellent vibration control and sharp tips.
- A small vibration even a sound, can distrub the tip and the sample together.
- Even a single dust particle can damage the needle.
- STMs use highly specialized equipment that is fragile and expensive.
- The electronics require for STM are extremely sophisticated as well as very expensive.

Applications of STM

1. It is a powerful tool used in many research fields and industries to obtain atomic scale sample imaging and magnification.
2. One innovative applications of STM recently found is manipulation of atoms. For example, Iron atoms are placed on Cu surface and dragged by the STM tip and moves across the surface to a desired position.
3. It is used to analyze the electronic structures of the active sites at catalyst surfaces.

4. STM is used in the study of structure, growth, morphology, electronic structure of surface, thin films and nano structures.

Concept of Resonant Tunneling

An interesting phenomena occurs when two barriers of width a separated by a potential well of small distance L as shown in fig. 7.13. This leads to the concept of *resonant tunneling*.

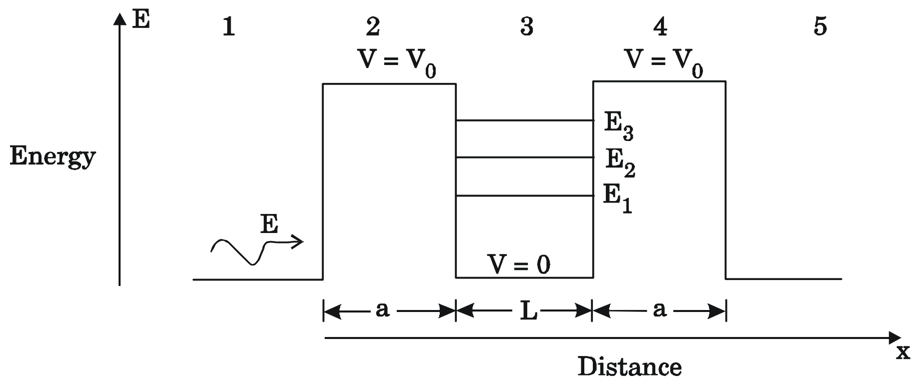


Fig. 7.13 Double barrier junction with no applied bias. E is the energy of the incident electron and E_1, E_2 are the energy level in quasi band states in the well.

For example, assume that incident electrons have energy E and that all the energy states E_n lie above E, as shown in fig. 7.14.

The barriers are sufficiently thin to allow tunneling and the well region between the two barriers is also sufficiently narrow to form discrete (quasi-bound) energy levels, as shown in fig. 7.13.

The transmission coefficient of the double symmetric barrier becomes unity (ie., $T = 1$), when the energy of the incomingelectronwave(E) coincides with the energy of one of the discrete states formed by the well.

$$\text{ie.,} \quad \boxed{E = E_n = \frac{n^2 h^2}{8\, m\, L^2}}$$

where $n = 1, 2, 3 \ldots$

Thus, transmission probability of the double symmetric barrier is maximum and hence, the tunneling current reaches peak value when the energy of electron wave is equal to quantised energy state of the well.

This phenomenon is known as resonance tunneling.

- The double barrier junction has important applications to a device known as Resonant Tunneling Diode (RTD)

7.4 RESONANT DIODE

- **It is a device that has two tunneling junctions. Its I-V characteristic shows negative differential resistance characteristic.**

Definition

A resonant tunneling diode (RTD) is a diode with resonant tunneling structure. The electrons can tunnel through some resonant states at certain energy levels.

Principle

When electron (wave) incident with energy equal to energy level of a potential well of thin barrier, then the tunneling reaches its maximum value. This is known as resonant tunneling.

Structure of RTD

A typical resonant tunneling diode structure is made by using *n*-type GaAs for the regions to the left and right of both barriers (regions 1 and 5) (Fig. 7.14).

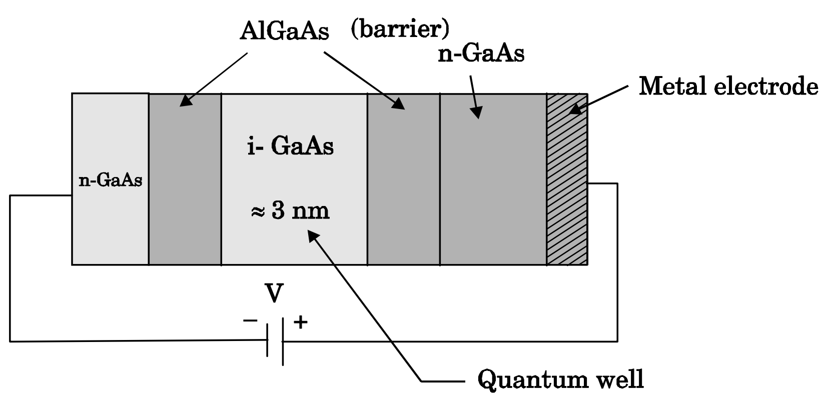


Fig. 7.14 Structure of Resonant Tunnel Diode (RTD)

The intrinsic GaAs is for the well region (region 3) and Al Ga As or Al As for the barrier material (regions 2 and 4).

Tunneling is controlled by applying a bias voltage across the device.

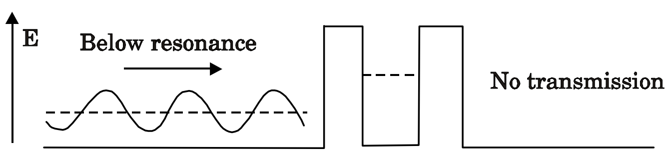
Working

Tunneling control

Tunneling is controlled by applying a bias voltage across the device.

Without applied bias

For the case of no applied bias, the energy band diagram is shown in fig. 7.15(a).



(a)

Practically it is very difficult to control the barrier height as well as the width of the potential well to match with the energy of the electron. This energy matching and hence resonant tunneling could be achieved by biasing the potential barriers.

With applied bias

When voltage is applied, the band diagram shifts and if the voltage is varied until the quantized discrete energy level corresponding to the potential well matches with the energy of the electron wave, resonant tunneling occurs. Such behavior is shown in fig. 7.15(b).

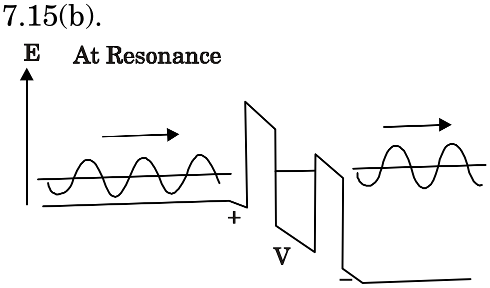


Fig. 7.15 (b) Resonant tunneling by proper biasing of the potential barriers Double barrier function under the action of an applied bias

Current - Energy characteristic for a resonant tunneling diode

When the incident electron energy E is very different from the energy of a discrete state E_n, transmission is low. As E tends to E_n, transmission will increase, becoming a maximum when $E = E_n$.

As E increases, tunneling will increase, reaching a peak when $E = E_1$. After that point, a further increase in E will result in a decreasing current, as shown in Fig. 7.16.

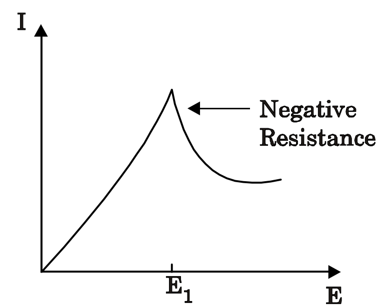


Fig. 7.16 Current - energy characteristic for a resonant tunneling junction, where E is the energy of the incident electron and E_1 is the energy of the first quasi - bound state in the well

This decrease of current with an increase of bias is called **negative resistance**. Further peaks and valleys will occur as E approaches, and then moves across other discrete energy states.

Application and uses of Resonant Tunneling Diodes (RTD)

1. One area or active application is building oscillators and switching devices that operate at **tera hertz** frequencies.
2. RTDs are very good rectifiers.

3. They are used in digital logic circuits.
4. They also used in inverters, memory cells and transistors (Resonant Tunneling Transistors (RTTs)

Advantages

- Resonant Tunneling diodes are very compact.
- They are capable of ultra-high-speed operations because the quantum tunneling effect through the very thin layers is a very fast process.

7.5 PARTICLE IN A FINITE POTENTIAL WELLS (Qualitative)

- Consider a particle of mass m moving with velocity v along the x-direction between $x = 0$ and $x = a$.
- The walls of the box are not rigid. Hence it is represented by a potential well of finite depth.

Step I: Let E be the total energy of particle inside the box and V be its P.E. The potential energy which is assumed to be zero within the box and its value outside the box is finite say V_o and $V_o > E$.

The variation of potential with x is shown in fig. 7.17.

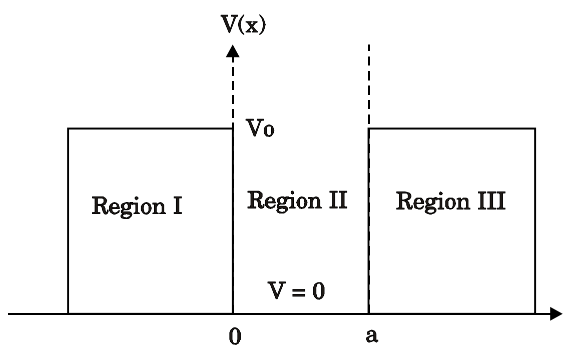


Fig. 7.17 Particle in a finite potential well with potential $V = 0$ in the Region II enclosed between the Regions I and III with $V = V_o$ of finite height

$V(x) = V_o \quad x \leq 0$ Region I

$V(x) = 0 \quad 0 < x < a$ Region II

and $V(x) = V_o \quad x \geq a$ Region III

Classically, the particle with energy $E < V_0$ cannot be present in regions I and III outside the box.

Consider the quantum mechanical picture of the particle in one dimension. If ψ is the wave function associated with the particle then Schrodinger's time independent equation for it is,

$$\boxed{\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}(E - V)\psi = 0} \quad \text{... (1)}$$

Step II: Consider the three regions I, II, III separately and let $\psi_I, \psi_{II}, \psi_{III}$ be the wave functions in them respectively.

We have for **region I.**

$$\boxed{\frac{d^2\psi_I}{dx^2} + \frac{2m}{\hbar^2}(E - V_0)\psi_I = 0} \quad \text{... (2)}$$

For **region II,**

$$\boxed{\frac{d^2\psi_{II}}{dx^2} + \frac{2mE}{\hbar^2}\psi_{II} = 0} \quad \text{... (3)}$$

$(\because V = 0)$

and **for region III,**

$$\boxed{\frac{d^2\psi_{III}}{dx^2} + \frac{2m}{\hbar^2}(E - V_0)\psi_{III} = 0} \quad \text{... (4)}$$

Let $\frac{2mE}{\hbar^2} = k^2$ and $\frac{2m(E - V_0)}{\hbar^2} = -k'^2$ (as $E < V_0$) ... (5)

Then the equation in the three regions is written as,

$$\frac{d^2 \psi_I}{dx^2} - k'^2 \psi_I = 0$$

$$\frac{d^2 \psi_{II}}{dx^2} + k^2 \psi_{II} = 0$$

$$\frac{d^2 \psi_{III}}{dx^2} - k'^2 \psi_{III} = 0 \quad \text{... (6)}$$

Step III: The solutions of these equations are of the form.

$$\psi_I = Ae^{k'x} + Be^{-k'x} \qquad \text{for } x < 0$$

$$\psi_{II} = P \cdot e^{ikx} + Q \cdot e^{-ikx} \qquad \text{for } 0 < x < a$$

and

$$\psi_{III} = C \cdot e^{k'x} + De^{-k'x} \qquad \text{for } x > a$$

Step IV: As $x \to \pm\infty, \psi$ should not become infinite. Hence $B = 0$ and $C = 0$.

Hence the wave functions in three regions are

$$\psi_I = Ae^{k'x}$$

$$\psi_{II} = P \cdot e^{ikx} + Q \cdot e^{-ikx}$$

and

$$\psi_{III} = D \cdot e^{-k'x}$$

Step V: The constants A, P, Q and D can be determined by applying the boundary conditions. The wave function ψ and its derivative $\frac{d\psi}{dx}$ should be continuous in the region where ψ is defined.

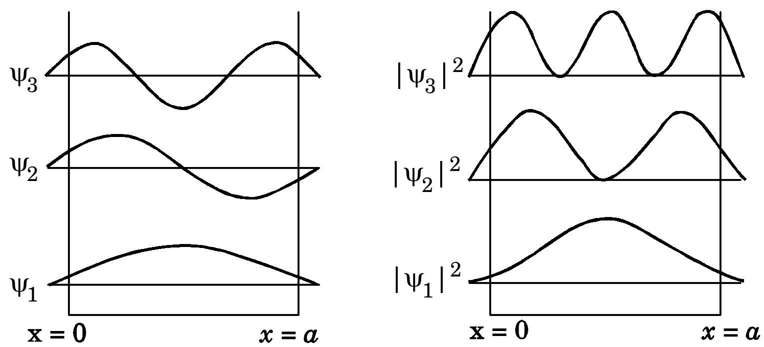
$$\psi_I(0) = \psi_{II}(0)$$

$$\left[\frac{d\,\psi_{I}}{dx}\right]_{x\,=\,0} = \left[\frac{d\,\psi_{II}}{dx}\right]_{x\,=\,0}$$

$$\psi_{II}\,(a) = \psi_{III}\,(a)$$

$$\left[\frac{d\,\psi_{II}}{dx}\right]_{x\,=\,a} = \left[\frac{d\,\psi_{III}}{dx}\right]_{x\,=\,a} \quad \ldots (8)$$

- Using these four conditions, we get four equations from which the four constants $A,\ P, Q, D$ can be determined. Thus the wave functions can be known completely.
- The first three wave functions and probability densities when plotted against x are as shown in fig. 7.18.



(a) Wave functions

(b) Probability densities inside non - rigid box

Fig. 7.18

- **The eigen functions are similar in appearance to those of infinite well except that they extend a little outside the box**.
- **Even though the particle energy E is less than the P.E. V_0, there is a definite probability that the particle is found outside the box.**
- **The particle energy is not enough to break through the walls of the box but it can penetrate the walls and leak out.**

- **This shows penetration of the particle into the classically forbidden region.**

- The energy levels of the particle are still discrete but there are a finite number of them. Such a limit exists because, soon the particle energy becomes equal to V_0.

For energies higher than this the particle energy is not quantised but may have any value above V_0.

- **These predictions are unique in quantum mechanics and shows different behaviour from that expected in classical physics**.

Band theory of solids (Zone theory)

The free electron theory explains the properties like thermal conductivity, electrical conductivity and specific heat of most of the metals.

But, it fails to explain why some solids are conductors, some are insulators and others are semiconductors.

A solution to this problem was given by band theory of solids and is called zone theory.

According to free electron theory, the potential energy of the electron inside the crystal through which an electron moves is supposed to be constant (zero). So it is completely free to move about in the crystal, restrained only by the surface of the crystal.

Postulates

1. According to band theory, potential energy of electron within the crystal is periodic due to periodicity of the crystal i.e., free electrons move inside periodic lattice field.
2. The potential energy of the solid varies periodically with the periodicity of space lattice ***'a'*** which is nothing but interatomic spacing.

Inside a real crystal, the electrons (–ve charge) move through periodic arrangement of positively charged holes (+ve charge) as shown in the fig. 7.19(a)

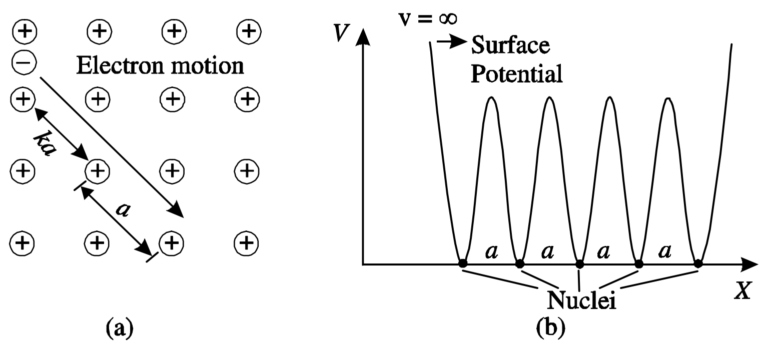


Fig. 7.19 One dimensional periodic potential distribution for a crystal

Fig. 7.19(b) shows one dimensional periodic potential distribution for a crystal. It is assumed that the potential energy of the electron at the positive ion site is zero and it is maximum when it is half way between the adjacent nuclei.

7.6 BLOCH'S THEOREM FOR PARTICLES IN A PERIODIC POTENTIAL

The motion of electron inside the lattice is not free as expected, but the electron experiences a periodic potential variation. The potential energy on the electron is maximum between adjacent ions and gradually decrease as the electron moves towards ions as shown in fig.7.20.

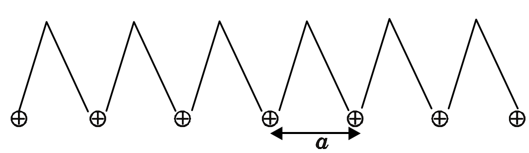


Fig. 7.20 One-dimensional periodic potential distribution on electron in crystal lattice

Bloch Theorem

It is a mathematical statement regarding the form of one electron wave function for a perfectly periodic potential.

Statement

If an electron in a linear lattice of lattice constant 'a' characterised by potential function $V(x) = V(x + a)$ satisfies the Schrodinger equation

$$\frac{d^2 \psi(x)}{dx^2} + \frac{2m}{\hbar^2} [E - V(x)] \psi(x) = 0 \qquad \dots (1)$$

then the wave functions $\psi(x)$ of electron (with energy E) is obtained as a solution of Schrodinger equation are of the form

$$\psi(x) = u_k(x) e^{ikx} \qquad \dots (2)$$

where $$u_k(x) = u_k(x + a) \qquad \dots (3)$$

Here $u_k(x)$ is also periodic with lattice periodicity.

The potential $V(x)$ is periodic as $V(x) = V(x + a)$ where a is a lattice constant.

From the Block theorem, we can say that the free electron is modulated by the periodic function $u_k(x) e^{ikx}$

In other words the solutions are plane waves modulated by the function $u_k(x)$ which has the same periodicity as the lattice. This theorem is known as ***Bloch Theorem***. The functions of the type (2) are called ***Bloch functions***.

Proof

If equation (1) has the solution with the property of equation (2), we can write the property of the Bloch functions i.e., equation (3) as

$$\psi(x + a) = e^{ik(x+a)} u_k(x + a)$$

(or) $$\psi(x + a) = e^{ikx} \cdot e^{ika} u_k(x + a)$$

Since $u_k(x + a) = u_k(x)$, we can write the above equation as

$$\psi(x + a) = e^{ikx} e^{ika} \cdot u_k(x) \quad \text{... (4)}$$

Since $\psi(x) = e^{ikx} u_k(x)$, we can write the above equation as

$$\boldsymbol{\psi(x + a) = e^{ika} \cdot \psi(x)} \quad \text{... (5)}$$

(or) $$\boldsymbol{\psi(x + a) = Q\,\psi(x)} \quad \text{... (6)}$$

where $$\boldsymbol{Q} = e^{ika}$$

If $\psi(x)$ is a single-valued function, then

we can write $\boldsymbol{\psi(x) = \psi(x+a)}$ Thus Bloch theorem is proved.

This equation is similar to that of eqn (2) and eqn (4) i.e., If the potential is a function of 'x' and 'a', then the wave function is also a function of 'x' and 'a'.

7.7 BASICS OF KRONIG PENNY MODEL

The essential feature of the behaviour of electronic potential is studied by considering a periodic rectangular well structure in one dimension. It was first discussed by **Kronig** and **Penny** in the year 1931.

The potential energy of an electron, when it moves in one dimensional perfect crystal lattice is assumed in the form of rectangular wells as shown in fig. 7.21.

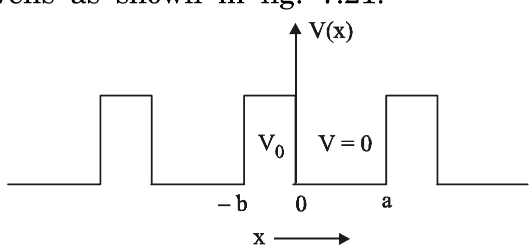


Fig. 7.21 One dimensional periodic potential (Kronig and Penny model)

In region where $0 < x < a$, the potential energy is **zero** and in the region $-b < x < 0$, the potential energy is V_o.

The one dimensional Schrodinger wave equations for two regions are written as

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}[E - 0]\,\psi = 0 \quad \text{for} \quad 0 < x < a \qquad \text{... (1)}$$

$$\text{(or)} \qquad \frac{d^2\psi}{dx^2} + \alpha^2\psi = 0 \qquad \text{... (2)}$$

$$\text{where } \alpha^2 = \frac{2mE}{\hbar^2}$$

and

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}(E - V_o)\,\psi = 0 \qquad \text{for } -b < x < 0 \qquad \text{... (3)}$$

$$\text{(or)} \qquad \frac{d^2\psi}{dx^2} - \beta^2\psi = 0 \qquad \text{... (4)}$$

$$\text{where } \beta^2 = \frac{2m}{\hbar^2}(V_o - E)$$

For both the regions, the appropriate solution suggested by Bloch is of the from

$$\psi = e^{ikx}\,U_K(x) \qquad \text{... (5)}$$

Differentiating equation (5) and substituting in equations (2) and (4), then further solving it under boundary conditions, we get

$$\frac{P \sin \alpha a}{\alpha a} + \cos \alpha a = \cos ka \qquad \text{... (6)}$$

$$\text{where } \alpha = \frac{\sqrt{2mE}}{\hbar} \quad \text{and} \quad P = \frac{mV_o ba}{\hbar^2}$$

The term P is called as Scattering power of the potential barrier. It is a measure of strength with which the electrons are attracted by the positive ions.

The equation (6) is analysed by drawing a plot between αa and $\left[\dfrac{P \sin \alpha a}{\alpha a} + \cos \alpha a\right]$ as shown in fig. 7.22.

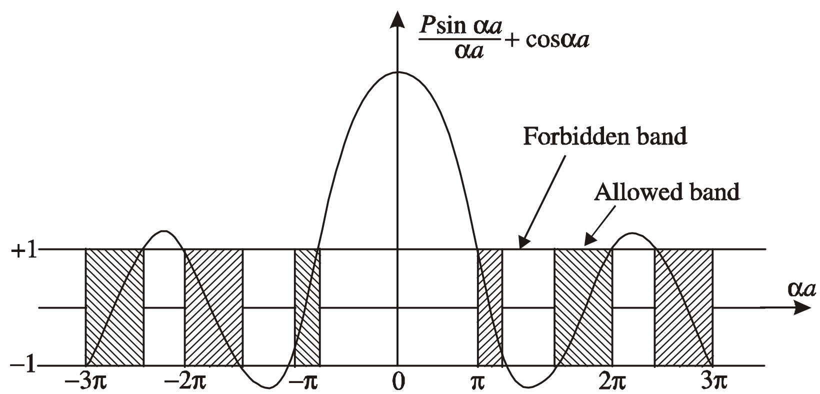


Fig. 7.22 A plot of αa versus $\left(\dfrac{P \sin \alpha a}{\alpha a} + \cos \alpha a\right)$

From the graph, we conclude that

1. The energy spectrum of an electron consists of a large number of allowed and forbidden energy bands.
2. The width of allowed energy band (shaded portion) increases with increase of energy values i.e., increasing the values of αa.

   This is because the first term of equation $\dfrac{P \sin \alpha a}{\alpha a}$ decreases with increase of αa.
3. In the limit $P \to \infty$ the allowed energy band reduces to one single energy level corresponding to the discrete energy level of an isolated atom. (ie., $a \to \infty$)

4. In the other extreme case, when $P \to 0$

$$\cos \alpha a = \cos ka$$

Thus, $\alpha = k$

$$\alpha^2 = k^2$$

$$\frac{2mE}{\hbar^2} = k^2 \qquad \left[\because \hbar = \frac{h}{2\pi}\right]$$

$$E = \frac{\hbar k^2}{2\,mE}$$

$$\boxed{E = \frac{h^2 k^2}{8\pi^2 m}}$$

which corresponds to free electron model.

This indicates that the particle is completely free and no energy levels exist. Thus by varying P from 0 to ∞ we find that the completely free electron becomes completely bound.

E - K curve

The energy of the electron in the periodic lattice is given by

$$\boxed{E = \frac{h^2 k^2}{8\pi m} \cdot k^2}$$

From the above equation as k changes, the corresponding energy (E) also changes. For a free electron, the energy curve is continuous as shown by dotted parabola.

But for the electron in the periodic lattice, the energy curve is not a continuous parabola and discontinuity occur at $k = \frac{n\pi}{a}$.

The zone between $+\frac{\pi}{a}$ and $-\frac{\pi}{a}$ is known as first Brillouin zone and the second zone has two parts from $+\frac{\pi}{a}$ to $+\frac{2\pi}{a}$, and $-\frac{\pi}{a}$ to $-\frac{2\pi}{a}$. These zones are the allowed energy bands separated by forbidden energy bands as shown in fig. 7.23.

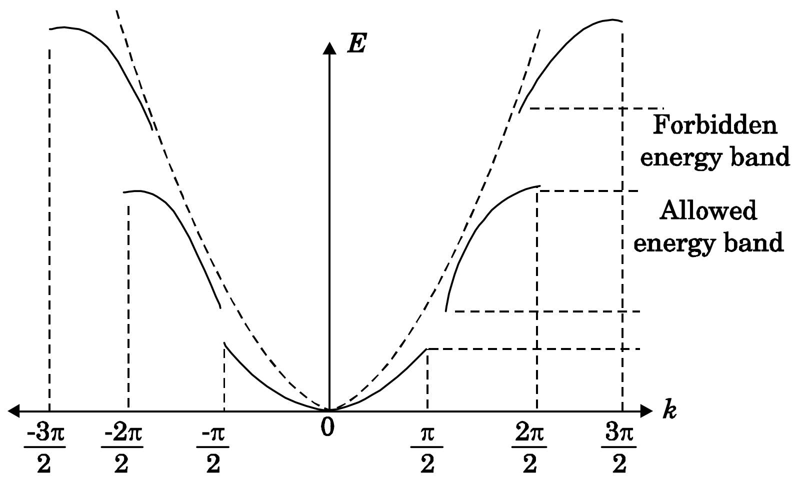


Fig. 7.23 Plot of energy Vs. wave vector in one dimensional lattice

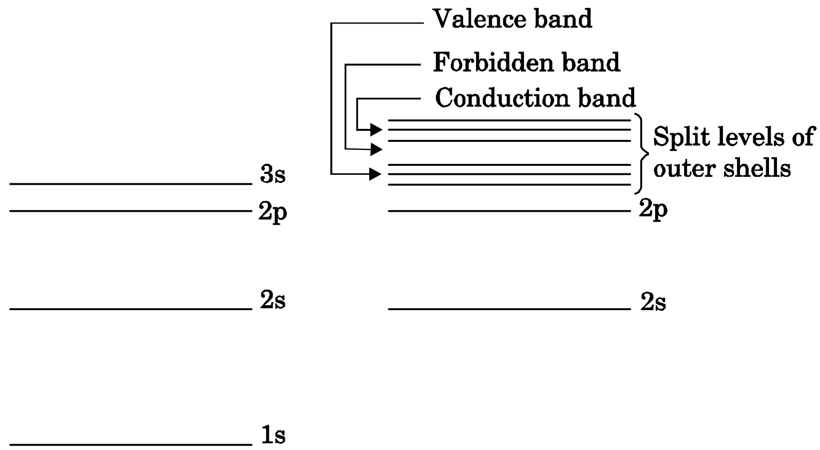
7.8 ORIGIN OF ENERGY BANDS

A solid contains an enormous number of atoms packed closely together. In the case of a single isolated atom, there are discrete energy levels, 1*s*, 2*s*, 2*p*, 3*s* …. These energy levels can be occupied by the electrons of the atom, as shown in fig. 7.24 (a).

All the atoms of a solid, if assumed isolated from one another, can have completely identical electronic schemes of their energy levels. Then the electrons fill the levels in each atom independently.

When the atoms come close together, they strongly interact and the outer electron orbitals overlap with each other.

Hence, the interactions of large number of atoms form closely spaced energy levels known as permitted energy band. The permitted energy bands are separated by energy gap E_g. The lower completely filled band is **valence band** and upper unfilled band is called **conduction band** (Fig. 7.24(b)).



(a) Isolated atom (b) For atoms in a solid

Fig. 7.24 Energy bands in a solid

Definition

A set of such closely spaced energy levels is called an energy band.

Concept of valence band, conduction band and forbidden band

- The energy bands in a solid correspond to the energy levels in an atom. An electron in a solid can have only those *discrete energies* that lie within these energy bands. These bands are, therefore, called ***allowed energy bands***.
- These (allowed) energy bands are, in general, separated by some gaps which have no allowed energy levels. These gap (regions) are known as forbidden energy bands.
- Band corresponding to valence electrons is called *valence band* and the band beyond forbidden band is called *conduction band*, into which, the electrons pass, and move freely.

- The electrons in the outermost shell are called valence electrons. **The band formed by a series of energy level containing the valence electrons is known as Valence Band.**
- Valence band is also defined as a band which is occupied by the valence electrons. The valence band may be partially or completely filled up depending on the nature of the material.
- The next higher permitted band is the **conduction band**. The energy levels occupying this band is defined as the **lowest unfilled energy band**. This band may be empty or partially filled. In conduction band, the electrons can move freely.
- Both conduction band and valence bands are separated by a region or gap known as forbidden band or gap which is shown in the fig. 7.25. This band is collectively formed by a series of energy levels above top of the valence band and below the bottom of the conduction band.
- **The energy gap between the valence band and conduction band is called the forbidden energy gap or forbidden band**

It should be noted that no electron can exist in this band. When an electron in the valence band absorbs enough energy, it crosses the forbidden gap and enters into the conduction band. (Fig. 7.26)

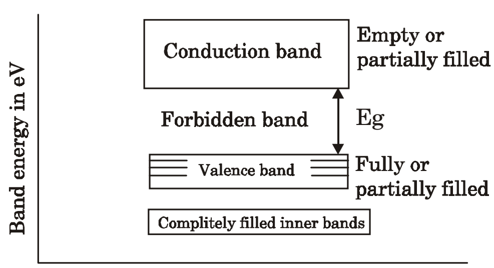


Fig. 7.25 Energy bands in solids

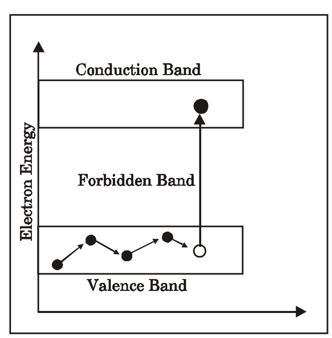


Fig. 7.26 Valence band, conduction band and forbidden gap

Classification of Metals, Semiconductors and Insulators

On the basis of width of forbidden gap valence and conduction band the solids are classified into insulators, semiconductors and conductors.

Insulators

- The band structure of insulators is as shown in fig. 7.27.
- The energy gap between conduction band and valence band is very high and is about 10 eV.
- The forbidden energy band is very wide. Due to this, electrons cannot jump from valence band to conduction band. In insulator, the valence electrons are bound very tightly to their parent atoms.

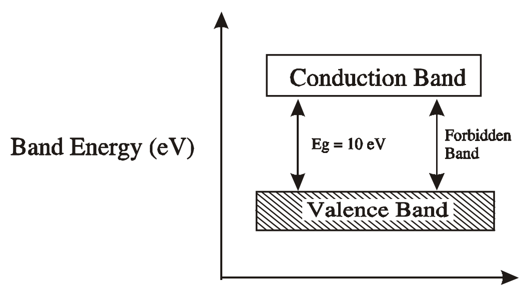


Fig. 7.27 Energy band for insulator

- The conduction band is completely vacant and valence band is completely filled.
- Even at high electric field, no electron will jump from valence band to the conduction band because of large energy gap. Hence, the electrical conductivity is zero.

Semiconductors

- The band structure of semiconductors is as shown in fig. 7.28.
- The forbidden gap is very small. Germanium and Silicon are the best examples of semiconductors.

- The energy gap between conduction band and valence band is very small. It is about 0.5 eV to 1 eV.

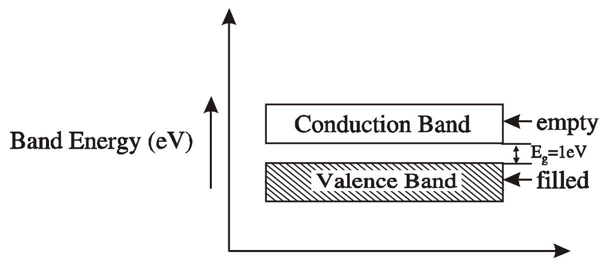


Fig. 7.28 Energy band for semiconductors

- As temperature increases, the bonds in the valence band break up and the created electrons move from valence band to the conduction. The vacancies created in the valence band due to breaking of bonds are termed as holes.
- Hence, conduction band is partially filled and valence band is partially vacant. These electrons and holes are responsible for electrical conduction.

Conductor

The band structure of conductors is as shown in fig. 7.29.

There is no forbidden gap, both valence and conduction bands overlap each other.

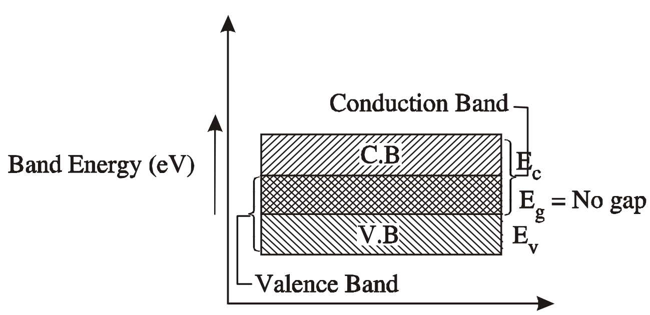


Fig. 7.29 Energy band diagram for conductor

- The electrons free to move within the conductor are responsible for electrical conduction.
- As temperature increases, the electrical conduction decreases, because mobility decreases due to large number of collisions with ions.
- The most important fact in conductors is that due to the absence of forbidden gap, there is no structure to establish holes. The total current in conductors is due to only the flow of electrons.

Part - 'A'
'2' Marks Questions with answers

1. What is a harmonic oscillator?

A particle undergoing simple harmonic motion is called a harmonic oscillator.

2. Give examples for harmonic oscillator.

Familiar examples are; a simple pendulum, an object floating in a liquid, a diatomic molecule and an atom in a crystal lattice.

3. What is significance of zero point energy is a harmonic oscillator?

For lowest (ground) state, $n = 0$

$$\boxed{E_o = \frac{1}{2} h\nu}$$

This is the lowest value of energy, called **zero point energy**. Even it the temperature reduces to absolute zero, the oscillator would still have an amount of energy $\frac{1}{2} h\nu$.

In old quantum mechanics, the energy of n^{th} level.

$$E_n = nh\nu$$

whereas in wave mechanics

$$E_n = \left(n + \frac{1}{2}\right)h\nu$$

4. Define barrier penetration.

The transmission of electrons through the barrier is known as **barrier penetration**. The phenomena is also termed as the **tunnel effect**. The phenomena of barrier penetration is entirely due to the wave nature of matter.

5. What is quantum tunneling?

The phenomenon of transmission of a particle through a potential barrier of finite width and height, even when its energy is less that the barrier height is called quantum tunneling.

6. What are the significance of tunneling effect?

1. Tunneling is a very important physical phenomena which occurs in certain semiconductor diodes. In such diodes electrons pass through potential barriers even though their kinetic energies are smaller than the barrier heights.
2. The tunneling effect also occurs in the case of the alpha particles. The kinetic energy of alpha particle is only a few MeV but it is able to escape from a nucleus whose potential wall is perhaps 25 MeV high.
3. The ability of electrons to tunnel through a potential barrier is used in the Scanning Tunneling Microscope (STM) to study surfaces on an atomic scale of size.

7. What is an electron microscope?

It is a microscope which uses electron beam to illuminate a specimen and it produces an enlarged image of the specimen.

It has very high magnification power and resolving power when compared to optical microscope.

8. What are the types of electrons microscopes.

There are four types of electron microscopes. They are

1. Transmission Electron Microscope (TEM)
2. Scanning Electron Microscope (SEM)
3. Scanning Transmission Electron Microscope (STEM)
4. Scanning Tunneling Microscope (STM).

9. What is scanning tunneling microscope?

It is an instrument used for imaging surfaces at the atomic level.

In STM, good resolution is considered to be 0.1 nm lateral resolution and 0.01 nm (10 pm) depth resolution. With this resolution, individual atoms within materials are routinely imaged and manipulated.

10. What is principle behind scanning tunneling microscope.

STM is based on the concept of quantum barrier tunneling.

When a conducting tip is brought very near to the surface to be examined, a bias (voltage difference) is applied between the two can allow electrons to tunnel through the vacuum between them.

Information is acquired by monitoring the current as the tip's position scans across the surface, and it is usually displayed in image form.

11. Mention few application of STM.

1. The STM shows the positions of atoms - or more precisely, the positions of some of the electrons.
2. Uses of STM to study metals and semiconductors surface can provide non-trivial real space information.
3. One innovative applications of STM recently found is manipulation of atoms.

4. To analyze the electronic structures of the active sites at catalyst surfaces.
5. STM is used in the study of structure, growth, morphology, electronic structure of surface, thin flims and nano structures.

12. State disadvantages of STM.

- A small vibration even a sound can disturb the tip and the sample together.
- A single dust particle can damage the needle.

13. What is resonant diode?

A resonant tunneling diode (RTD) is a diode with resonant tunneling structure. The electrons can tunnel through some resonant states at certain energy levels.

14. Define resonant tunneling.

The transmission probability of the double symmetric barrier is maximum. The tunneling current reaches peak value when energy of electron wave is equal to quantised energy state of the well.

This phenomenon is known as resonance tunneling.

15. State Block Theorem.

If an electron in a linear lattice of lattice constant 'a' characterised by potential function $V(x) = V(x + a)$ satisfies the Schoredinger equation

$$\frac{d^2 \psi(x)}{dx^2} + \frac{2m}{\hbar^2} [E - V(x)] \psi(x) = 0$$

then the wave functions $\psi(x)$ of electron (with energy E) obtained as a solution of Schrodinger equation are of the form

$$\psi(x) = u_k(x) e^{\pm ika}$$

$$\text{with} \quad u_k(x) = u_k(x + a)$$

16. What is an energy band?

A set of closely spaced energy levels is called an energy **band**.

17. What is valence band?

The electrons in the outermost shell are called valence electrons. The band formed by a series of energy level containing the valence electrons is known as Valence Band.

18. What is conduction band?

The band formed by a series of energy level containing the conduction electrons is known as conduction band.

The energy levels occupying this band is defined as the **lowest unfilled energy band**. This band may be empty or partially filled. In conduction band, the electrons can move freely.

19. What is forbidden gap?

Both conduction band and valence band are separated by a region or gap is known as forbidden band or gap.

Part - B '16' Marks Questions

1. Obtain an expression for the energy levels of the harmonic oscillator for applying Schrodinger wave equation.
2. Discuss barrier penetration and quantum tunneling.
3. What is the principle of scanning tunneling microscope. Explain the construction and working scanning tunneling microscope with a suitable diagram.
4. Write a note on resonant diode.
5. Discuss a particle in a finite potential well starting from Schrodinger wave equation.
6. Explain Bloch's theorem for particles in a periodic potential.
7. Discuss of Kraning penney model.
8. Describe origin of energy bands in solid.