

Unit - I

Partial Differential Equations

Defn ::

This is an equation involving a function of two or more variables and some of its partial derivatives.

Notations ::

$$\begin{array}{l} P = \frac{\partial z}{\partial x} \\ S = \frac{\partial^2 z}{\partial x \partial y} \end{array} \left| \begin{array}{l} q = \frac{\partial z}{\partial y} \\ t = \frac{\partial^2 z}{\partial y^2} \end{array} \right| \quad r = \frac{\partial^2 z}{\partial x^2}$$

Formation of P.D.E by Elimination of Arbitrary

Type: 1

Constants

No. of a.c = A.C $\rightarrow$ arbitrary Constant
No. of I.V = I.V $\rightarrow$ Independent variable
we use p & q only.

- ① Form the p.d.e by eliminating the arbitrary constants a & b from $z = ax + by$.

Soln ::

Given $z = ax + by \rightarrow$ ①

Diff. Partially w.r.to x & y

$$p = \frac{\partial z}{\partial x} = a \Rightarrow \boxed{p = a}$$

$$q = \frac{\partial z}{\partial y} = b \Rightarrow \boxed{q = b}$$

$$\textcircled{1} \Rightarrow \boxed{z = px + qy}$$

- ② Form the p.d.e by eliminating a.c a & b from $z = (x^2 + a^2)(y^2 + b^2)$

Soln ::

Given $z = (x^2 + a^2)(y^2 + b^2) \rightarrow$ ①
Diff P.w.r.to x & y .
 $P = \frac{\partial z}{\partial x} = 2x(y^2 + b^2)$

$$P = 2x(y^2 + b^2)$$

$$\boxed{\frac{P}{2x} = y^2 + b^2} \rightarrow (2)$$

$$Q = \frac{\partial Z}{\partial y} = 2y(x^2 + a^2)$$

$$Q = 2y(x^2 + a^2)$$

$$\boxed{\frac{Q}{2y} = x^2 + a^2} \rightarrow (3)$$

Sub (2) & (3) in (1),

$$Z = \left(\frac{Q}{2y}\right) \left(\frac{P}{2x}\right)$$

$$Z = \frac{PQ}{4xy}$$

$$\boxed{4xyz = PQ}$$

(3) Form the P.d.e by eliminating a & b from

$$(x-a)^2 + (y-b)^2 = z^2 \cot^2 \alpha$$

Soln: Given $\downarrow \rightarrow (1)$

Diff (1) p.w.r. to x & y .

$$2(x-a) = 2z \frac{\partial Z}{\partial x} \cot^2 \alpha$$

$$(x-a) = z p \cot^2 \alpha \rightarrow (2)$$

$$2(y-b) = 2z \frac{\partial Z}{\partial y} \cot^2 \alpha$$

$$y-b = z q \cot^2 \alpha \rightarrow (3)$$

Sub (2) & (3) in (1),

$$(zp \cot^2 \alpha)^2 + (zq \cot^2 \alpha)^2 = z^2 \cot^2 \alpha$$

$$z^2 p^2 \cot^4 \alpha + z^2 q^2 \cot^4 \alpha = z^2 \cot^2 \alpha$$

$$\div z \cot^2 \alpha$$

$$p^2 \cot^2 \alpha + q^2 \cot^2 \alpha = 1$$

$$\cot^2 \alpha [p^2 + q^2] = 1$$

$$p^2 + q^2 = \frac{1}{\cot^2 \alpha}$$

$$\boxed{\cot^2 \alpha = \frac{\cos \alpha}{\sin \alpha}}$$

$$p^2 + q^2 = \frac{\sin^2 \alpha}{\cos^2 \alpha}$$

$$p^2 + q^2 = \tan^2 \alpha$$

$$\frac{\sin \alpha}{\cos \alpha} = \tan \alpha$$

4) Form the P.d.e by eliminating the constants a & b from $z = ax^n + by^n$.

Soln:

Given $z = ax^n + by^n \rightarrow (1)$

Diff (1) p.w.r.to x & y

$$x^n = nx^{n-1}$$

$$p = \frac{\partial z}{\partial x} = anx^{n-1}$$

$$P = anx^n \cdot x^{-1}$$

$$P = \frac{anx^n}{x}$$

$$xP = anx^n \rightarrow (2)$$

Sub (2) & (3) in (1)

$$q = \frac{\partial z}{\partial y} = bny^{n-1}$$

$$q = bny^n \cdot y^{-1}$$

$$q = \frac{bny^n}{y}$$

$$yq = bny^n \rightarrow (3)$$

$$\frac{yq}{n} = by^n \rightarrow (3)$$

$$Z = \frac{xP}{n} + \frac{yq}{n}$$

$$Z = \frac{xP + yq}{n}$$

$$nz = xP + yq$$

① plane $\Rightarrow ax + by + cz + d = 0$ (Ans) $Z = Px + Qy$

② Sphere $\Rightarrow (0, 0, c) \Rightarrow x^2 + y^2 + (z - c)^2 = k^2$ (Ans) $qx = py$

③ Sphere $\Rightarrow (a, b, 0) \Rightarrow (x - a)^2 + (y - b)^2 + z^2 = c^2$ (Ans) $z^2(P^2 + Q^2) =$

④ plane $\Rightarrow \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ (Ans) $P = Q$

Type: 2

A.f = 1, p.d.e order = 1

① Eliminate f from $z = f(x^2 + y^2)$.

Soln:

Given $z = f(x^2 + y^2) \rightarrow ①$

Diff ① p.w.r.to x & y .

$$\frac{\partial z}{\partial x} = f'(x^2 + y^2) (2x)$$

$$P = f'(x^2 + y^2) (2x) \rightarrow ②$$

$$\frac{\partial z}{\partial y} = f'(x^2 + y^2) (2y)$$

$$Q = f'(x^2 + y^2) (2y) \rightarrow ③$$

$$\frac{②}{③} \Rightarrow \frac{P}{Q} = \frac{f'(x^2 + y^2) (2x)}{f'(x^2 + y^2) (2y)}$$

$$\frac{P}{Q} \hat{=} \frac{x}{y}$$

$$P_y = x Q$$

$$\boxed{P_y - x Q = 0}$$

② Eliminate the a.f f from $z = f\left(\frac{y}{x}\right)$ and form a p.d.e.

Soln:

Given $z = f\left(\frac{y}{x}\right) \rightarrow ①$

Diff ① p.w.r.to x & y

$$P = \frac{\partial z}{\partial x} = f'\left(\frac{y}{x}\right) \left(-\frac{y}{x^2}\right) \rightarrow ②$$

$$P = f'\left(\frac{y}{x}\right) (-y x^{-2})$$

$$P = f'\left(\frac{y}{x}\right) (-y x^{-2})$$

$$P = f'\left(\frac{y}{x}\right) \left(-\frac{y}{x^2}\right) \rightarrow ②$$

$$Q = \frac{\partial z}{\partial y} = f'\left(\frac{y}{x}\right) \left(\frac{1}{x}\right) \rightarrow ③$$

$$\frac{②}{③} \Rightarrow \frac{P}{Q} = \frac{f'\left(\frac{y}{x}\right) \left(-\frac{y}{x^2}\right)}{f'\left(\frac{y}{x}\right) \left(\frac{1}{x}\right)}$$

$$\frac{P}{Q} = -\frac{y}{x} \Rightarrow \boxed{Px + Qy = 0}$$

③ Eliminate f from $z = x + y + f(xy)$.

Soln:

Given $z = x + y + f(xy) \rightarrow (1)$

diff p.w.r. to x & y .

$$p = \frac{\partial z}{\partial x} = 1 + f'(xy) y$$

$$p - 1 = f'(xy) y$$

$$p - 1 = f'(xy) y \rightarrow (2)$$

$$q = \frac{\partial z}{\partial x} = 1 + f'(xy) x$$

$$q - 1 = f'(xy) x$$

$$q - 1 = f'(xy) x \rightarrow (3)$$

$$\frac{(2)}{(3)} \Rightarrow \frac{p-1}{q-1} = \frac{f'(xy) y}{f'(xy) x}$$

$$\frac{p-1}{q-1} = \frac{y}{x}$$

$$x(p-1) = y(q-1)$$

$$xp - x = yq - y$$

$$\boxed{xp - yq = x - y}$$

Type: 3

a.f = 2, p.d.e order = 2

① Form the p.d.e by eliminating the a-fs f & ϕ from $z = f(x+ct) + \phi(x-ct)$.

Soln:

$$\text{Given } z = f(x+ct) + \phi(x-ct) \rightarrow (1)$$

$$\frac{\partial z}{\partial x} = f'(x+ct) + \phi'(x-ct)$$

$$\frac{\partial^2 z}{\partial x^2} = f''(x+ct) + \phi''(x-ct) \rightarrow (2)$$

$$\frac{\partial z}{\partial t} = c f'(x+ct) - c \phi'(x-ct)$$

$$\frac{\partial^2 z}{\partial t^2} = c^2 f''(x+ct) + c^2 \phi''(x-ct)$$

$$\frac{\partial^2 z}{\partial t^2} = c^2 [f''(x+ct) + \phi''(x-ct)] \rightarrow (3)$$

$$\frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2} \text{ by (2)}$$

② Form a p.d.e by eliminating a-f from $z = x f(2x+y) + g(2x+y)$.

Soln:

Singular Integrals : Solution of Standard Types of 1st order P.d.e.

Define Singular integrals

$$\text{Let } f(x, y, z, \underline{p}, \underline{q}) = 0$$

$$\text{Let the complete integral be } \phi(x, y, z, \underline{a}, \underline{b}) = 0$$

Diff ① p.w.r.to a & b

$$\frac{\partial \phi}{\partial a} = 0 \rightarrow ②$$

$$\frac{\partial \phi}{\partial b} = 0 \rightarrow ③$$

The elimination of a & b from the above three equations ①, ② & ③ is called singular integral problems

① Solve $\frac{\partial z}{\partial x} = \sin x$

Soln Given $\frac{\partial z}{\partial x} = \sin x$

Integrating p.w.r.to x.

$$\int \frac{\partial z}{\partial x} = \int \sin x dx$$

$$z = -\cos x + f(y)$$

$$\therefore f(y) \Rightarrow a \cdot f$$

② Find the general solution of $\frac{\partial^2 z}{\partial y^2} = 0$

Soln

$$\text{Given } \frac{\partial^2 z}{\partial y^2} = 0$$

$$\frac{\partial}{\partial y} \left(\frac{\partial z}{\partial y} \right) = 0$$

$$\int \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial y} \right) = 0$$

$$\frac{\partial z}{\partial y} = f(x)$$

$$\int \frac{\partial z}{\partial y} = \int f(x) dy$$

$$z = f(x) y + F(x)$$

f(x) & F(x) $\Rightarrow$ arbitrary.



③ DEC-2008
APR-17 Solve $\frac{\partial^2 z}{\partial x \partial y} = 0$

$$\frac{\partial}{\partial x} \left(\frac{\partial z}{\partial y} \right) = 0$$

Integrating p.w.r.to x.

$$\frac{\partial z}{\partial y} = f(y)$$

$$\int \frac{\partial z}{\partial y} = \int f(y) dy + \phi(x)$$

$$z = F(y) + \phi(x)$$

④ APR-07
APR-11 Solve $\frac{\partial^2 z}{\partial x^2} = \sin y$

Soln.

$$\frac{\partial}{\partial x} \left(\frac{\partial z}{\partial x} \right) = \sin y$$

$$\frac{\partial z}{\partial x} = (\sin y)x + f(y)$$

$$z = (\sin y) \frac{x^2}{2} + f(y)x + F(y)$$

$f(y)$ & $F(y) \Rightarrow$ arbitrary functions.

Solution of 1st order P.D.E

Type: 1 $F(p, q) = 0$

APR-19
APR-28/16

① Solve $\sqrt{p} + \sqrt{q} = 1$

Soln:

Given $\sqrt{p} + \sqrt{q} = 1 \rightarrow \textcircled{1}$

Let the type $F(p, q) = 0$.

Let $z = ax + by + c \rightarrow \textcircled{2}$

Diff $\textcircled{1}$ p.w.r to x & y .

$$\frac{\partial z}{\partial x} = a \quad \left| \quad \frac{\partial z}{\partial y} = b\right.$$

$$p = a \quad \left| \quad q = b\right.$$

$$\textcircled{1} \Rightarrow \sqrt{a} + \sqrt{b} = 1$$

$$\sqrt{b} = 1 - \sqrt{a}$$

Power 2x

$$(b)^{\frac{1}{2}})^2 = (1 - \sqrt{a})^2$$

$$\boxed{b = (1 - \sqrt{a})^2}$$

From $\textcircled{2}$,

$$z = ax + by + c.$$

$$\text{Put } c = f(a) \Rightarrow z = ax + (1 - \sqrt{a})^2 y + f(a).$$

② Find the complete solution of $p^2 + q^2 = 4$

Soln: Given $p^2 + q^2 = 4 \rightarrow \textcircled{1}$

$$\text{Let } z = ax + by + c$$

$$\frac{\partial z}{\partial x} = a \quad \left| \quad \frac{\partial z}{\partial y} = b\right.$$

$$p = a \quad \left| \quad q = b\right.$$

$$\textcircled{1} \Rightarrow a^2 + b^2 = 4$$

$$b^2 = 4 - a^2$$

$$\boxed{b = \pm \sqrt{4 - a^2}}$$

From $\textcircled{2}$

$$z = ax + by + c$$

$$\text{Put } c = f(a); z = ax \pm \sqrt{4 - a^2} y + f(a)$$

③ write the complete integral of $PQ=16$
Soln: Given $PQ=16 \rightarrow$ ①
let $z = ax + by + c \rightarrow$ ②

$$\frac{\partial z}{\partial x} = a \quad \left| \quad \frac{\partial z}{\partial y} = b\right. \\ p = a \quad \left| \quad q = b$$

$$\text{①} \Rightarrow ab = 16$$

$$b = \frac{16}{a}$$

from ②

$$z = ax + by + c$$

$$\text{put } c = f(a) \Rightarrow z = ax + \frac{16}{a} y + f(a)$$

④ Solve $P+Q=PQ$

Soln: Given $P+Q=PQ \rightarrow$ ①

$$\text{let } z = ax + by + c \rightarrow$$
 ②

$$\frac{\partial z}{\partial x} = a \quad \left| \quad \frac{\partial z}{\partial y} = b\right. \\ p = a \quad \left| \quad q = b$$

$$\text{①} \Rightarrow a + b = ab$$

$$a = ab - b$$

$$a = b(a-1)$$

$$b = \frac{a}{a-1}$$

$$\text{②} \Rightarrow z = ax + \frac{a}{a-1} y + c$$



1.4) Lagrange's Linear Equation

This equation $Pp + Qq = R$ is called Lagrange's linear equation, where P, Q, R are functions of x, y, z ,

$$\therefore \text{This equations } \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

Method of Grouping:

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$u = c_1, \quad v = c_2$$

$$f\left(\frac{c_1}{c_2}\right) = 0$$

Problems

① Solve $px + qy = z$

Soln:

Given $px + qy = z$

Lagrange's Method is

The Equation is

$$Pp + Qq = R$$

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$$

$$\frac{dx}{x} = \frac{dz}{z}$$

$$\int \frac{dx}{x} = \int \frac{dz}{z}$$

$$\log x = \log z + \log c_2$$

$$\log x = \log(zc_2)$$

$$x = zc_2$$

$$c_2 = \frac{x}{z}$$

$$\boxed{\frac{x}{z} = c_2}$$

$$\log ab = \log a + \log b$$

$$\log \frac{a}{b} = \log a - \log b$$

$$\frac{dx}{x} = \frac{dy}{y}$$

$$\int \frac{dx}{x} = \int \frac{dy}{y}$$

$$\log x = \log y + \log c_1$$

$$\log x = \log(y c_1)$$

$$x = y c_1$$

$$c_1 = \frac{x}{y}$$

$$\boxed{\frac{x}{y} = c_1}$$



$$\therefore f(x, y) = 0$$

$$f\left(\frac{x}{y}, \frac{x}{z}\right) = 0$$

② Solve $p \tan x + q \tan y = \tan z$

Soln:-

$$\text{Given } p \tan x + q \tan y = \tan z$$

Lagrange's type is

$$\text{Method:- } p + q = R$$

The Equation is:-

$$\frac{dx}{p} = \frac{dy}{q} = \frac{dz}{R}$$

$$\frac{dx}{\tan x} = \frac{dy}{\tan y} = \frac{dz}{\tan z}$$

$$\int \frac{dx}{\tan x} = \int \frac{dy}{\tan y}$$

$$\int \cot x dx = \int \cot y dy$$

$$\log(\sin x) = \log(\sin y) + \log c_1$$

$$\log(\sin x) = \log(\sin y c_1)$$

$$\sin x = \sin y c_1$$

$$c_1 = \frac{\sin x}{\sin y}$$

$$u = \frac{\sin x}{\sin y}$$

$$\int \frac{dy}{\tan y} = \int \frac{dz}{\tan z}$$

$$\int \cot y dy = \int \cot z dz$$

$$\log(\sin y) = \log(\sin z) + \log c_2$$

$$\log(\sin y) = \log(\sin z c_2)$$

$$\sin y = \sin z c_2$$

$$c_2 = \frac{\sin y}{\sin z}$$

$$v = \frac{\sin y}{\sin z}$$

$$f(x, y) = 0$$

$$f\left(\frac{\sin x}{\sin y}, \frac{\sin y}{\sin z}\right) = 0$$

③ Solve $Pzx + Qzy = xy$

$$\text{Method: } Pp + Qq = R$$

Equation:-

$$\frac{dx}{p} = \frac{dy}{q} = \frac{dz}{R}$$



$$\frac{dx}{zx} = \frac{dy}{zy} = \frac{dz}{xy}$$

$$\frac{dx}{zx} = \frac{dy}{zy}$$

$$\int \frac{dx}{x} = \int \frac{dy}{y}$$

$$\log x = \log y + \log c_1$$

$$\log x = \log(y c_1)$$

$$x = y c_1$$

$$c_1 = \frac{x}{y}$$

$$\frac{dy}{zy} = \frac{dz}{xy}$$

$$\frac{dy}{z} = \frac{dz}{x}$$

$$\int \frac{dy}{z} = \int \frac{dz}{x}$$

$$\log z = \log x + \log c_2$$

$$\log z = \log(x c_2)$$

$$z = x c_2$$

$$c_2 = \frac{z}{x}$$

$$f(c_1, c_2) = 0$$

$$f\left(\frac{x}{y}, \frac{z}{x}\right) = 0$$

$$\textcircled{1} \text{ Sol/no } P \cot x + Q \cot y = \cot z$$

Soln

$$\text{Method: } Pp + Qq = R$$

$$\text{Equation: } \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{\cot x} = \frac{dy}{\cot y} = \frac{dz}{\cot z}$$

$$\frac{dx}{\cot x} = \frac{dy}{\cot y}$$

$$\frac{1}{\cot x} = \tan x$$

$$\tan x dx = \tan y dy$$

$$\log \sec x = \log \sec y + \log c_1$$

$$\log \sec x = \log(\sec y c_1)$$

$$\sec x = \sec y c_1$$

$$c_1 = \frac{\sec x}{\sec y}$$

$$\frac{dy}{\cot y} = \frac{dz}{\cot z}$$

$$\tan y dy = \tan z dz$$

$$\log \sec y = \log \sec z + \log c_2$$

$$\log \sec y = \log(\sec z c_2)$$

$$\sec y = \sec z c_2$$

$$c_2 = \frac{\sec y}{\sec z}$$

$$f(c_1, c_2) = 0$$

$$f\left(\frac{\sec x}{\sec y}, \frac{\sec y}{\sec z}\right) = 0$$

$$x^2 + y^2 = 1$$

$$\text{Method: } P(x, y) = 2x$$

$$\text{Equation: } \frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$$

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$$

$$\frac{dx}{x} = \frac{dy}{y}$$

$$\int \frac{dx}{x} = \int \frac{dy}{y} \quad \left(\int u^n du = \frac{u^{n+1}}{n+1} \right)$$

$$\int x^{-1} dx = \int y^{-1} dy$$

$$\frac{x^{0+1}}{0+1} = \frac{y^{0+1}}{0+1} + C_1$$

$$\frac{x^1}{1} = \frac{y^1}{1} + C_1$$

$$\frac{x}{1} = \frac{y}{1} + C_1$$

$$\frac{x}{1} = \frac{y}{1} + C_1$$

$$\left(\frac{x}{1} - \frac{y}{1} = C_1 \right)$$

$$\frac{dy}{y^2} = \frac{dz}{-z^2}$$

$$\int y^{-2} dy = \int -z^{-2} dz$$

$$\frac{y^{-2+1}}{-2+1} = \frac{-z^{-2+1}}{-2+1} + C_2$$

$$\frac{y^{-1}}{-1} = \frac{-z^{-1}}{-1} + C_2$$

$$-\frac{1}{y} = -\frac{1}{z} + C_2$$

$$\frac{dy}{y} = \frac{dz}{z}$$

$$\int \frac{dy}{y} = \int \frac{dz}{z}$$

$$\ln y = \ln z + C_3$$

Method of multipliers:

① Solve $x(y-z)p + y(z-x)q = z(x-y)$

(or)
Solve $\left(\frac{y-z}{yz}\right)p + \left(\frac{z-x}{zx}\right)q = \frac{x-y}{xy}$

Soln:

Given $\uparrow$

Method: $Pp + Qq = R$

Equation: $\therefore \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)}$$

Multipliers

$\times 1, 1, 1$

$$\frac{dx + dy + dz}{x(y-z) + y(z-x) + z(x-y)} = \frac{d(x+y+z)}{0}$$

$= \cancel{xy - xz} + \cancel{yz - yx} + \cancel{zx - zy} = 0$

$$d(x+y+z) = 0$$

$$\int d(x+y+z) = 0$$

$$\boxed{x+y+z = a}$$

$\times \frac{1}{x}, \frac{1}{y}, \frac{1}{z}$

$$\frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{(y-z) + (z-x) + (x-y)} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$$

$$\frac{1}{x} dx + \frac{1}{y} dy + \frac{1}{z} dz = 0$$

Integrating, $\int \frac{1}{x} dx + \int \frac{1}{y} dy + \int \frac{1}{z} dz = 0$

$$\log x + \log y + \log z = \log b$$

$$\log(xyz) = \log b$$

$$\boxed{xyz = b}$$

$$\therefore f(a, b) = 0$$

$$f(x+y+z, xyz) = 0$$

where f is arbitrary.

② Solve $(mz - ny)p + (nx - lz)q = ly - mx$

Soln:

Given $\otimes$

Method: $Pp + Qq = R$

Equation: $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\frac{dx}{mz - ny} = \frac{dy}{nx - lz} = \frac{dz}{ly - mx}$$

X x, y, z ,

$$\frac{x dx + y dy + z dz}{x(mz - ny) + y(nx - lz) + z(ly - mx)}$$

$$= \frac{x dx + y dy + z dz}{0}$$

$$= \frac{x dx + y dy + z dz}{0}$$

$$\Rightarrow x dx + y dy + z dz = 0$$

$\Delta \Rightarrow$ Integrating

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \frac{a}{2}$$

$$\boxed{x^2 + y^2 + z^2 = a}$$

X l, m, n ,

$$\frac{l dx + m dy + n dz}{l(mz - ny) + m(nx - lz) + n(ly - mx)} = \frac{l dx + m dy + n dz}{0}$$

$$= \frac{l(mz - ny) + m(nx - lz) + n(ly - mx)}{0}$$

$$\Rightarrow l m z - l n y + m n x - m l z + n l y - n m x = 0$$



$$l dx + m dy + n dz = 0$$

$$\text{Integrating } \int l dx + \int m dy + \int n dz = 0$$

$$\boxed{lx + my + nz = b}$$

$$f(a, b) = 0$$

$$f(x^2 + y^2 + z^2, lx + my + nz) = 0$$

where f is arbitrary.

③ Solve $(3z - 4y)p + (4x - 2z)q = 2y - 3x$

Soln. Given $\otimes$

Nov-18
Nov-11
Nov-6, 3

Method: $Pp + Qq = R$

Equation: $\therefore \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\frac{dx}{3z - 4y} = \frac{dy}{4x - 2z} = \frac{dz}{2y - 3x}$$

X x, y, z ,

$$\begin{aligned} & \frac{x dx + y dy + z dz}{x(3z - 4y) + y(4x - 2z) + z(2y - 3x)} \\ &= \frac{x dx + y dy + z dz}{3xz - 4xy + 4xy - 2y^2 + 2zy - 3zx} \\ &= \frac{x dx + y dy + z dz}{0} \end{aligned}$$

$$x dx + y dy + z dz = 0$$

Integrating $\int x dx + \int y dy + \int z dz = 0$

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \frac{a}{2}$$

$$\boxed{x^2 + y^2 + z^2 = a}$$

X 2, 3, 4,

$$\begin{aligned} & \frac{2dx + 3dy + 4dz}{2(3z - 4y) + 3(4x - 2z) + 4(2y - 3x)} \\ &= \frac{2dx + 3dy + 4dz}{6z - 8y + 12x - 12z + 8y - 12x} \end{aligned}$$



$$\begin{aligned} 2dx + 3dy + 4dz &= 0 \\ 2dx + 3dy + 4dz &= b \\ \boxed{2x + 3y + 4z &= b} \end{aligned}$$

$$\therefore f(a, b) = 0$$

$$f(x^2 + y^2 + z^2, 2x + 3y + 4z) = 0$$

where f is arbitrary.

(4) Find the general solution of

$$x(y^2 - z^2)p + y(z^2 - x^2)q = z(x^2 - y^2)$$

(or)

$$x(z^2 - y^2)p + y(x^2 - z^2)q = z(y^2 - x^2)$$

Soln:

Method $\therefore Pp + Qq = R$

Equation:- $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\frac{dx}{x(z^2 - y^2)} = \frac{dy}{y(x^2 - z^2)} = \frac{dz}{z(y^2 - x^2)}$$

X x, y, z

$$= \frac{x dx + y dy + z dz}{x^2(z^2 - y^2) + y^2(x^2 - z^2) + z^2(y^2 - x^2)}$$

$$= \frac{x dx + y dy + z dz}{x^2 z^2 - x^2 y^2 + y^2 x^2 - y^2 z^2 + z^2 y^2 - z^2 x^2}$$

$$= \frac{x dx + y dy + z dz}{0}$$

$$x dx + y dy + z dz = 0$$

$$\int x dx + \int y dy + \int z dz = 0$$

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \frac{a^2}{2}$$

$$\frac{x^2 + y^2 + z^2}{2} = \frac{a^2}{2}$$

$$x^2 + y^2 + z^2 = a^2$$

$$\times \frac{1}{x}, \frac{1}{y}, \frac{1}{z}$$

$$= \frac{1}{x} dx + \frac{1}{y} dy + \frac{1}{z} dz$$

$$z^2 - y^2 + x^2 - z^2 + y^2 - x^2 = 0$$

$$\frac{1}{x} dx + \frac{1}{y} dy + \frac{1}{z} dz = 0$$

$$\int \frac{1}{x} dx + \int \frac{1}{y} dy + \int \frac{1}{z} dz = 0$$

$$\log x + \log y + \log z = \log b$$

$$xyz = b$$

$$\therefore f(a, b) = 0$$

$$f(x^2 + y^2 + z^2, xyz) = 0$$

Where f is arbitrary

⑤ solve

$$(x - 2z)p + (2z - y)q = y - x$$

Soln:

Given

$$\text{Method: } Pp + Qq = R$$

Equation

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{x - 2z} = \frac{dy}{2z - y} = \frac{dz}{y - x}$$

$$\times 1, 1, 1$$

$$dx + dy + dz$$

$$x - 2z + 2z - y + y - x = 0$$

$$= \frac{dx + dy + dz}{0}$$

$$dx + dy + dz = 0$$

$$\int d(x + y + z) = a$$

$$x + y + z = a$$

$$\times y, x, 2z$$

$$= \frac{y dx + x dy + 2z dz}{0}$$

$$y(x - 2z) + x(2z - y) + 2z(y - x)$$

$$yx - 2yz + 2xz - xy + 2zy - 2zx = 0$$

$$= \frac{y dx + x dy + 2z dz}{0}$$

$$y dx + x dy + 2z dz = 0$$

$$\int y dx + \int x dy + 2 \int z dz = 0$$

$$\frac{y^2}{2} + \frac{x^2}{2} + \frac{2z^2}{2}$$

$$d(xy) + 2z dz = 0$$

$$\int d(xy) + 2 \int z dz = a$$

$$xy + \frac{2z^2}{2} = b$$

$$xy + z^2 = b$$

Solution:

$$f(a, b) = 0$$

$$f(x + y + z, xy + z^2) = 0$$

where f is arbitrary.

⑥ solve

$$x(y^2+z^2)p + y(z^2+x^2)q = z(y^2-x^2)$$

Soln:

Method $Pp + Qq = R$

Equation:

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{x(y^2+z^2)} = \frac{dy}{y(z^2+x^2)} = \frac{dz}{z(y^2-x^2)}$$

$x, -\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$

$$= \frac{-\frac{1}{x} dx + \frac{1}{y} dy + \frac{1}{z} dz}{-y^2 - z^2 + z^2 + x^2 + y^2 - x^2} = 0$$

$$= \frac{-\frac{1}{x} dx + \frac{1}{y} dy + \frac{1}{z} dz}{0}$$

$$-\frac{1}{x} dx + \frac{1}{y} dy + \frac{1}{z} dz = 0$$

$$-\int \frac{1}{x} dx + \int \frac{1}{y} dy + \int \frac{1}{z} dz = 0$$

$$-\log x + \log y + \log z = \log a$$

$$\log \left(\frac{yz}{x} \right) = \log a$$

$$\boxed{\frac{yz}{x} = a}$$

$x, -y, z$

$$= \frac{xdx - ydy + zdz}{0}$$

$$\frac{x^2 y^2 + x^2 z^2 - y^2 z^2 - y^2 x^2 + z^2 y^2 - z^2 x^2}{0} = 0$$

$$xdx - ydy + zdz = 0$$

$$\int x dx - \int y dy + \int z dz = 0$$

$$\frac{x^2}{2} - \frac{y^2}{2} + \frac{z^2}{2} = \frac{a}{2}$$

$$\boxed{x^2 - y^2 + z^2 = a}$$

Solution

$$f(a, b) = 0$$

$$f\left(\frac{yz}{x}, x^2 - y^2 + z^2\right) = 0$$

where f is arbitrary

⑦ solve

$$x(y^2+z^2)p + y(x^2+z^2)q = z(x^2-y^2)$$

Multipliers ① $\frac{1}{x}, -\frac{1}{y}, \frac{1}{z}$

② $x, -y, -1$

2 marks &
8 marks

1.5) Linear Partial P.E of 2nd & Higher Order with constant coefficients of Both Homogeneous And Non-Homogeneous Types.

To find the Complementary Functions :-

Roots	C.F
Roots are different $m_1 \neq m_2 \neq m_3$	$\phi_1(y+m_1x) + \phi_2(y+m_2x) + \phi_3(y+m_3x)$
Two roots are equal $m_1 \neq m_2 = m_3$	$\phi_1(y+m_1x) + x\phi_2(y+m_2x) + \phi_3(y+m_3x)$
Three roots are equal $m_1 = m_2 = m_3$	$\phi_1(y+m_1x) + x\phi_2(y+m_2x) + x^2\phi_3(y+m_3x)$
$m_1, \alpha \pm i\beta$	$\phi_1(y+m_1x) + \phi_2(y+(\alpha+i\beta)x) + \phi_3(y+(\alpha-i\beta)x)$

Replace $D \rightarrow m$
 $D' \rightarrow 1$

Problems

APR-08

① Solve $(D^2 - 2DD' + D'^2)z = 0$

Put $D=m, D'=1$.

$$A.E \text{ is } m^2 - 2m + 1 = 0$$

$$m = 1, 1$$

C.F

$$z = \phi_1(y+x) + x\phi_2(y+x)$$

1
-1 -1
opposite sign
+1, 1

$$\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 & 0 \\ \hline 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 & 0 \end{array}$$



② Nov-09 Solve $(D^2 - 2DD')z = 0$

Put $D=m, D'=1$

A.E $m^2 - 2m = 0$

$m^2(m-2) = 0$

$m^2 = 0 ; m-2 = 0$

$m = 0, 0 \quad m = 2$

$z = \phi_1(y) + x\phi_2(y) + \phi_3(y+2x)$

H.W ① $(D^2 + 5DD' + 6D'^2)z = 0$

③ Nov-10

Find the P.I of $(D^2 - 2DD' + D'^2)z = e^{x-y}$

P.I = $\frac{e^{x-y}}{D^2 - 2DD' + D'^2}$
 $= \frac{e^{x-y}}{1+2+1}$
 $= \frac{e^{x-y}}{4}$

Put $D=1$
 $D'=-1$

④ Nov-12

Solve $(D-1)(D-D'+1)z = 0$

A.E $(m-1)(m-1+1) = 0$
 $(m-1)m = 0$
 $m = 0, 1$

Put $D=m$
 $D'=1$

$z = \phi_1(y) + \phi_2(y+x)$

⑤ Solve $4 \frac{\partial^2 z}{\partial x^2} - 12 \frac{\partial^2 z}{\partial x \partial y} + 9 \frac{\partial^2 z}{\partial y^2} = 0$

$(4D^2 - 12DD' + 9D'^2)z = 0$

$4m^2 - 12m + 9 = 0$

$m = \frac{3}{2}, \frac{3}{2}$

$z = \phi_1(y + \frac{3}{2}x) + x\phi_2(y + \frac{3}{2}x)$

36
 $\wedge$
 $-6 \quad -6$
 $-\frac{6}{4} \quad -\frac{6}{4}$
 $-3/2, -3/2$

$D = \frac{\partial}{\partial x} \quad D = m$
 $D' = \frac{\partial}{\partial y} \quad D' = 1$

opposite sign
 $3/2, 3/2$

⑥ Find the particular integral of

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} = \sin(x-y)$$

$$P.I = \frac{1}{D^2 - DD'} \sin(ax+by)$$

$$= \frac{1}{-1-41} \sin(x-y)$$

$$= \frac{1}{-2} \sin(x-y)$$

$$\begin{aligned} D^2 &= -a^2 = -1 \\ DD' &= -1 \times 1 = +1 \end{aligned}$$

Rule

$$P.I = \frac{1}{f(D,D')} \sin(ax+by) \text{ (or) } \cos(ax+by)$$

$$D^2 = -a^2$$

$$D'^2 = -b^2$$

$$DD' = -ab$$

Particular Integral :

$$P.I = \frac{1}{f(D,D')} F(x,y)$$

S.No	F(x,y)	P.I
1.	e^{ax+by}	$P.I = \frac{1}{f(D,D')} e^{ax+by}$ $= e^{ax+by} \frac{1}{f(a,b)} ; f(a,b) \neq 0$ $= x e^{ax+by} \frac{1}{f'(a,b)} \left\{ \begin{array}{l} f(a,b) = 0 \\ f'(a,b) \neq 0 \end{array} \right.$
2 ✓ (X)	$\sin(ax+by) \text{ (or) } \cos(ax+by)$	$P.I = \frac{1}{f(D,D')} [\sin(ax+by) \text{ (or) } \cos(ax+by)]$ $\left. \begin{array}{l} D^2 = -a^2 \\ D'^2 = -b^2 \\ DD' = -ab \end{array} \right\} \text{ (X) }$

Type: 1

① Solve $\frac{\partial^2 z}{\partial x^2} - 4 \frac{\partial^2 z}{\partial x \partial y} + 4 \frac{\partial^2 z}{\partial y^2} = e^{2x+y}$

Soln
A.E is $(D^2 - 4DD + 4D^2)z = e^{2x+y}$

$$m^2 - 4m + 4 = 0$$

$$m = 2, 2$$

$$C.F = \phi_1(y+mx) + x \phi_2(y+mx)$$

$$C.F = \phi_1(y+2x) + x \phi_2(y+2x)$$

4
-2 -2
opposite sign
m=2, 2

$$P.I = \frac{1}{D^2 - 4DD + 4D^2} e^{2x+y}$$

$$4 - 4(2)(1) + 4 = 8 - 8 = 0$$

ordinary rules fails.

$$= \frac{1}{D^2 - 4D + 4} e^{2x+y}$$

$$= \frac{x}{2D - 4} e^{2x+y}$$

$$2 \times 2 - 4 = 4 - 4 = 0 \text{ (ordinary rules fails)}$$

$$P.I = \frac{x^2}{2} e^{2x+y}$$

Solution:

$$z = C.F + P.I$$

$$z = \phi_1(y+2x) + x \phi_2(y+2x) + \frac{x^2}{2} e^{2x+y}$$

② Solve $[D^2 - 2DD' + D'^2]z = 8e^{x+2y}$

Soln

A.E is $m^2 - 2m + 1 = 0$

C.F = $\phi_1(y+m_1x) + \phi_2(y+m_2x)$

C.F = $\phi_1(y+x) + \phi_2(y+x)$

Opposite sign
 $m=1, 1$

P.I = $\frac{1}{D^2 - 2DD' + D'^2} 8e^{x+2y}$

= $\frac{8}{1 - 2(1)(2) + 4} e^{x+2y}$

= $\frac{8}{(1-4+4)} e^{x+2y}$

P.I = $8e^{x+2y}$

Solution

$z = C.F + P.I$

$z = \phi_1(y+x) + \phi_2(y+x) + 8e^{x+2y}$

③ Solve $(D^2 + 2DD' + D'^2)z = e^{x-y}$

Soln

A.E is

$m^2 + 2m + 1 = 0$

C.F = $\phi_1(y+m_1x) + \phi_2(y+m_2x)$

C.F = $\phi_1(y-x) + \phi_2(y-x)$

P.I = $\frac{1}{D^2 + 2DD' + D'^2} e^{x-y}$

= $\frac{1}{1 - 2 + 1} e^{x-y}$

= $\frac{1}{2D + 2D'} e^{x-y}$

= $\frac{1}{2 - 2} e^{x-y}$ ordinary rules are fails

$D=1$
 $D' \rightarrow -1$

= $\frac{x^2}{2} e^{x-y}$

Solution

$z = C.F + P.I$

$z = \phi_1(y-x) + \phi_2(y-x) + \frac{x^2}{2} e^{x-y}$



Type-II

- ① $\sin A \cos B = \frac{1}{2} [\sin(A+B) + \sin(A-B)]$
 ② $\cos A \sin B = \frac{1}{2} [\sin(A+B) - \sin(A-B)]$
 ③ $\cos A \cos B = \frac{1}{2} [\cos(A+B) + \cos(A-B)]$
 ④ $\sin A \sin B = \frac{1}{2} [\cos(A-B) - \cos(A+B)]$

① Solve $(D^2 - 2DD' + D^2)z = \cos(x-3y)$

Soln:

A.E

$$m^2 - 2m + 1 = 0$$

$$m = 1, 1$$

1
-1 -1
opposite
sign
1, 1

$$\boxed{D=m}$$

$$\boxed{D'=1}$$

$$C.F = \phi_1(y+x) + x \phi_2(y+x)$$

$$P.I = \frac{1}{D^2 - 2DD' + D^2} \cos(x-3y)$$

$$= \frac{1}{-1 - 2(3) + (-9)} \cos(x-3y)$$

$$P.I = \frac{\cos(x-3y)}{-16}$$

Solution

$$z = C.F + P.I$$

$$z = \phi_1(y+x) + x \phi_2(y+x) - \frac{\cos(x-3y)}{16}$$

$$a=1; b=-3$$

$$D^2 = -a^2 = -1$$

$$D'^2 = -(b^2) = -3^2 = -9$$

$$DD' = -(ab) = -(-3) = 3$$

② Solve $(D^2 - 2DD' + 2D^2)z = \sin(x-y)$

$$D^2 = -a^2 = -1$$

$$D'^2 = -(b^2) = -(-1)^2$$

$$DD' = -ab = -(-1)(1) = 1$$

$$\sin(x-y)$$

A.E

$$m^2 - 2m + 2 = 0$$

$$a=1; b=-2; c=2$$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{2 \pm \sqrt{4 - 8}}{2}$$

$$= \frac{2 \pm \sqrt{-4}}{2}$$

$$= \frac{2 \pm 2i}{2}$$

Solution: $m = 1 \pm i$

$$C.F = \phi_1(y + (1+i)x) + \phi_2(y + (1-i)x)$$

$$P.I = \frac{1}{D^2 - 2DD' + 2D^2}$$

$$= \frac{\sin(x-y)}{-1 - 2(1) + 2(-1)}$$

$$P.I = -\frac{\sin(x-y)}{5}$$

$$z = C.F + P.I$$

$$z = \phi_1(y + (1+i)x) + \phi_2(y + (1-i)x) - \frac{\sin(x-y)}{5}$$

① Solve $(D^2 - DD' - 20D'^2)Z = e^{5x+y} + \sin(4x-y)$

Soln:

A.E

$D = m$
 $D' = 1$

$m^2 - m - 20 = 0$

$m = -4, 5$

-20



$-4 \quad 5 = -1$

opposite sign

C.F = $\phi_1(y-4x) + \phi_2(y+5x)$

P.I = $\frac{1}{D^2 - DD' - 20D'^2}$

$\frac{e^{5x+y}}{25 - 5 - 20 = 0}$

$D = 5,$
 $D' = 1$

$D^2 - D - 20$

$\frac{x e^{5x+y}}{2D - D'}$

$\frac{x e^{5x+y}}{2 \times 5 - 1}$

P.I = $\frac{x}{9} e^{5x+y}$

$a = 4; b = -1$

$D^2 = -a^2 = -(4)^2 = -16$

$D'^2 = -b^2 = -(-1)^2 = -1$

$DD' = -ab = -4 \times (-1) = 4$

P.I = $\frac{1}{D^2 - DD' - 20D'^2} \sin(4x-y)$

$\frac{\sin(4x-y)}{-16 - 4 - 20(-1) = 0}$

$\frac{x \sin(4x-y)}{2D - D'}$

$D = \frac{\partial}{\partial x} \quad D' = \frac{\partial}{\partial y}$

$\frac{x}{2D - D'} \times \frac{2D + D'}{2D + D'} \sin(4x-y)$

$(a+b)(a-b) = a^2 - b^2$

$\frac{x}{4D^2 - D'^2} [2D(\sin(4x-y)) + D'(\sin(4x-y))]$

$\frac{x}{4(-16) - (-1)} [2 \times 4 \cos(4x-y) - \cos(4x-y)]$

$-64 + 1 = -63$

$\frac{x}{-63} [8 \cos(4x-y) - \cos(4x-y)]$



Solution

$$P.I = C.F + P.I$$

$$= \phi_1(y-4x) + \phi_2(y+5x) + \frac{2}{9} e^{5x+y} - \frac{x}{63} [8 \cos(4x-y) - \cos(4x-y)]$$

NOV-13

$$(2) \text{ Solve } (D^2 - DD' - 2D'^2) = 2x + 3y + e^{3x+4y}$$

A.E $m^2 - m - 2 = 0$
 $m = -1, 2$

$\begin{matrix} -2 \\ \wedge \\ 1-2 \\ \text{opposite sign} \\ -1, 2 \end{matrix}$

$\begin{matrix} D = m \\ D' = 1 \end{matrix}$

$$C.F = \phi_1(y-x) + \phi_2(y+2x)$$

$$P.I = \frac{1}{D^2 - DD' - 2D'^2} (2x + 3y)$$

$$= \frac{1}{D^2 \left[1 - \left(\frac{D'}{D} + \frac{2D'^2}{D^2} \right) \right]} (2x + 3y)$$

$$= \frac{1}{D^2} \left[1 - \left(\frac{D'}{D} + \frac{2D'^2}{D^2} \right) \right]^{-1} (2x + 3y)$$

$(1-x)^{-1} = 1 + x + x^2 + x^3 + \dots$

$$= \frac{1}{D^2} \left[1 + \frac{D'}{D} + \frac{2D'^2}{D^2} \right] (2x + 3y)$$

[omit D'^2 & higher power]

y-mu m-cis

$$= \frac{1}{D^2} \left[2x + 3y + \frac{1}{D} D'(2x + 3y) \right]$$

$$= \frac{1}{D^2} \left[2x + 3y + \frac{1}{D} (3) \right]$$

$D'(2x + 3y) = 0 + 3 = 3$

$D' = \frac{\partial}{\partial y}$

$$= \frac{1}{D^2} (2x + 3y + 3x)$$

$$= \frac{1}{D^2} (5x + 3y)$$

$$= \frac{1}{D} \left(\frac{5x^2}{2} + 3yx \right)$$

$\frac{1}{D} (5x) = 5 \int x dx = \frac{5x^2}{2}$

$\frac{1}{D^2} (5x) = 5 \int \frac{x^2}{2} dx = \frac{5x^3}{6}$

$$P.I = \frac{5x^3}{6} + \frac{3yx^2}{2}$$

$$P.I = \frac{1}{D^2 - DD' - 2D'^2} e^{3x+4y}$$

$\begin{matrix} D = 3 \\ D' = 4 \end{matrix}$

$$= \frac{e^{3x+4y}}{9 - 12 - 32} = -\frac{1}{35} e^{3x+4y}$$

Solution

$$Z = C.F + P.I$$

$$Z = \phi_1(y-x) + \phi_2(y+2x) + \frac{5x^3}{6} + \frac{3yx^2}{2} - \frac{1}{35} e^{3x+4y}$$



③ Solve $[D^2 + 3DD' + 2D'^2]Z = x+y$

A.E

$$m^2 + 3m + 2 = 0$$

$$m = -1, -2$$

$$\begin{matrix} D = m \\ D' = 1 \end{matrix}$$

2
1 2
opposite sign
-1, -2

$$C.F = \phi_1(y-x) + \phi_2(y-2x)$$

$$P.I = \frac{1}{D^2 + 3DD' + 2D'^2} (x+y)$$

$$= \frac{1}{D^2 \left[1 + \left(\frac{3D'}{D} + \frac{2D'^2}{D^2} \right) \right]} (x+y)$$

$$= \frac{1}{D^2} \left[1 + \left(\frac{3D'}{D} + \frac{2D'^2}{D^2} \right) \right]^{-1} (x+y)$$

$$= \frac{1}{D^2} \left(1 - \frac{3D'}{D} - \frac{2D'^2}{D^2} \right) (x+y)$$

$$(1+x)^{-1} = 1 - x + x^2 - \dots$$

$$= \frac{1}{D^2} \left[x+y - \frac{3D'(x+y)}{D} \right]$$

omit D'^2 & higher power.

$$= \frac{1}{D^2} \left[x+y - \frac{3}{D}(1) \right]$$

$$D'(x+y) = 0 + 1 = 1$$

$$= \frac{1}{D^2} (x) + \frac{1}{D^2} (y) - \frac{3}{D^3}$$

$$= \frac{x^3}{6} + \frac{yx^2}{2} - \frac{3x^3}{6}$$

$$= \frac{yx^2}{2} - \frac{2x^3}{3}$$

$$P.I = \frac{yx^2}{2} - \frac{x^3}{3}$$

Solution:

$$Z = C.F + P.I$$

$$Z = \phi_1(y-x) + \phi_2(y-2x) + \frac{yx^2}{2} - \frac{x^3}{3}$$

$$\frac{1}{D}(x) = \int x dx = \frac{x^2}{2}$$

$$\frac{1}{D^2}(x) = \int \frac{x^2}{2} dx = \frac{x^3}{6}$$

$$\frac{1}{D}(1) = \int dx = x$$

$$\frac{1}{D^2}(1) = \int x dx = \frac{x^2}{2}$$

$$\frac{1}{D^3}(1) = \int \frac{x^2}{2} dx = \frac{x^3}{6}$$

① Solve $(D^3 - 2D^2D')z = 0$

Soln:

A.E is

$$m^3 - 2m^2 = 0$$

$$m^2(m-2) = 0$$

$$m^2 = 0 \quad | \quad m-2 = 0$$

$$m = 0, 0 \quad | \quad \boxed{m = 2}$$

$$\boxed{\begin{matrix} D = m \\ D' = 1 \end{matrix}}$$

$$C.F = \phi_1(y + m_1x) + x \phi_2(y + m_2x) + \phi_3(y + m_3x)$$

$$= \phi_1(y + 0x) + x \phi_2(y + 0x) + \phi_3(y + 2x)$$

$$C.F = \phi_1(y) + x \phi_2(y) + \phi_3(y + 2x)$$

② Find the General Solution of

Soln: $(D^2 - 5DD' + 6D'^2)z = 0$

A.E is

$$m^2 - 5m + 6 = 0$$

$$m = 3, 2$$

$$\boxed{\begin{matrix} D = m \\ D' = 1 \end{matrix}}$$

$$C.F = \phi_1(y + m_1x) + \phi_2(y + m_2x)$$

$$C.F = \phi_1(y + 3x) + \phi_2(y + 2x)$$

opposite sign
3 2

③ Find the particular integral of

$$(D^2 + 2DD' + D'^2)z = e^{x-y}$$

Soln:

$$P.I = \frac{1}{D^2 + 2DD' + D'^2}$$

$$= \frac{1}{1 + 2(1)(-1) + (-1)^2} e^{x-y}$$

$$= \frac{1}{1 - 2 + 1} = \frac{1}{0}$$

$$\boxed{\begin{matrix} D = 1 \\ D' = -1 \end{matrix}}$$

$$P.I = \frac{x e^{x-y}}{2D + 2D'}$$

$$D=1$$

$$D'=-1$$

$$2(1) + 2(-1) = 2 - 2 = 0$$

$$P.I = \frac{x^2 e^{x-y}}{2}$$

④ Solve $(D^2 - 3DD' + 2D'^2)z = 0$

A.E is

$$m^2 - 3m + 2 = 0$$

$$m = 1, 2$$

$$\boxed{D = m}$$

$$\boxed{D' = 1}$$

$$z = \phi_1(y + m_1 x) + \phi_2(y + m_2 x) \quad -1 \quad -2$$

$$\boxed{z = \phi_1(y + x) + \phi_2(y + 2x)}$$

opposite
sign
1, 2

⑤ Find the particular integral of $(D^2 - 2DD' + D'^2)z = e^{x-y}$

Soln:

$$\textcircled{P} \textcircled{D'} \\ x-y$$

$$P.I = \frac{1}{D^2 - 2DD' + D'^2} e^{x-y}$$

$$\textcircled{D=1}$$

$$\textcircled{D'=-1}$$

$$= \frac{1}{1 - 2(1)(-1) + 1} e^{x-y}$$

$$= \frac{1}{1 + 2 + 1} e^{x-y}$$

$$\boxed{P.I = \frac{1}{4} e^{x-y}}$$



⑥ Solve $(D^4 - D'^4) \cdot z = 0$

Soln

A.E is

$$\begin{matrix} D = m \\ D' = 1 \end{matrix}$$

$$m^4 - 1 = 0$$

$$a^2 - b^2 = (a+b)(a-b)$$

$$(m^2 - 1)(m^2 + 1) = 0$$

$$m^2 - 1 = 0 \quad \left| \quad m^2 + 1 = 0 \right.$$

$$m^2 = 1$$

$$m^2 = -1$$

$$m = \pm 1$$

$$m = \pm i$$

$$z = \phi_1(y + m_1 x) + x \phi_2(y + m_2 x) + x^2 \phi_3(y + m_3 x) + \phi_4(y + m_4 x)$$

$$z = \phi_1(y + x) + x \phi_2(y - x) + x^2 \phi_3(y + ix) + \phi_4(y - ix)$$

⑦ Solve the equation $(D - D')^3 z = 0$

Soln

$$(m - 1)^3 = 0$$

$$\begin{matrix} D = m \\ D' = 1 \end{matrix}$$

$$(m - 1)(m - 1)(m - 1) = 0$$

$$m - 1 = 0 ; m = 1, m = 1$$

$$m = 1, 1, 1$$

$$z = \phi_1(y + m_1 x) + x \phi_2(y + m_2 x) + x^2 \phi_3(y + m_3 x)$$

$$z = \phi_1(y + x) + x \phi_2(y + x) + x^2 \phi_3(y + x)$$

⑧ Solve $(D^2 - 7DD' + 6D'^2) = 0$

Soln: A.E is $m^2 - 7m + 6 = 0$

$$m_1 = 1; m_2 = 6$$

$$z = \phi_1(y + m_1 x) + \phi_2(y + m_2 x)$$

$$z = \phi_1(y + x) + \phi_2(y + 6x)$$

$$\begin{matrix} D = m \\ D' = 1 \end{matrix}$$

$\begin{matrix} b \\ \swarrow \searrow \\ -1 \quad -6 \\ \text{opposite sign} \\ 1, 6 \end{matrix}$



① obtain the P.d.e by eliminating arbitrary constants a & b from $(x+a)^2 + (y-b)^2 = z$.

Soln ∴

$$\text{Given } z = (x+a)^2 + (y-b)^2 \rightarrow \textcircled{1}$$

$$P = \frac{\partial z}{\partial x} = 2(x+a)$$

$$2(x+a) = P$$

$$\boxed{x+a = \frac{P}{2}}$$

$$q = \frac{\partial z}{\partial y} = 2(y-b)$$

$$2(y-b) = q$$

$$\boxed{y-b = \frac{q}{2}}$$

From ①,

$$z = (x+a)^2 + (y-b)^2$$

$$= \left(\frac{P}{2}\right)^2 + \left(\frac{q}{2}\right)^2$$

$$= \frac{P^2}{4} + \frac{q^2}{4}$$

$$z = \frac{P^2 + q^2}{4}$$

$$\boxed{4z = P^2 + q^2}$$

② Form the Pde by eliminating the arbitrary constants a & b from the relation,

$$z = (x+a)(y+b)$$

Soln

$$\text{Given } z = (x+a)(y+b) \rightarrow \textcircled{1}$$

$$P = \frac{\partial z}{\partial x} = (1)(y+b)$$

$$\boxed{P = y+b} \rightarrow \textcircled{2}$$

$$q = \frac{\partial z}{\partial y} = (x+a)(1)$$

$$\boxed{q = (x+a)} \rightarrow \textcircled{3}$$

Sub ② & ③ in ①

$$z = (x+a)(y+b)$$

$$\boxed{z = Pq}$$

Unit-II

Fourier Series

Harmonic Analysis

$$a_0 = \frac{2}{N} \sum y$$

$$a_n = \frac{2}{N} \sum y \cos nx$$

$$b_n = \frac{2}{N} \sum y \sin nx$$

$$y = \frac{a_0}{2} + \sum (a_n \cos nx + b_n \sin nx)$$

$$y = \frac{a_0}{2} + [a_1 \cos x + b_1 \sin x] + [a_2 \cos 2x + b_2 \sin 2x] + \dots$$

Type: 1

Given Data Are In (π) Form

① Compute the first two harmonics of the fourier series of $f(x)$ given the following data

x	0	$\frac{\pi}{3}$	$2\frac{\pi}{3}$	π	$\frac{4\pi}{3}$	$5\frac{\pi}{3}$	2π
$f(x)$	1.0	1.4	1.9	1.7	1.5	1.2	1.0

Soln:

$$N = \text{No. of Intervals} = 6.$$

[1st & Last value same].
(omit the last value).

x	y	$y \cos x$	$y \cos 2x$	$y \sin x$	$y \sin 2x$
0	1.6	1	1	0	0
60'	1.4	0.7	-0.7	1.21	1.21
120'	1.9	-0.95	-0.95	1.64	-1.64
180'	1.7	-0.75 -1.7	1.7	0	0
240'	1.5	-0.75	-0.75	-1.29	1.29
300'	1.2	0.6	-0.6	-1.03	-1.03
Total	8.7	-1.1	-0.3	0.53	-0.17

$$a_0 = \frac{2}{N} \sum y$$

$$= \frac{2}{6} (8.7)$$

$$\boxed{a_0 = 2.9}$$

$$a_n = \frac{2}{N} \sum y \cos nx$$

$$a_1 = \frac{2}{6} \sum y \cos x$$

$$= \frac{2}{6} (-1.1)$$

$$\boxed{a_1 = -0.36}$$

$$a_2 = \frac{2}{6} \sum y \cos 2x$$

$$= \frac{2}{6} (-0.3)$$

$$\boxed{a_2 = -0.1}$$

$$b_n = \frac{2}{N} \sum y \sin nx$$

$$b_1 = \frac{2}{6} (0.53)$$

$$\boxed{b_1 = 0.176}$$

$$b_2 = \frac{2}{6} \sum y \sin 2x$$

$$= \frac{2}{6} (-0.17)$$

$$\boxed{b_2 = -0.056}$$

$$y = \frac{a_0}{2} + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + \dots$$

$$= \frac{2.9}{2} + (-0.36) \cos x + (0.176) \sin x$$

$$+ ((0.1) \cos 2x + (-0.056) \sin 2x) + \dots$$

Type: 2 Given Data Are In $\textcircled{T}$ Form

$$\theta = \frac{2\pi x}{T}$$

① The values of x & the corresponding values of $f(x)$ over a period T are given below. Show that

$$f(x) = 0.75 + 0.37 \cos \theta + 1.004 \sin \theta. \text{ Where}$$

$$\theta = \frac{2\pi x}{T}$$

x	0	$T/6$	$T/3$	$T/2$	$2T/3$	$5T/6$	T
$f(x)$	1.98	1.30	1.05	1.30	-0.88	-0.25	1.98

Soln:

$N = \text{No. of Intervals} = 6$

(1st & last value same, omit the last value)

x	y	$\theta = \frac{2\pi x}{T}$	$y \cos \theta$	$y \sin \theta$
0	1.98	$0 = 0^\circ$	1.98	0
$T/6$	1.30	$\pi/3 = 60^\circ$	0.65	1.1258
$T/3$	1.05	$2\pi/3 = 120^\circ$	-0.525	0.9093
$T/2$	1.30	$\pi = 180^\circ$	-1.3	0
$2T/3$	-0.88	$4\pi/3 = 240^\circ$	0.44	0.762
$5T/6$	-0.25	$5\pi/3 = 300^\circ$	-0.125	0.2165
Total	4.5		-1.12	3.0136

$$a_0 = \frac{2}{b} \sum y$$

$$= \frac{4.5}{3}$$

$$\boxed{a_0 = 1.5}$$

$$\frac{a_0}{2} = \frac{1.5}{2} = 0.75$$

$$a_1 = \frac{2}{b} \sum y \cos \theta$$

$$= \frac{2}{6} (1.12)$$

$$\boxed{a_1 = 0.3733}$$

$$b_1 = \frac{2}{b} \sum y \sin \theta$$

$$= \frac{2}{6} (3.0136)$$

$$\boxed{b_1 = 1.0045}$$

$$f(x) = \frac{a_0}{2} + a_1 \cos \theta + b_1 \sin \theta$$

$$f(x) = 0.75 + 0.373 \cos \theta + 1.0045 \sin \theta$$

Type: 3

Given Data Are In Degree Form

① The displacement y of a part of a mechanism is tabulated with corresponding angular movement x° of the crank. Express y as a Fourier Series neglecting the harmonic above third

x°	0	30	60	90	120	150	180	210	240	270	300
y	1.80	1.10	0.30	0.16	1.50	1.30	2.16	1.25	1.30	1.52	1.76

Soln:-

$$a_0 = \frac{2}{n} (\sum y) = \frac{2}{12} (15.15) = 2.525$$

$$a_1 = \frac{2}{n} (\sum y \cos x) = \frac{2}{12} (0.246) = 0.041$$

$$a_2 = \frac{2}{n} (\sum y \cos 2x)$$

Type: 1

Given Data Are In 2 form

Formula:

$$\phi = \frac{\pi x}{l}$$

① Find the first two harmonics of the Fourier series from the following table.

x	0	1	2	3	4	5
y	9	18	24	28	26	20

Soln:

$$2l = 6 \quad ; \quad n = \text{no. of intervals} = 6$$

$$l = \frac{6}{2}$$

$$l = 3$$

x	$\phi = \frac{\pi x}{3}$	y	$y \cos 0$	$y \cos 20$	$y \sin 0$	$y \sin 20$
0	0°	9	9	9	0	0
1	60°	18	9	-9		
2	120°	24	-12	-12	15.588	15.5885
3	180°	28	-28	28	20.785	-20.7846
4	240°	26	-13	-13	0	0
5	300°	20	10	-10	-22.517	22.517
Total		125	-25	-7	-3.465	0.0004

$$a_0 = \frac{2}{n} \sum y = 2 \left(\frac{125}{6} \right) = 41.67 \quad ; \quad \frac{a_0}{2} = \frac{41.67}{2} = 20.84$$

$$a_1 = \frac{2}{n} \sum y \cos 0 = 2 \left(\frac{-25}{6} \right) = -8.33$$

$$a_2 = \frac{2}{n} \sum y \cos 20 = \frac{2}{6} (-7) = -2.333$$



$$b_1 = \frac{a}{n} \sum y \sin \theta = \frac{2}{6} (-3.465) = -1.16$$

$$b_2 = \frac{a}{n} \sum y \sin 2\theta = \frac{2}{6} (0.0004) = 0.00013$$

$$y = \frac{a_0}{2} + a_1 \cos \theta + b_1 \sin \theta + a_2 \cos 2\theta + b_2 \sin 2\theta$$

$$y = 20.84 - 8.33 \cos \theta - 1.16 \sin \theta - 2.33 \cos 2\theta + 0.00013 \sin 2\theta$$

$$\therefore \theta = \frac{\pi x}{3}$$



Example

Parseval's Relation (or) Theorem (or) Identity

Statement

Let $f(x)$ be a periodic function with period 2π defined in the interval $(-\pi, \pi)$. Then

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} (f(x))^2 dx = \frac{a_0^2}{4} + \frac{1}{2} \sum_{n=1}^{\infty} [a_n^2 + b_n^2]$$

Where a_0, a_n & b_n are Fourier coefficient of $f(x)$

Problems

① Find the Sine series for $f(x) = x$ in $0 < x < \pi$.

Using R.M.S value. Show that

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots$$

Soln ∴

Given $b_n = \frac{-2}{n} (-1)^n$

$$b_n^2 = \frac{4}{n^2} (-1)^{2n} = \frac{4}{n^2}$$

Given $f(x) = x$
 $f(-x) = -x$ | $f(x) \neq f(-x)$
 This is an odd function

$$\left. \begin{array}{l} f(x) = x \\ (f(x))^2 = x^2 \end{array} \right\}$$

$$\therefore f(x) = x$$

$$(f(x))^2 = x^2$$

$$\int_0^{\pi} (f(x))^2 dx = \int_0^{\pi} x^2 dx = \left[\frac{x^3}{3} \right]_0^{\pi} = \frac{\pi^3}{3} - 0 = \frac{\pi^3}{3}$$

Using R.M.S value..

$$\frac{2}{\pi} \int_0^{\pi} (f(x))^2 dx = \sum_{n=1}^{\infty} b_n^2$$

$$\frac{2}{\pi} \left(\frac{\pi^3}{3} \right) = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{2\pi^2}{3 \times 2}$$

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6}$$

Hence the proof

② Find the fourier series x^2 in $(-\pi, \pi)$. Use parsevals identity to prove

$$\frac{\pi^4}{90} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \dots$$

Soln:- $a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) dx = \frac{2}{\pi} \int_0^{\pi} x^2 dx = \frac{2}{\pi} \left[\frac{x^3}{3} \right]_0^{\pi} = \frac{2}{\pi} \left(\frac{\pi^3}{3} \right) = \frac{2\pi^2}{3}$

Given

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx = \frac{2}{\pi} \int_0^{\pi} x^2 \cos nx dx$$

$$a_0 = \frac{2\pi^2}{3}$$

$$a_n = \frac{4}{n^2} (-1)^n$$

$$b_n = 0$$

$$a_0^2 = \frac{4\pi^4}{9}$$

$$a_n^2 = \frac{16}{n^4}$$

$$b_n^2 = 0$$

$$f(x) = x^2$$

$$(f(x))^2 = x^4$$

Given
 $f(x) = x^2$
 $f(-x) = (-x)^2 = x^2$
 $f(x) = f(-x)$
 this is an even function

$$\begin{aligned} \int_{-\pi}^{\pi} (f(x))^2 dx &= \int_{-\pi}^{\pi} x^4 dx = 2 \int_0^{\pi} x^4 dx \\ &= 2 \left[\frac{x^5}{5} \right]_0^{\pi} \\ &= \frac{2}{5} \pi^5 \end{aligned}$$

By Parseval's theorem,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} (f(x))^2 dx = \frac{a_0^2}{4} + \frac{1}{2} \sum_{n=1}^{\infty} [a_n^2 + b_n^2]$$

$$\frac{1}{2\pi} \left[\frac{x}{5} \pi^5 \right] = \frac{\pi^4}{9} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{16}{n^4}$$

$$\frac{\pi^4}{5} = \frac{\pi^4}{9} + 8 \sum_{n=1}^{\infty} \frac{16}{n^4}$$

$$8 \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{5} - \frac{\pi^4}{9} = \pi^4 \left(\frac{9-5}{45} \right)$$

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

$$\therefore \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

$$\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \dots = \frac{\pi^4}{90}$$

Hence the proof

Expand



Odd function: $\frac{2}{\pi} \int_0^{\pi} (f(x))^2 dx = \sum_{n=1}^{\infty} b_n^2 \quad (\because a_0=0, a_n=0)$

Even function: $\frac{2}{\pi} \int_0^{\pi} (f(x))^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 \quad (\because b_n=0)$

Half Range Cosine Series: $\frac{2}{\pi} \int_0^{\pi} (f(x))^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2$

Half Range Sine Series: $\frac{2}{\pi} \int_0^{\pi} (f(x))^2 dx = \sum_{n=1}^{\infty} b_n^2$

or odd function

Half-Range Sine Series (odd)

$f(x) = f(-x) \Rightarrow$ even
 $f(x) \neq f(-x) \Rightarrow$ odd

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$\text{where } b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx$$

① Find the fourier expansion of $f(x) = x$
 in $-\pi < x < \pi$

Soln:

$$\text{Given } f(x) = x$$

$$f(-x) = -x$$

$$\therefore f(x) \neq f(-x)$$

This is a odd function

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} \frac{x}{u} \frac{\sin nx}{v} \, dx$$

$$= \frac{2}{\pi} \left[x \left(-\frac{\cos nx}{n} \right) \right]_0^{\pi} - \left[(1) \left(-\frac{\sin nx}{n^2} \right) \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\pi \frac{\cos n\pi}{n} - 0 \left(\frac{1}{n} \right) \right]$$

$$= \frac{2}{n} (-(-1)^n)$$

$$b_n = \frac{2}{n} (-1)^{n+1}$$

$$\int \sin x \, dx = -\cos x$$

$$\int \cos x \, dx = \sin x$$

Diff	Integration
$u = x$	$v = \sin nx$
$u' = 1$	$v_1 = -\frac{\cos nx}{n}$
$u'' = 0$	$v_2 = -\frac{\sin nx}{n^2}$

Bernoulli's formula
 $\int u v \, dx = u v_1 - u' v_2 + u'' v_3 - \dots$

$\cos(-\theta) = \cos \theta$	$\cos n\pi = (-1)^n$
$\sin(-\theta) = -\sin \theta$	$\cos 2n\pi = 1$
$-(-1)^n$	$\cos \pi = -1$
$= (-1)^{n+1}$	$\sin \pi = 0$
$= (-1)^{n-1}$	$\sin n\pi = 0$
$\cos 0 = 1$	$(-1)^{2n} = 1$
$\sin 0 = 0$	



$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$= \sum_{n=1}^{\infty} \frac{2}{n} (-1)^{n+1} \sin nx$$

$$f(x) = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx$$

② Find the Fourier Series for $f(x) = x^2$ in $-\pi \leq x \leq \pi$

(i) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6}$

(ii) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \dots = \frac{\pi^2}{12}$

(iii) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$

Soln:

Given $f(x) = x^2$

$f(-x) = (-x)^2 = x^2 = f(x)$

$\therefore f(x) = f(-x)$

This is an even function

Formula:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) dx$$

$$a_n = \frac{2}{n} \int_0^{\pi} f(x) \cos nx dx$$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) dx = \frac{2}{\pi} \int_0^{\pi} x^2 dx = \frac{2}{\pi} \left[\frac{x^3}{3} \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left(\frac{\pi^3}{3} \right) = \frac{2\pi^2}{3}$$



$$a_0 = \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx \, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} \frac{x^2}{u} \frac{\cos nx}{v} \, dx$$

$$= \frac{2}{\pi} \left[x^2 \left(\frac{\sin nx}{n} \right) - 2x \left(\frac{-\cos nx}{n^2} \right) + 2 \left(\frac{-\sin nx}{n^3} \right) \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left(\frac{\pi^2 (-1)^n}{n^2} \right)$$

$$= \frac{4}{n^2} (-1)^n$$

$$a_n = \frac{4}{n^2} (-1)^n$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$= \frac{2\pi^2}{3} \times \frac{1}{2} + \sum_{n=1}^{\infty} \frac{4}{n^2} (-1)^n \cos nx$$

$$f(x) = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx$$

Diff	Integration
$u = x^2$	$v = \cos nx$
$u' = 2x$	$v_1 = \frac{\sin nx}{n}$
$u'' = 2$	$v_2 = -\frac{\cos nx}{n^2}$
$u''' = 0$	$v_3 = -\frac{\sin nx}{n^3}$

Bernoulli's formula

$$\int u v \, dx =$$

$$\int u v \, dx =$$

$$u v_1 - u' v_2 + u'' v_3 - \dots$$

$$\sin n\pi = 0;$$

$$\cos n\pi = (-1)^n$$

$$\sin 0 = 0$$

$$\cos 0 = 1$$

Deduction

(i) Put $x = \pi$

$$f(x) \text{ at } x = \pi \left\{ \begin{array}{l} f(-\pi) + f(\pi) = \frac{(-\pi)^2 + \pi^2}{2} = \frac{2\pi^2}{2} = \pi^2 \\ \text{or } x^2 = \pi^2 \end{array} \right.$$

$$\Rightarrow \pi^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{1}{n^2} (-1)^n (-1)^n \quad (\cos n\pi = (-1)^n)$$

$$\pi^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$4 \sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2 - \frac{\pi^2}{3}$$

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{1}{4} \left(\frac{3\pi^2 - \pi^2}{3} \right) = \frac{1}{6} (2\pi^2)$$

$$\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \pi^2 \right] \rightarrow \textcircled{2}$$



Deduction : (ii)

Put $x = 0$.

$$\left. \begin{array}{l} f(x) \\ \text{at } x=0 \end{array} \right\} = 0$$

$$\textcircled{1} \Rightarrow 0 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{1}{n^2} (-1)^n \frac{\cos 0}{\textcircled{1}}$$

$$-\frac{\pi^2}{3} = 4 \sum_{n=1}^{\infty} \frac{1}{n^2} (-1)^n$$

$$\frac{-\pi^2}{3 \times 4} = -\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \dots$$

$\times \Rightarrow$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \dots \rightarrow \textcircled{2}$$

Deduction (iii)

Add $\textcircled{2} + \textcircled{3}$

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} + \dots = \textcircled{*}$$

$$2 \left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right] = \frac{\pi^2}{6} + \frac{\pi^2}{12}$$

$$\begin{aligned} \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots &= \frac{1}{2} \left[\frac{\pi^2 + 2\pi^2}{12} \right] \\ &= \frac{1}{2} \left(\frac{3\pi^2}{12} \right) \\ &= \frac{\pi^2}{8} \end{aligned}$$

$$\boxed{\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}}$$

Half - Range Cosine Series

①

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{l} \int_0^{\pi} f(x) dx$$

$$a_n = \frac{2}{l} \int_0^{\pi} f(x) \cos nx$$

① Find the half range cosine series of

$$f(x) = x \text{ in } 0 < x < \pi$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \boxed{l = \pi}$$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) dx$$

$$= \frac{2}{\pi} \int_0^{\pi} x dx$$

$$= \frac{2}{\pi} \left[\frac{x^2}{2} \right]_0^{\pi}$$

$$= \frac{\pi^2}{\pi}$$

$$\boxed{a_0 = \pi}$$

Given $f(x) = x$
 $f(-x) = -x$
 $f(x) \neq f(-x)$
 This is an odd function.

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx$$

$$= \frac{2}{\pi} \int_0^{\pi} x \cos nx dx$$

$$= \frac{2}{\pi} \left[x \left(\frac{\sin nx}{n} \right) - (1) \left(\frac{-\cos nx}{n^2} \right) \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\frac{\sin n\pi}{n} + \frac{\cos n\pi}{n^2} - 0 - \frac{1}{n^2} \right]$$

$$= \frac{2}{\pi} \left[0 + \frac{(-1)^n}{n^2} - \frac{1}{n^2} \right]$$

Diff	Integral
$u = x$	$v = \cos nx$
$u' = 1$	$v_1 = \frac{\sin nx}{n}$
$u'' = 0$	$v_2 = \frac{-\cos nx}{n^2}$

Bernoulli's formula

$$\int u dv = uv_1 - u'v_2 + u''v_3 - \dots$$

$$\begin{aligned} \sin 0 &= 0 \\ \cos 0 &= 1 \\ \sin n\pi &= 0 \\ \cos n\pi &= (-1)^n \end{aligned}$$



$$a_n = \frac{2}{n^2 \pi} [(-1)^n - 1]$$

$$\therefore a_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{-4}{n^2 \pi} & \text{if } n \text{ is odd} \end{cases}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$= \frac{\pi}{2} + \sum_{n=\text{odd}} \frac{-4}{n^2 \pi} \cos nx$$

$$= \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=\text{odd}} \frac{1}{n^2} \cos nx$$

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos(2n-1)x$$

Replace n by $2n-1$

② Find the half-range cosine series of $f(x) = (\pi - x^2)$ in $(0, \pi)$.

Soln:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$l = \pi$$

$$a_0 = \frac{2}{l} \int_0^l f(x) dx$$

$$= \frac{2}{\pi} \int_0^{\pi} (\pi - x^2) dx$$

$$= \frac{2}{\pi} \left[\pi x - \frac{x^3}{3} \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left(\pi^2 - \frac{\pi^3}{3} \right)$$

$$a_0 = \frac{2}{\pi} \left(\frac{3\pi^2 - \pi^3}{3} \right)$$

$$= \frac{2}{3\pi} \pi (3\pi - \pi^2)$$

$$a_0 = \frac{2}{3} (3\pi - \pi^2)$$



$$a_n = \frac{2}{\pi} \int_0^{\pi} \underbrace{(\pi - x^2)}_u \underbrace{\cos nx}_{v} dx$$

Diff	Integratio
$u = \pi - x^2$	$v = \cos nx$
$u' = -2x$	$v_1 = \frac{\sin nx}{n}$
$u'' = -2$	$v_2 = -\frac{\cos nx}{n^2}$
$u''' = 0$	$v_3 = -\frac{\sin nx}{n^3}$

$$= \frac{2}{\pi} \left[\underbrace{(\pi - x^2)}_u \underbrace{\left(\frac{\sin nx}{n}\right)}_{v_1} - (-2x) \underbrace{\left(-\frac{\cos nx}{n^2}\right)}_{v_2} + (-2) \underbrace{\left(-\frac{\sin nx}{n^3}\right)}_{v_3} \right]_0^{\pi}$$

$$a_n = \frac{2}{\pi} \left[\left(-\frac{2\pi}{n^2}\right) (-1)^n \right]$$

$$a_n = -\frac{4}{n^2} (-1)^n$$

$$\int u v dx = u v_1 - u' v_2 + u'' v_3 - \dots$$

$\sin n\pi = 0$
 $\cos n\pi = (-1)^n$
 $\sin 0 = 0$
 $\cos 0 = 1$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$f(x) = \frac{(3\pi - \pi^2)}{3} + \sum_{n=1}^{\infty} \left(-\frac{4}{n^2}\right) (-1)^n \cos nx$$

③ Find the half range sine series of $f(x) = 1-x$ in $(0,1)$.

Soln:

$$l = 1$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin n\pi x$$

$$b_n = \frac{2}{1} \int_0^1 f(x) \sin n\pi x dx$$

$$= \frac{2}{1} \int_0^1 \underbrace{(1-x)}_u \underbrace{\sin n\pi x}_v dx$$

$$= 2 \left[(1-x) \left(-\frac{\cos n\pi x}{n\pi}\right) - (-1) \left(-\frac{\sin n\pi x}{n^2 \pi^2}\right) \right]_0^1$$

$$= 2 \left[0 - 0 - (1-0) \left(\frac{1}{n\pi}\right) - 0 \right]$$

$$= 2 \times \frac{1}{n\pi}$$

$$b_n = \frac{2}{n\pi}$$

$$\therefore f(x) = \sum_{n=1}^{\infty} b_n \sin n\pi x$$

$$f(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin n\pi x$$

Diff	Integratio
$u = 1-x$	$v = \sin n\pi x$
$u' = -1$	$v_1 = \frac{-\cos n\pi x}{n\pi}$
$u'' = 0$	$v_2 = \frac{-\sin n\pi x}{n^2 \pi^2}$

$$\int u v dx = u v_1 - u' v_2 + u'' v_3 - \dots$$

$\cos n\pi = (-1)^n$
 $\sin n\pi = 0$
 $\cos 0 = 1$
 $\sin 0 = 0$



Unit-II

2 marks

① Define Dirichlet's conditions.

(or)

State the condition for $f(x)$ to have Fourier series expansion.

Soln:

Any function $f(x)$ can be developed as a Fourier series $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$; where a_0, a_n, b_n are constants.

Conditions

- ① $f(x)$ is Periodic, Single valued & finite.
- ② $f(x)$ has a finite number of finite discontinuities in any one period & no infinite discontinuity.
- ③ $f(x)$ has at the most a finite number of maxima & minima.

② State Parseval's identity for Fourier transform.

Soln:

If $F(s)$ is the Fourier transform of $f(x)$, then

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(s)|^2 ds$$

③ write $x+x^3$ as a Fourier Series in $(-\pi, \pi)$.

Soln:

$$\text{Let } f(x) = x + x^3$$

$$f(-x) = (-x) + (-x)^3 = -x - x^3 = -(x + x^3) = -f(x)$$

$$\therefore f(-x) = -f(x).$$

This is an odd function

$$a_0 = 0 \text{ \& } a_n = 0$$

④ Find b_n in the expansion of x^2 as a Fourier Series in $(-\pi, \pi)$.

Solution...

Given $f(x) = x^2$ is an even function in the interval $(-\pi, \pi)$.

$$\therefore b_n = 0.$$

⑤ If $f(x)$ is an odd function defined in $(-l, l)$, what are the values of a_0 & a_n ?

Solution...

Given $f(x)$ is an odd function in the interval $(-l, l)$.

$$\therefore a_0 = 0; a_n = 0.$$

⑥ Find the Fourier constants b_n for $x \sin x$ in $(-\pi, \pi)$.

Solution...

$$\text{Given } f(x) = x \sin x$$

$$f(-x) = (-x) \sin(-x)$$

$$= (-x)(-\sin x)$$

$$= x \sin x.$$

$$\boxed{f(-x) = f(x)}$$

$\therefore f(x)$ is an even function.

$$\therefore b_n = 0.$$

⑦ State Parseval's identity for the half-range cosine expansion of $f(x)$ in $(0, l)$.

Soln...

$$\int_0^l |f(x)|^2 dx = \frac{a_0^2}{4} + \frac{1}{2} \sum_{n=1}^{\infty} a_n^2.$$



(8) Find the value of a_n in the cosine Series expansion of $f(x)=k$ in the interval $(0,10)$.

Soln:

Given $f(x)=k$ in $(0,10)$. Here $l=10$.

$$a_n = \frac{2}{l} \int_0^l f(x) \cos\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{2}{10} \int_0^{10} k \cos\left(\frac{n\pi x}{10}\right) dx$$

$$= \frac{k}{5} \int_0^{10} \cos\left(\frac{n\pi x}{10}\right) dx$$

$$= \frac{k}{5} \left[\frac{\sin\left(\frac{n\pi x}{10}\right)}{\frac{n\pi}{10}} \right]_0^{10}$$

$$= \frac{k}{5} \times \frac{10^2}{n\pi} \left(\sin\left(\frac{10n\pi}{10}\right) - \sin 0 \right)$$

$$= \frac{2k}{n\pi} (\sin(n\pi) - \sin 0)$$

$$= \frac{2k}{n\pi} (0 - 0)$$

$$\begin{aligned} \sin(n\pi) &= 0 \\ \sin 0 &= 0 \end{aligned}$$

$$\boxed{a_n = 0}$$

(9) Determine the value of a_n in the Fourier Series expansion of $f(x)=x^3$ in $-\pi < x < \pi$.

Soln:

$$\text{Given } f(x) = x^3$$

$$f(-x) = (-x)^3$$

$$= -x^3$$

$$f(-x) = -f(x)$$

$\therefore f(x)$ is an odd function

$$\therefore a_0 = 0 \text{ \& } a_n = 0$$

(10) If $f(x)=2x$ in the interval $(0,4)$, then find the value of a_2 in the Fourier series expansion.

Soln: Here $2l=4$; $l=\frac{4}{2}$
 $\boxed{l=2}$



$$a_n = \frac{1}{2} \int_0^{2\pi} f(x) \cos\left(\frac{n\pi x}{l}\right) dx$$

$$a_2 = \frac{1}{2} \int_0^4 2x \cos\left(\frac{2\pi x}{2}\right) dx$$

$$= \left[x \left[\frac{\sin \pi x}{\pi} \right] - (1) \left(-\frac{\cos \pi x}{\pi^2} \right) \right]_0^4$$

$$= \left(0 + \frac{1}{\pi^2} \right) - \left(0 + \frac{1}{\pi^2} \right)$$

$$a_2 = 0$$

Diff	Integration
$u=x$	$v=\cos \pi x$
$u'=1$	$v_1=\frac{\sin \pi x}{\pi}$
$u''=0$	$v_2=-\frac{\cos \pi x}{\pi^2}$

$$\int u v dv = u v_1 - u' v_2 + u'' v_3 - \dots$$

ii) If $f(x) = 3x - 4x^3$, defined in the interval $(-2, 2)$, then find the value of a_1 in the Fourier series expansion.

Soln:

$$\text{Given } f(x) = 3x - 4x^3$$

$$f(-x) = 3(-x) - 4(-x)^3 = -3x + 4x^3$$

$$= -(3x - 4x^3)$$

$$f(-x) = -f(x)$$

This is an odd function.

$$a_0 = 0 \text{ \& } a_n = 0$$

$\therefore$ The value of $a_1 = 0$

Unit - III Solution of one Dimensional wave Equation

Problem on vibrating string with Zero Initial velocity

wave equation :-

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

conditions :-

$$\left. \begin{array}{l} \text{(i)} \quad y(0, t) = 0, \quad \forall t > 0 \\ \text{(ii)} \quad y(l, t) = 0, \quad \forall t > 0 \end{array} \right\} \text{Boundary condition}$$

$$\left. \begin{array}{l} \text{(iii)} \quad \frac{\partial y}{\partial t}(x, 0) = 0 : 0 < x < l \\ \text{(iv)} \quad \frac{\partial y}{\partial t}(x, 0) = \frac{f(x)}{a} : 0 < x < l \end{array} \right\} \text{Initial condition}$$

Suitable solution :-

$$y(x, t) = (c_1 \cos px + c_2 \sin px) (c_3 \cos pat + c_4 \sin pat) \quad \rightarrow \textcircled{1}$$

Apply conditions

$$\begin{array}{l} \text{(i)} \Rightarrow c_1 = 0 \\ \text{(ii)} \quad p = \frac{n\pi}{l} \\ \text{(iii)} \quad c_4 = 0 \end{array}$$

General Solution :-

$$y(x, t) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{l} \cos \frac{n\pi at}{l} \quad \rightarrow \textcircled{2}$$

Apply condition

$$\text{(iv)} \Rightarrow y(x, 0) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{l} = f(x)$$

Where $c_n = b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx$.
 Substitute c_n in (2), we get the general solution.

① A string is stretched and fastened to two points $x=0$ & $x=l$ apart. motion is started by

① A tightly stretched string with fixed ends points $x=0$ and set vibrating giving each point a velocity $\lambda x(l-x)$. find the displacement $y(x,t)$ of the string at any time t and at a distance x from the zero end.

Soln : Given $f(x) = \lambda x(l-x)$

wave Equation

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \rightarrow (1)$$

Conditions

- (i) $y(0,t) = 0, \forall t > 0$
 (ii) $y(l,t) = 0, \forall t > 0$ } Boundary ~~Apply~~ Conditions
- (iii) $y(x,0) = 0, 0 < x < l$
 (iv) $\frac{\partial y}{\partial t}(x,0) = \lambda x(l-x), 0 < x < l$ } Initial conditions

Suitable solution :

$$y(x,t) = (c_1 \cos px + c_2 \sin px) \cdot (c_3 \cos pat + c_4 \sin pat) \rightarrow (2)$$

(i) $c_1 = 0$

$$y(0,t) = (c_1 \cos 0 + c_2 \sin 0) (c_3 \cos pat + c_4 \sin pat) = 0$$

$\cos 0 = 1$
 $\sin 0 = 0$

$$\Rightarrow c_1 (c_3 \cos pat + c_4 \sin pat) = 0$$

$$\therefore c_1 = 0 ; c_3 \cos pat + c_4 \sin pat \neq 0.$$

From (2)

$$y(x,t) = (c_2 \sin px) (c_3 \cos pat + c_4 \sin pat)$$

Condition

(ii) $y(1,t) = 0$

$\rightarrow (3)$

$$y(1,t) = (c_2 \sin pl) (c_3 \cos pat + c_4 \sin pat) = 0$$

$\rightarrow (4)$

$$c_2 \sin pl = 0 \quad ; \quad c_3 \cos pat + c_4 \sin pat \neq 0$$

$$\boxed{c_2 = 0} ; \sin pl = 0$$

$$\therefore \sin pl = 0$$

$$\boxed{\sin n\pi = 0}$$

$$\sin pl = \sin n\pi$$

$$pl = n\pi$$

$$\boxed{p = \frac{n\pi}{l}}$$

$n \rightarrow \text{Integer}$

From (3),

$$y(x,t) = c_2 \sin \frac{n\pi x}{l} \left[c_3 \cos \frac{n\pi at}{l} + c_4 \sin \frac{n\pi at}{l} \right]$$

By condition (iii),

$\rightarrow (5)$

$$y(x,0) = c_2 \sin \frac{n\pi x}{l} [c_3 \cos 0 + c_4 \sin 0] = 0$$

$$\cancel{y(x,0)} \Rightarrow c_2 \sin \frac{n\pi x}{l} (c_3) = 0$$

$$\boxed{\cos 0 = 1, \sin 0 = 0}$$

$$\Rightarrow c_2 c_3 \sin \frac{n\pi x}{l} = 0$$

$$\therefore c_2 \neq 0 ; \boxed{c_3 = 0} ; \sin \frac{n\pi x}{l} \neq 0$$

From (5),

$$y(x,t) = c_2 \sin\left(\frac{n\pi x}{l}\right) \left[(0) \cos \frac{n\pi at}{l} + c_4 \sin \frac{n\pi at}{l} \right]$$

$$= c_2 \sin\left(\frac{n\pi x}{l}\right) c_4 \sin\left(\frac{n\pi at}{l}\right)$$

$$= c_2 c_4 \sin\left(\frac{n\pi x}{l}\right) \sin\left(\frac{n\pi at}{l}\right)$$

$$y(x,t) = c_n \sin\left(\frac{n\pi x}{l}\right) \sin\left(\frac{n\pi at}{l}\right) \quad \boxed{\therefore c_n = c_2 c_4}$$

General Solution :

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{l}\right) \sin\frac{n\pi at}{l} \rightarrow (b)$$

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{l}\right) \cos\left(\frac{n\pi at}{l}\right) \left(\frac{n\pi a}{l}\right)$$

condition (iv)

$$\frac{\partial y}{\partial t}(x,0) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{l}\right) \cos 0 \left(\frac{n\pi a}{l}\right) = \lambda x(l-x)$$

$\cos 0 = 1$

$$= \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{l}\right) \left(\frac{n\pi a}{l}\right) = \lambda x(l-x)$$

$$= \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) = \lambda x(l-x)$$

$$\therefore b_n = c_n \left(\frac{n\pi a}{l}\right)$$

$$b_n = \frac{2}{l} \int_0^l f(x) \sin\frac{n\pi x}{l} dx$$

$$= \frac{2}{l} \int_0^l \lambda x(l-x) \sin\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{2\lambda}{l} \int_0^l (lx - x^2) \sin\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{2\lambda}{l} \left[(lx - x^2) \left(\frac{\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}} \right) - (l-2x) \left(\frac{-\sin \frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2} \right) + (-2) \frac{\cos \left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^3} \right]_0^l$$

(2-1=0)

$$= \frac{2\lambda}{l} \left[\left(0 + 0 - 2 \cos\left(\frac{n\pi x}{l}\right) \left(\frac{l}{n\pi}\right)^3 \right) - \left(0 + 0 - 2 \cos 0 \left(\frac{l}{n\pi}\right)^3 \right) \right]$$

$$= \frac{2\lambda}{l} \left[-2 \left(\frac{l}{n\pi}\right)^3 (-1)^n + 2 \left(\frac{l}{n\pi}\right)^3 \right]$$

$$= \frac{2\lambda}{l} 2 \left(\frac{l}{n\pi}\right)^3 [1 - (-1)^n]$$

Points
 $l=l$ 0 to l
 $f(x) = \lambda x(l-x)$

Diff	Integration
$u = lx - x^2$	$v = \sin \frac{n\pi x}{l}$
$u' = l - 2x$	$v_1 = \frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}}$
$u'' = -2$	$v_2 = \frac{-\sin \frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}$
$u''' = 0$	$v_3 = \frac{\cos \frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^3}$
$\int u v dx = u v_1 - u' v_2 + u'' v_3$ $\sin n\pi = 0 \quad \cos n\pi = (-1)^n$ $\sin 0 = 0 \quad \cos 0 = 1$	

$$b_n = \frac{4\lambda l^2}{n^3 \pi^3} [1 - (-1)^n]$$

$$\therefore b_n = c_n \left(\frac{n\pi a}{l} \right)$$

$$c_n \left(\frac{n\pi a}{l} \right) = \frac{4\lambda l^2}{n^3 \pi^3} [1 - (-1)^n]$$

$$c_n = \left(\frac{l}{n\pi a} \right) \left(\frac{4\lambda l^2}{n^3 \pi^3} \right) [1 - (-1)^n]$$

$$c_n = \frac{4\lambda l^3}{a n^4 \pi^4} [1 - (-1)^n]$$

$$\therefore c_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{8\lambda l^3}{a n^4 \pi^4} & \text{if } n \text{ is odd} \end{cases}$$

From (6)

$$y(x,t) = \sum_{n=\text{odd}}^{\infty} \frac{8\lambda l^3}{a n^4 \pi^4} \sin\left(\frac{n\pi x}{l}\right) \sin\left(\frac{n\pi at}{l}\right)$$

$$= \frac{8\lambda l^3}{a \pi^4} \sum_{\text{odd}} \frac{1}{n^4} \sin\left(\frac{n\pi x}{l}\right) \sin\left(\frac{n\pi at}{l}\right)$$

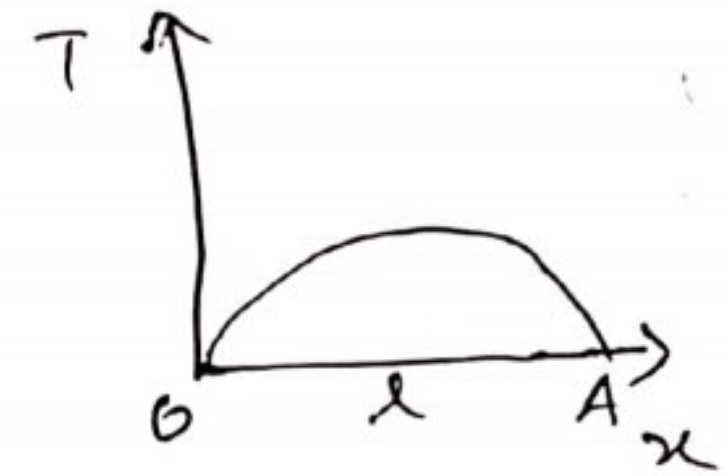
$$y(x,t) = \frac{8\lambda l^3}{a \pi^4} \sum_{\text{odd}} \frac{1}{(2n-1)^4} \sin\left(\frac{(2n-1)\pi x}{l}\right) \sin\left(\frac{(2n-1)\pi at}{l}\right) \quad \because n = 2n-1$$

② A string is stretched and fastened to two points l apart. Motion is started by displacing the string into the form $y = k(lx - x^2)$ from which it is released at time $t=0$. Find the displacement of any point of the string at a distance x from one end at any time t .

Soln:-

Wave Equation:-

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$



[Ans ① sum-mu ~~Unit~~ $\frac{1}{a^2}$]

③ If a string of length l is initially at rest in its equilibrium position and each of its points is given by a velocity v such that

$$v = \begin{cases} cx & \text{for } 0 < x < \frac{l}{2} \\ c(l-x) & \text{for } \frac{l}{2} < x < l \end{cases}$$

Show that the displacement at any time t is given by

$$y(x,t) = \frac{4l^2c}{\pi^3a} \left[\frac{1}{1^3} \sin \frac{\pi x}{l} \sin \frac{\pi at}{l} - \frac{1}{3^3} \sin \frac{3\pi x}{l} \sin \frac{3\pi at}{l} + \dots \right]$$

Soln:-

Wave Equation:-

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

Conditions

(i) $y(0,t) = 0 \quad \forall t \geq 0$

(ii) $y(l,t) = 0 \quad \forall t \geq 0$

(iii) $y(x,0) = 0$

(iv) $\left(\frac{\partial y}{\partial t} \right)_{(x,0)} = \begin{cases} cx & \text{if } 0 \leq x \leq \frac{l}{2} \\ c(l-x) & \text{if } \frac{l}{2} \leq x \leq l \end{cases}$

Type: 2

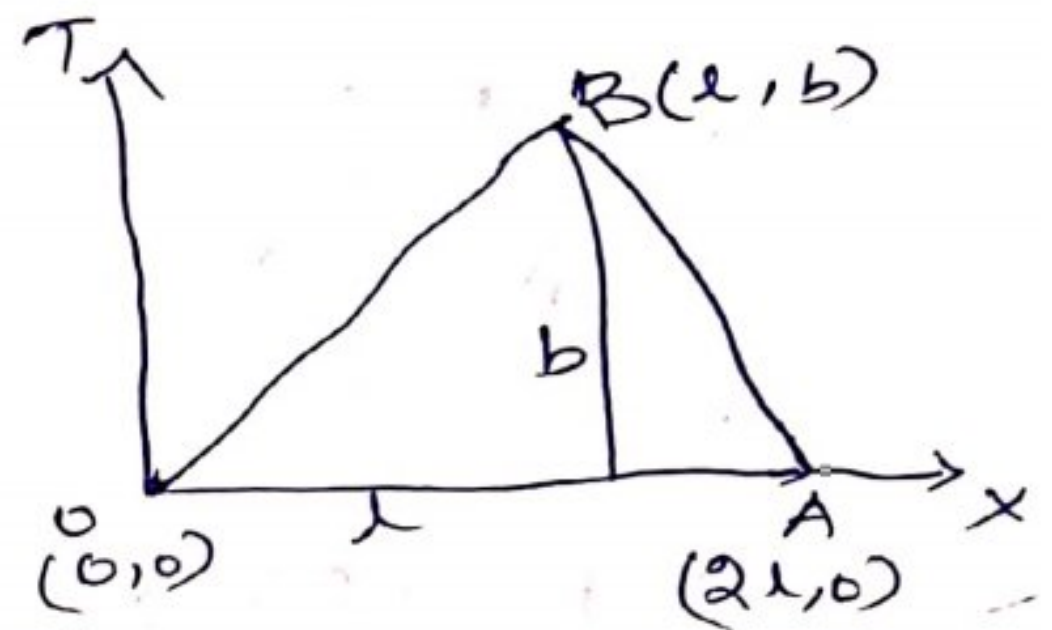
- ① A tightly stretched string of length $2l$ is fixed at both ends. The midpoint of the string is displaced by a distance " b " transversely and the string is released from rest in this position. Find the displacement of any point of the string at any subsequent time.

Soln:

Wave Equation: $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$

Conditions

- ① $y(0, t) = 0, \forall t \geq 0$
- ② $y(2l, t) = 0, \forall t > 0$
- ③ $\frac{\partial y}{\partial t}(x, 0) = 0; 0 < x < 2l$



Equation of OB is

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

$$\frac{y - 0}{b - 0} = \frac{x - 0}{l - 0}$$

$$\frac{y}{b} = \frac{x}{l}$$

$$\boxed{y = \frac{b}{l}x}$$

$O(x_1, y_1) \& B(x_2, y_2)$

Equation of BA is

$B(x_2, y_2) \& A(x_1, y_1)$

$$\frac{y - b}{0 - b} = \frac{x - l}{2l - l}$$

$$\underline{y - b} = \frac{x - l}{l}(-b)$$

$$y = b + \frac{bx}{l} - \frac{bx}{l}$$

$$= 2b - \frac{bx}{l}$$

$$\boxed{y = b\left(\frac{2l - x}{l}\right)}$$

$$y(x, 0) = \begin{cases} \frac{bx}{l} & ; 0 < x < l \\ \frac{b(2l - x)}{l} & ; l < x < 2l \end{cases}$$

$$y(x,t) = (c_1 \cos px + c_2 \sin px) (c_3 \cos pat + c_4 \sin pat)$$

⊗ 1st - sum - ~~over~~ 'L' and 'T' $\rightarrow$ (1)

General solution:

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{2l} \cos \frac{n\pi at}{2l} \rightarrow (5)$$

Condition (iv)

$$y(x,0) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{2l} \rightarrow (6)$$

put
 $l=2l$

$$y(x,0) = \sum_{n=1}^{\infty} b_n \sin \left(\frac{n\pi x}{l} \right) \rightarrow (7)$$

$$(6) = (7)$$

$$b_n = c_n$$

$$c_n = \frac{2}{(2l)} \int_0^{2l} f(x) \sin \frac{n\pi x}{2l} dx$$

$$= \frac{1}{l} \left[\int_0^l \frac{bx}{l} \sin \left(\frac{n\pi x}{2l} \right) dx + \int_l^{2l} \frac{b(2l-x)}{l} \sin \frac{n\pi x}{2l} dx \right]$$

$$= \frac{b}{l^2} \left[\int_0^l x \sin \left(\frac{n\pi x}{2l} \right) dx + \int_l^{2l} (2l-x) \sin \left(\frac{n\pi x}{2l} \right) dx \right]$$

$$= \frac{b}{l^2} \left[\left[x \left(\frac{\cos \frac{n\pi x}{2l}}{\frac{n\pi}{2l}} \right) - (1) \left(-\sin \left(\frac{n\pi x}{2l} \right) \right) \right]_0^l + \left[(2l-x) \left(\frac{-\cos \left(\frac{n\pi x}{2l} \right)}{\frac{n\pi}{2l}} \right) - (-1) \left(\frac{\sin \left(\frac{n\pi x}{2l} \right)}{\left(\frac{n\pi}{2l} \right)^2} \right) \right]_l^{2l} \right]$$

$$C_n = \frac{b}{l^2} \left[-\frac{2l^2}{n\pi} \cos \frac{n\pi}{2} + \frac{4l^2}{n^2\pi^2} \sin \frac{n\pi}{2} + \frac{2l^2}{n\pi} \cos \frac{n\pi}{2} + \frac{4l^2}{n^2\pi^2} \sin \frac{n\pi}{2} \right]$$

$$= \frac{b}{l^2} \left[\frac{8l^2}{n^2\pi^2} \sin \left(\frac{n\pi}{2} \right) \right]$$

$$C_n = \frac{8b}{n^2\pi^2} \sin \left(\frac{n\pi}{2} \right)$$

$$C_n = \begin{cases} 0 & ; n \text{ is even} \\ \frac{8b}{n^2\pi^2} \sin \left(\frac{n\pi}{2} \right) & ; n \text{ is odd} \end{cases}$$

From (5),

$$y(x,t) = \sum_{n=\text{odd}}^{\infty} \frac{8b}{n^2\pi^2} \sin \left(\frac{n\pi}{2} \right) \sin \left(\frac{n\pi x}{2} \right) \cos \left(\frac{n\pi a t}{2} \right)$$

Unit - III

Two marks

① State the suitable solution of the one dimensional heat equation $\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$

Soln:

$$u(x,t) = (A \cos px + B \sin px) e^{-c^2 p^2 t}$$

② A string is stretched and fastened to two points l apart. Motion is started by displacing the string into the form $y = y_0 \sin \frac{\pi x}{l}$ from which it is released at time $t=0$. Formulate this problem as the boundary value problem.

Soln:

wave equation

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

Conditions:

$$① y(0,t) = 0 \quad \forall t \geq 0$$

$$② y(l,t) = 0 \quad \forall t \geq 0$$

$$③ \frac{\partial y}{\partial t}(x,0) = 0$$

$$④ y(x,0) = f(x) = y_0 \sin\left(\frac{\pi x}{l}\right)$$

③ what are the various solutions of

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

Soln:

wave equation $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$

various possible solutions.

$$y(x,t) = (A_1 e^{Px} + A_2 e^{-Px}) (A_3 e^{Pat} + A_4 e^{-Pat})$$

$$y(x,t) = (A_5 \cos Px + A_6 \sin Px) (A_7 \cos Pat + A_8 \sin Pat)$$

$$y(x,t) = (A_9 x + A_{10}) (A_{11} t + A_{12})$$

④ write down the diffusion problem in one-dimension as a boundary value problem in two different forms.

Soln:

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} \quad \text{one dimensional heat flow,}$$

Here $a^2 = \frac{k}{\rho c}$ is called the diffusivity.

In the steady state $\frac{d^2 u}{dx^2} = 0$.

⑤ State any two laws which are assumed to derive one dimensional heat equation
(or)
State Fourier law of heat conduction

Soln:

① Heat flows from higher to lower temp.

② The rate at which heat flows across any area is proportional to the area and to the temp. gradient normal to the surface. This constant of proportionality is known as the thermal conductivity (k) of the material. It is known as Fourier law of heat conduction.

⑥ In Steady State conditions derive the Solution of one dimensional heat flow equation.

Soln:

The P.d.e of unsteady one dimensional heat flow is $\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} \rightarrow (1)$

Not time t , $\frac{\partial u}{\partial t} = 0$.

From (1),

$$\therefore \frac{\partial^2 u}{\partial x^2} = 0.$$

Solution:

$u = ax + b$; where a & b are arbitrary.

⑦ State one-dimensional heat equation with the initial and boundary conditions

Soln:

The one dimensional heat equation is

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}.$$

Conditions:

① $u(0, t) = k_1^\circ C \quad \forall t \geq 0$

② $u(l, t) = k_2^\circ C \quad \forall t \geq 0$

③ $u(x, 0) = f(x); \quad 0 < x < l$

⑧ write the boundary conditions and initial conditions for solving the vibration of string equation, if the string is subjected to initial displacement $f(x)$ & initial velocity $g(x)$.

Soln: wave equation $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$

① $y(0, t) = 0$

② $y(l, t) = 0$

③ $\frac{\partial y}{\partial t}(x, 0) = g(x)$

④ $y(x, 0) = f(x).$

⑨ write down the governing equation of two dimensional steady state heat conduction.

Soln : The required equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

one dimensional Wave Equation

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

$$a^2 = \frac{T}{m} = \frac{\text{Tension}}{\text{Mass Per unit length}}$$

$$\frac{T}{m} = \text{positive,}$$

various solution are,

① $y(x,t) = (A_1 e^{px} + A_2 e^{-px}) (A_3 e^{pat} + A_4 e^{-pat})$
(non-periodic solution).

② $y(x,t) = (A_5 \cos px + A_6 \sin px) (A_7 \cos pat + A_8 \sin pat)$
(vibrating string)

③ $y(x,t) = (A_9 x + A_{10}) (A_{11} t + A_{12})$

one dimensional Heat Equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

$$\alpha^2 = \frac{k}{c\rho}$$

$\alpha^2 \rightarrow$ diffusivity

$\rho \rightarrow$ density

$c \rightarrow$ specific heat

$k \rightarrow$ Thermal Conductivity

Various solution are

① $u(x,t) = e^{\alpha^2 p^2 t} (A_1 e^{px} + A_2 e^{-px})$
(non transient state solution)

② $u(x,t) = e^{-\alpha^2 p^2 t} (A_3 \cos px + A_4 \sin px)$
(Transient state solution)

② $u(x, t) = A_5 x + A_6$ (Steady State Solution)

∴ The P.d.e of one dimensional heat flow in steady state $\frac{d^2 u}{dx^2} = 0$.

Two dimensional steady State heat equation

Laplace equation is $\nabla^2 u = 0$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Polar form is

$$r^2 \frac{\partial^2 u}{\partial r^2} + r \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0.$$

Various solutions are

① $u(x, y) = (A_1 e^{px} + A_2 e^{-px}) \cdot (A_3 \cos py + A_4 \sin py)$

② $u(x, y) = (A_5 \cos px + A_6 \sin px) (A_7 e^{py} + A_8 e^{-py})$

③ $u(x, y) = (A_9 x + A_{10}) (A_{11} y + A_{12})$.

classification of P.d.e of the Second order

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + f\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0$$

$B^2 - 4AC < 0$ Elliptic equation

$B^2 - 4AC = 0$ Parabolic equation

$B^2 - 4AC > 0$ Hyperbolic equation.

① Find the nature of Pde

$$4u_{xx} + 4u_{xy} + u_{yy} + 2u_x - u_y = 0$$

Soln..

Given $4u_{xx} + 4u_{xy} + u_{yy} + 2u_x - u_y = 0$.

Formula..

$$A(x,y) \frac{\partial^2 u}{\partial x^2} + B(x,y) \frac{\partial^2 u}{\partial x \partial y} + C(x,y) \frac{\partial^2 u}{\partial y^2}$$

$$+ f(x,y,u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}) = 0$$

$$A=4; B=4, C=1$$

$$B^2 - 4AC = 16 - 4(4)(1)$$

$$B^2 - 4AC = 0$$

This is Parabolic Equation.

② Classify the equation $u_{xx} - y^4 u_{yy} = 2y^3 u_y$.

Soln..

$$\text{Given } u_{xx} - y^4 u_{yy} - 2y^3 u_y = 0$$

Formula..

$$A(x,y) \frac{\partial^2 u}{\partial x^2} + B(x,y) \frac{\partial^2 u}{\partial x \partial y} + C(x,y) \frac{\partial^2 u}{\partial y^2}$$

$$+ f(x,y,u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}) = 0$$

$$A=1; B=0, C=-1$$

$$B^2 - 4AC = 0 - 4(1)(-1)$$

$$B^2 - 4AC = 4 > 0$$

This is hyperbolic Equation

③ Classify the P.d.e $(1+x)^2 u_{xx} - 4x u_{yy} + u_{yy} = x$.

Formula:

$$A(x,y) \frac{\partial^2 u}{\partial x^2} + B(x,y) \frac{\partial^2 u}{\partial x \partial y} + C(x,y) \frac{\partial^2 u}{\partial y^2}$$

$$+ f(x,y,u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}) = 0$$

$$A = (1+x)^2 ; B = -4x ; C = 1$$

$$\begin{aligned} B^2 - 4AC &= 16x^2 - 4(1+x)^2 & (a+b)^2 &= a^2 + b^2 + 2ab \\ &= 16x^2 - 4(1+x^2+2x) \\ &= 16x^2 - 4 - 4x^2 - 8x \\ &= 12x^2 - 8x - 4 \end{aligned}$$

Put $x=0 \Rightarrow B^2 - 4AC = 12(0)^2 - 8(0) - 4$

$$B^2 - 4AC = -4$$

$\Rightarrow$ parabolic equation

Put $x = +ve \text{ or } -ve$; $B^2 - 4AC > 0$

Hyperbolic equation.

④ Classify $x^2 u_{xx} + 2xy u_{xy} + (1+y^2) u_{yy} - 2u_x = 0$

Soln:

Formula

$$A(x,y) \frac{\partial^2 u}{\partial x^2} + B(x,y) \frac{\partial^2 u}{\partial x \partial y} + C(x,y) \frac{\partial^2 u}{\partial y^2} + f(x,y,u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}) = 0$$

$$A = x^2 ; B = 2xy ; C = 1+y^2$$

$$\begin{aligned} B^2 - 4AC &= 4x^2 y^2 - 4(x^2)(1+y^2) \\ &= 4x^2 y^2 - 4x^2 - 4x^2 y^2 \end{aligned}$$

$$= -4x^2 < 0$$

$$B^2 - 4AC < 0$$

This is elliptic equation

⑤ Classify the P.d.e $3u_{xx} + 4u_{xy} + 3u_y - 2u_x = 0$

Formula

$$A(x,y) \frac{\partial^2 u}{\partial x^2} + B(x,y) \frac{\partial^2 u}{\partial x \partial y} + C(x,y) \frac{\partial^2 u}{\partial y^2}$$

$$+ f(x,y,u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}) = 0$$

$$A = 3; B = 4; C = 0$$

$$B^2 - 4AC = (4)^2 - 4(3)(0)$$

$$= 16 > 0$$

$B^2 - 4AC > 0$ This is hyperbolic equation.

① The ends A & B of a rod of length 10cm long have their temperatures kept at 20°C and 70°C . Find the steady state temperature distribution on the rod.

Soln:

The steady state equation of one dimensional heat flow is $\frac{\partial^2 u}{\partial x^2} = 0 \rightarrow \textcircled{1}$

General Solution: $u(x) = ax + b \rightarrow \textcircled{2}$

Boundary Conditions: $u(0) = 20$
 $u(1) = 70$

Put $x=0$ in $\textcircled{2}$

$$u(0) = a(0) + b$$

$$\boxed{20 = b}$$

From $\textcircled{2}$, $\boxed{l = 10\text{cm}}$

$$u(x) = ax + b$$

$$u(x) = \frac{50}{l}x + 20$$

Put $x=1$ in $\textcircled{2}$,

$$u(1) = a(1) + b$$

$$70 = a(1) + 20$$

$$a(1) = 70 - 20$$

$$a(1) = 50$$

$$\boxed{a = \frac{50}{1}}$$

$$u(x) = \frac{50}{10} x + 20$$

$$u(x) = 5x + 20$$

- ② A rod (50 cm) long with insulated sides has its ends A & B kept at 20°C & 70°C respectively. Find the steady state temperature distribution of the rod.

Soln:

1st sum-on unig'g गद्य 2/6.

$$u(x) = ax + 20 \quad ; \quad \frac{a = \frac{50}{50} = 1}{a=1} \quad l=50.$$

$$u(x) = 5x + 20$$

- ③ write any two solution of the Laplace equation $u_{xx} + u_{yy} = 0$ involving exponential terms in x & y .

Soln

$$u(x, y) = (c_1 e^{px} + c_2 e^{-px}) (c_3 \cos py + c_4 \sin py)$$

$$u(x, y) = (c_1 \cos px + c_2 \sin px) (c_3 e^{py} + c_4 e^{-py})$$

- ④ write down the three possible solutions of Laplace equation in two-dimensions.

Soln:

Laplace equation in two-dimensions is $\nabla^2 u = 0$.

$$① \quad u(x, y) = (c_1 e^{px} + c_2 e^{-px}) (c_3 \cos py + c_4 \sin py)$$

$$② \quad u(x, y) = (c_5 \cos px + c_6 \sin px) (c_7 e^{py} + c_8 e^{-py})$$

$$③ \quad u(x, y) = (c_9 x + c_{10}) (c_{11} y + c_{12})$$

5) A rod 20cm long with insulated sides has its ends A & B kept at 30°C & 90°C respectively. Find the steady state temperature distribution of the rod.

Soln: Formulae

Given $l = 20$

$$\frac{d^2u}{dx^2} = 0 \rightarrow (1)$$

$$u(x) = ax + b \rightarrow (2)$$

Boundary Conditions } $u(0) = 30$; $u(l) = 90$.

$x = 0$ in (2)

$$u(0) = a \cdot 0 + b$$

$$u(0) = b = 30$$

$$\therefore u(x) = ax + b$$

$$u(x) = 3x + 30$$

$x = l = 20$ in (2)

$$u(20) = a \cdot 20 + b$$

$$90 = a \cdot 20 + 30$$

$$20a = 90 - 30$$

$$20a = 60$$

$$a = \frac{60}{20} = 3$$

$$a = 3$$

6) The ends A & B of a rod 1 cm long have the temperature 40°C & 90°C until steady state prevails. Find the temperature in the rod at that state.

Soln

Given $l = 1$

Formulae

$$\frac{d^2u}{dx^2} = 0 \rightarrow (1)$$

$$u(x) = ax + b \rightarrow (2)$$

Boundary Conditions } $u(0) = 40$
 $u(1) = 90$

Put $x = 0$ in (2)

$$u(0) = a \cdot 0 + b$$

$$u(0) = b = 40$$

Put $x = l$ in (2)

$$u(1) = a \cdot 1 + b$$

$$90 = a(1) + 40$$

$$a = 90 - 40$$

$$a = 50$$

$$\therefore u(x) = ax + b$$

$$u(x) = 50x + 40$$

Unit - V
Z-Transforms and Difference Equations
Solving of Difference Equation Using



Z-Transform

Formula:

① $Z[y(n)] = F(z)$

② $Z[y(n+1)] = z \cdot F(z) - z y(0) \text{ (or) } z[F(z) - y(0)]$

③ $Z[y(n+2)] = z^2 F(z) - z^2 y(0) - z y(1) \text{ (or)}$

$z^2 \left[F(z) - y(0) - \frac{y(1)}{z} \right]$

④ $Z[y(n+3)] = z^3 F(z) - z^3 y(0) - z^2 y(1) - z y(2) \text{ (or)}$

⑤ $Z[a^n] = \frac{z}{z-a}$

$z^3 \left[F(z) - y(0) - \frac{y(1)}{z} - \frac{y(2)}{z^2} \right]$

⑥ $Z[y_{n-1}] = z^{-1} F(z)$

⑦ $Z[a^n n] = \frac{az}{(z-a)^2}$

① Solve $y_{n+1} - 2y_n = 0$ given $y_0 = 3$

Soln:

Given $y_{n+1} - 2y_n = 0$ & $y_0 = 3$.

XZ

$$z[y_{n+1}] - 2z(y_n) = z(0)$$

$$z(0) = 0$$

$$[zF(z) - zy(0)] - 2F(z) = 0$$

$$zF(z) - z(3) - 2F(z) = 0$$

$$y(0) = 3$$

$$(z-2)F(z) - 3z = 0$$

$$(z-2)F(z) = 3z$$

$$F(z) = \frac{3z}{z-2}$$

formula

$$z(y_n) = \frac{3z}{z-2}$$

$$y_n = z^{-1}\left(\frac{3z}{z-2}\right)$$

$$= 3z^{-1}\left(\frac{z}{z-2}\right)$$

$$y_n = 3(2^n)$$

$$z(y_n) = F(z)$$

$$z(a^n) = \frac{z}{z-a}$$

$$a^n = z^{-1}\left(\frac{z}{z-a}\right)$$

$$a=2$$

② Solve the difference equation

$$y(n+3) - 3y(n+1) + 2y(n) = 0 \text{ given that}$$

$$y(0) = 4, y(1) = 0 \text{ & } y(2) = 8.$$

Soln:

Given $y(n+3) - 3y(n+1) + 2y(n) = 0$

$$z(0) = 0$$

XZ $z[y(n+3)] - 3z[y(n+1)] + 2z[y(n)] = z(0)$

$$[z^3 F(z) - z^3 y(0) - z^2 y(1) - zy(2)]$$

$$- 3[zF(z) - zy(0)] + 2F(z) = 0$$

④

$$\begin{aligned} & (z^3 F(z) - 4z^3 - 8z) \\ & - 3(z F(z) - 4z) \\ & + 2F(z) = 0 \end{aligned}$$

$$\begin{aligned} y(0) &= 4; y(1) = 0 \\ y(2) &= 8 \end{aligned}$$

$$(z^3 - 3z + 2) F(z) - 4z^3 - 8z + 12z = 0$$

$$(z^3 - 3z + 2) F(z) - 4z^3 + 4z = 0$$

$$(z-1)^2 (z+2) F(z) = 4z^3 - 4z$$

$$\begin{aligned} F(z) &= \frac{4z^3 - 4z}{(z-1)^2 (z+2)} \\ &= \frac{4z(z^2 - 1)}{(z-1)^2 (z+2)} \quad a^2 - b^2 = (a+b)(a-b) \end{aligned}$$

$$F(z) = \frac{4z(z+1)(z-1)}{(z-1)^2 (z+2)}$$

$$\frac{F(z)}{z} = \frac{4(z+1)}{(z-1)(z+2)} \quad \text{--- (4)}$$

$$\begin{array}{r|rrrr} & 1 & 0 & -3 & 2 \\ +1 & 0 & +1 & 1 & -2 \\ \hline 1 & 1 & +1 & -2 & 0 \\ & 0 & 1 & 2 & \\ \hline & 1 & 2 & 0 & \\ & (z-1)(z-1)(z+2) & & & \\ & (z-1)^2(z+2) & & & \end{array}$$

$$\begin{aligned} z^2 - 1^2 &= (z+1)(z-1) \\ a^2 - b^2 &= (a+b)(a-b) \end{aligned}$$

$$\text{Let } \frac{F(z)}{z} = \frac{4(z+1)}{(z-1)(z+2)} = \frac{A}{z-1} + \frac{B}{z+2} \quad \text{--- (1)}$$

$$\frac{4(z+1)}{(z-1)(z+2)} = \frac{A(z+2) + B(z-1)}{(z-1)(z+2)}$$

$$4(z+1) = A(z+2) + B(z-1)$$

Put $z=1$,

$$4(1+1) = A(1+2) + B(0)$$

$$8 = 3A$$

$$\boxed{A = \frac{8}{3}}$$

Put $z=-2$,

$$-4 = -3B$$

$$\boxed{B = \frac{4}{3}}$$

from ①

$$\frac{F(z)}{z} = \frac{8/3}{z-1} + \frac{4/3}{z+2}$$

Formulas

$$Z(a^n) = \frac{z}{z-a}$$

$$a^n = z^{-1} \left(\frac{z}{z-a} \right)$$

Formulas

$$F(z) = \frac{8}{3} \left(\frac{z}{z-1} \right) + \frac{4}{3} \left(\frac{z}{z+2} \right)$$

$$Z[y(n)] = \frac{8}{3} \left(\frac{z}{z-1} \right) + \frac{4}{3} \left(\frac{z}{z+2} \right)$$

$$y(n) = \frac{8}{3} z^{-1} \left(\frac{z}{z-1} \right) + \frac{4}{3} z^{-1} \left(\frac{z}{z+2} \right)$$

$$y(n) = \frac{8}{3} (1)^n + \frac{4}{3} (-2)^n$$

$$y(n) = \frac{4}{3} [2 + (-2)^n]$$

③ Solve $u_{n+2} + 4u_{n+1} + 3u_n = 3^n$ with $u_0 = 0, u_1 = 1$ using z-transform.

Soln:

Given $u_{n+2} + 4u_{n+1} + 3u_n = 3^n$ & $u_0 = 0; u_1 = 1$.

XZ

$$Z(u_{n+2}) + 4Z(u_{n+1}) + 3Z(u_n) = Z(3^n)$$

$$\begin{aligned} & [z^2 F(z) - z^2 u(0) - z u(1)] + 4[z F(z) - z u(0)] + 3 F(z) = \frac{z}{z-3} \\ & z^2 F(z) - z + 4z F(z) + 3 F(z) = \frac{z}{z-3} \end{aligned}$$

$$z^2 F(z) - z + 4z F(z) + 3 F(z) = \frac{z}{z-3}$$

$$(z^2 + 4z + 3) F(z) = \frac{z}{z-3} + z$$

$$(z+1)(z+3) F(z) = \frac{z + z(z-3)}{(z-3)}$$

$$F(z) = \frac{z + z^2 - 3z}{(z+1)(z+3)(z-3)}$$

$$\begin{array}{c} z^2 + 4z + 3 \\ \swarrow \quad \searrow \\ 3 \\ \swarrow \quad \searrow \\ 1 \quad 3 \\ (z+1) \quad (z+3) \end{array}$$

$$F(z) = \frac{z^2 - 2z}{(z+1)(z+3)(z-3)}$$

$$F(z) = \frac{z(z-2)}{(z+1)(z+3)(z-3)}$$

$$\frac{F(z)}{z} = \frac{(z-2)}{(z+1)(z+3)(z-3)}$$

$$\frac{F(z)}{z} \frac{z-2}{(z+1)(z+3)(z-3)} = \frac{A}{z+1} + \frac{B}{(z+3)} + \frac{C}{(z-3)} \rightarrow (1)$$

$$\frac{z-2}{(z+1)(z+3)(z-3)} = \frac{A(z+3)(z-3) + B(z+1)(z-3) + C(z+1)(z+3)}{(z+1)(z+3)(z-3)}$$

$$z-2 = A(z+3)(z-3) + B(z+1)(z-3) + C(z+1)(z+3)$$

Put $z=3$,

$$3-2 = A(0) + B(0) + C(4)(6)$$

$$1 = 24C$$

$$\boxed{C = \frac{1}{24}}$$

Put $z=-3$

$$-3-2 = A(0) + B(-3+1)(-3-3) + C(0)$$

$$-5 = B(-2)(-6)$$

$$-5 = 12B$$

$$\boxed{B = -\frac{5}{12}}$$

Put $z=-1$

$$-1-2 = A(-1+3)(-1-3) + B(0) + C(0)$$

$$-3 = A(2)(-4)$$

$$-3 = -8A$$

$$\boxed{A = \frac{3}{8}}$$

from (1),

$$\frac{F(z)}{z} = \frac{3}{8} \left(\frac{1}{z+1} \right) - \frac{5}{12} \left(\frac{1}{z+3} \right) + \frac{1}{24} \left(\frac{1}{z-3} \right)$$

$$F(z) = \frac{3}{8} \left(\frac{z}{z+1} \right) - \frac{5}{12} \left(\frac{z}{z+3} \right) + \frac{1}{24} \left(\frac{z}{z-3} \right)$$

$$z y(n) = \frac{3}{8} \left(\frac{z}{z+1} \right) - \frac{5}{12} \left(\frac{z}{z+3} \right) + \frac{1}{24} \left(\frac{z}{z-3} \right)$$

$$y(n) = \frac{3}{8} z^{-1} \left(\frac{z}{z+1} \right) - \frac{5}{12} z^{-1} \left(\frac{z}{z+3} \right) + \frac{1}{24} z^{-1} \left(\frac{z}{z-3} \right)$$

$$y(n) = \frac{3}{8} (1)^n - \frac{5}{12} (3)^n + \frac{1}{24} (-3)^n$$

(4) solve $y_{n+2} + 6y_{n+1} + 9y_n = 2^n$ given

$$y_0 = y_1 = 0$$

$$\therefore z[an] = \frac{z}{z-a}$$

Soln:

Given $y_{n+2} + 6y_{n+1} + 9y_n = 2^n$

$\times z$ $z[y_{n+2}] + 6z[y_{n+1}] + 9z[y_n] = z(2^n)$

$$\left[z^2 F(z) - z^2 y(0) - z y(1) \right] + 6 \left[z F(z) - z y(0) \right] + 9 F(z) = \frac{z}{z-2}$$

$$z^2 F(z) + 6z F(z) + 9 F(z) = \frac{z}{z-2}$$

$$[z^2 + 6z + 9] F(z) = \frac{z}{z-2}$$

$$\therefore (z+3)^2 F(z) = \frac{z}{z-2}$$

$$F(z) = \frac{z}{(z-2)(z+3)^2}$$

$$\begin{array}{c} z^2 + 6z + 9 \\ \swarrow \quad \searrow \\ 3 \quad 3 \\ (z+3)(z+3) \\ (z+3)^2 \end{array}$$

$$\frac{F(z)}{z} = \frac{1}{(z-2)(z+3)^2}$$

$$\frac{F(z)}{z} = \frac{1}{(z-2)(z+3)^2} = \frac{A}{z-2} + \frac{B}{z+3} + \frac{C}{(z+3)^2} \rightarrow \textcircled{1}$$

$$1 = A(z+3)^2 + B(z-2)(z+3) + C(z-2)$$

Put $z=2$,

$$1 = A(2+3)^2 + B(0) + C(0)$$

$$1 = A(5)^2$$

$$1 = 25A$$

$$\boxed{A = \frac{1}{25}}$$

put $z=-3$

$$1 = A(0) + B(0) + C(-3-2)$$

$$1 = -5C$$

$$\boxed{C = -\frac{1}{5}}$$

! Equating z^2 coefficient,

$$0 = A + B$$

$$B = -A$$

$$\boxed{B = -\frac{1}{25}}$$

From ①,

$$\frac{F(z)}{z} = \frac{1}{25} \left(\frac{1}{z-2} \right) - \frac{1}{25} \left(\frac{1}{z+3} \right) - \frac{1}{5} \frac{1}{(z+3)^2}$$

$$F(z) = \frac{1}{25} \left(\frac{z}{z-2} \right) - \frac{1}{25} \left(\frac{z}{z+3} \right) + \frac{1}{5} \left(\frac{z}{(z+3)^2} \right)$$

$$z y(n) = \frac{1}{25} \left(\frac{z}{z-2} \right) - \frac{1}{25} \left(\frac{z}{z+3} \right) + \frac{1}{5} \left(\frac{z}{(z+3)^2} \right)$$

$$y(n) = \frac{1}{25} z^{-1} \left(\frac{z}{z-2} \right) - \frac{1}{25} z^{-1} \left(\frac{z}{z+3} \right) + \frac{1}{5} z^{-1} \left(\frac{z}{(z+3)^2} \right)$$

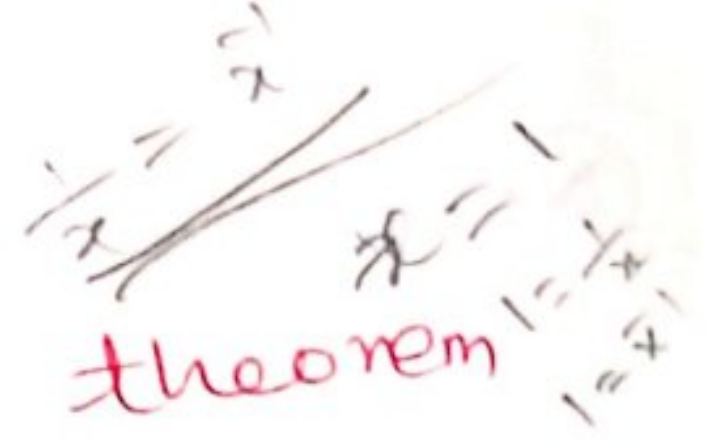
$$\boxed{y(n) = \frac{1}{25} (2^n) - \frac{1}{25} (-3)^n + \frac{1}{15} (-3)^n n}$$

Formulas

$$z[a^n] = \frac{z}{z-a}$$

$$z[a^n n] = \frac{az}{(z-a)^2}$$

Convolution Theorem



① State and prove Convolution theorem on z-transform.

Statement ::

If $\bar{z}^{-1}[x(z)] = x_n$ & $\bar{z}^{-1}[y(z)] = y_n$, then

$$\bar{z}^{-1}[x(z)y(z)] = \sum_{m=0}^n x_m y_{n-m} = x_n * y_n \dots$$

where $*$ denotes the convolution operation.

Proof ::

We know that,

$$x(z) = \sum_{n=0}^{\infty} x_n z^{-n} \quad \&$$

$$y(z) = \sum_{n=0}^{\infty} y_n z^{-n}$$

$$x(z)y(z) = \left(\sum_{n=0}^{\infty} x_n z^{-n} \right) \left(\sum_{n=0}^{\infty} y_n z^{-n} \right)$$

$$= (x_0 + x_1 z^{-1} + x_2 z^{-2} + \dots + x_n z^{-n} + \dots)$$

$$(y_0 + y_1 z^{-1} + y_2 z^{-2} + \dots + y_n z^{-n} + \dots)$$

$$\sum_{n=0}^{\infty} z^{-n} = z$$

$$= \sum_{n=0}^{\infty} [x_0 y_n + x_1 y_{n-1} + x_2 y_{n-2} + \dots + x_n y_0] z^{-n}$$

$$x(z)y(z) = \sum_{n=0}^{\infty} [x_0 y_n + x_1 y_{n-1} + \dots + x_n y_0] z^{-n}$$

$$\bar{z}^{-1}[x(z)y(z)] = x_0 y_n + x_1 y_{n-1} + \dots + x_n y_0$$

$$= \sum_{m=0}^n x_m y_{n-m}$$

$$= x_n * y_n$$

(By convolution Definition)

② Find $Z^{-1} \left[\frac{z^2}{(z-a)(z-b)} \right]$

Soln.

$z^2 = z \cdot z$

$$Z^{-1} \left[\frac{z^2}{(z-a)(z-b)} \right] = Z^{-1} \left[\frac{z}{z-a} \cdot \frac{z}{z-b} \right]$$

$$= Z^{-1} \left[\frac{z}{z-a} \right] * Z^{-1} \left[\frac{z}{z-b} \right]$$

$$= a^n * b^n$$

$$= a \sum_{m=0}^n a^m b^{n-m} \quad \left[\text{By Convolution theorem} \right]$$

$$= \sum_{m=0}^n a^m \cdot b^n \cdot b^{-m}$$

$$= b^n \sum_{m=0}^n a^m \left(\frac{1}{b^m} \right)$$

$$= b^n \sum_{m=0}^n \left(\frac{a}{b} \right)^m$$

$\frac{a}{b} = \frac{1}{1}$

$$= b^n \left[1 + \left(\frac{a}{b} \right) + \left(\frac{a}{b} \right)^2 + \dots + \left(\frac{a}{b} \right)^n \right]$$

$$= b^n \left[\frac{1 - \left(\frac{a}{b} \right)^{n+1}}{1 - \frac{a}{b}} \right]$$

$r = \frac{a}{b}$

$$= b^n \left[\frac{1 - \frac{a^{n+1}}{b^{n+1}}}{1 - \frac{a}{b}} \right]$$

$b^{n+1} = b^n \cdot b$

$$= b^n \left[\frac{b^{n+1} - a^{n+1}}{b^{n+1} \cdot \left(\frac{b-a}{b} \right)} \right]$$

$$= b^n \left[\frac{b^{n+1} - a^{n+1}}{b^n \cdot b \cdot (b-a)} \times \frac{b}{(b-a)} \right]$$

$$= \frac{b^{n+1} - a^{n+1}}{b-a}$$

Formula

$$a + ar + ar^2 + \dots + ar^n$$

$= \frac{a[1 - r^{n+1}]}{1 - r}$ if $r < 1$

Here $a=1$

③ Using Convolution theorem evaluate inverse z-transform of $\left[\frac{z^2}{(z-1)(z-3)} \right]$.

Soln:

We first prove that

$$z^{-1} \left[\frac{z^2}{(z-a)(z-b)} \right] = \frac{b^{n+1} - a^{n+1}}{b-a}$$

$$\frac{z^2}{(z-1)(z-3)} = \frac{3^{n+1} - 1^{n+1}}{3-1}$$

$$= \frac{3^{n+1} - 1}{2}$$

$$\boxed{a=1}$$

$$\boxed{b=3}$$

④ Find $z^{-1} \left[\frac{z^2}{(z-3)(z-4)} \right]$

We first prove that

$$z^{-1} \left[\frac{z^2}{(z-a)(z-b)} \right] = \frac{b^{n+1} - a^{n+1}}{b-a}$$

$$z^{-1} \left[\frac{z^2}{(z-3)(z-4)} \right] = \frac{4^{n+1} - 3^{n+1}}{4-3}$$

$$= 4^{n+1} - 3^{n+1}$$

$$\boxed{a=3}$$

$$\boxed{b=4}$$

⑤ Find $z^{-1} \left[\frac{z^2}{(z-a)^2} \right]$.

$$z^{-1} \left[\frac{z^2}{(z-a)^2} \right] = z^{-1} \left[\frac{z}{(z-a)} \cdot \frac{z}{(z-a)} \right]$$

$$= z^{-1} \left[\frac{z}{z-a} \right] * z^{-1} \left[\frac{z}{z-a} \right]$$

$$= a^n * a^n$$

$$= \sum_{m=0}^{\infty} a^{n-m} a^m \quad \left[\text{By Convolution theorem} \right]$$

$$= \sum_{m=0}^{\infty} a^n \cdot \underbrace{a^{-m} a^m}_{=1}$$

$$= \sum_{m=0}^{\infty} a^n (1)^m$$

$$= a^n [1^0 + 1^1 + 1^2 + \dots + 1^n]$$

$$= a^n (n+1)$$

$$\underbrace{a^{-m+m}}_{=1} = a^0 = 1$$

or $\frac{1}{a^n} \times a^n = 1$

⑥ Find $z^{-1} \left[\frac{z^2}{(z+a)^2} \right]$

We first prove that

$$z^{-1} \left[\frac{z^2}{(z-a)^2} \right] = (n+1) a^n$$

$$z^{-1} \left[\frac{z^2}{(z+a)^2} \right] = (n+1) \underbrace{(-a)}_{\text{Sub}}^n$$

~~$$z^{-1} \left[\frac{z^2}{(z+a)^2} \right] =$$~~

Inverse Z-Transform Using Partial Fraction Method

① Find $\mathcal{Z}^{-1} \left[\frac{10z}{(z-1)(z-2)} \right]$

Soln:

Let $F(z) = \frac{10z}{(z-1)(z-2)}$

$$\frac{F(z)}{z} = \frac{10}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2} \rightarrow \textcircled{1}$$

$$\frac{10}{\cancel{(z-1)}\cancel{(z-2)}} = \frac{A(z-2) + B(z-1)}{\cancel{(z-1)}\cancel{(z-2)}}$$

$$10 = A(z-2) + B(z-1)$$

Put $z=1$,

$$10 = A(1-2) + B(0)$$

$$10 = A(-1)$$

$$\boxed{A = -10}$$

Put $z=2$,

$$10 = A(0) + B(2-1)$$

$$\boxed{10 = B}$$

$$\frac{F(z)}{z} = \frac{-10}{z-1} + \frac{10}{z-2}$$

$$F(z) = -10 \left(\frac{z}{z-1} \right) + 10 \left(\frac{z}{z-2} \right)$$

$$z y(n) = -10 \left[\frac{z}{z-1} \right] + 10 \left(\frac{z}{z-2} \right)$$

$$y(n) = -10 \mathcal{Z}^{-1} \left[\frac{z}{z-1} \right] + 10 \mathcal{Z}^{-1} \left(\frac{z}{z-2} \right)$$

$$\boxed{y(n) = -10(1)^n + 10(2^n)}$$

$\mathcal{Z}[a^n] = \frac{z}{z-a}$
 $\therefore \mathcal{Z} y(n) = F(z)$

② Find $\bar{z}^{-1} \left[\frac{z(z^2 - z + 2)}{(z+1)(z-1)^2} \right]$

Soln:

Let $F(z) = \frac{z(z^2 - z + 2)}{(z+1)(z-1)^2}$

$$\frac{F(z)}{z} = \frac{z^2 - z + 2}{(z+1)(z-1)^2} = \frac{A}{z+1} + \frac{B}{z-1} + \frac{C}{(z-1)^2} \rightarrow \textcircled{1}$$

$$\frac{z^2 - z + 2}{(z+1)(z-1)^2} = \frac{A(z-1)^2 + B(z+1)(z-1) + C(z+1)}{(z+1)(z-1)^2}$$

$$z^2 - z + 2 = A(z-1)^2 + B(z+1)(z-1) + C(z+1)$$

Put $z = -1$

$$1 + 1 + 2 = A(-1-1)^2 + B(0) + C(0)$$

$$4 = 4A$$

$$\boxed{A = 1}$$

Put $z = 1$

$$1 - 1 + 2 = A(0) + B(0) + C(1+1)$$

$$2 = 2C$$

$$\boxed{C = 1}$$

Put $z = 0$,

$$0 - 0 + 2 = A(-1)^2 + B(1)(-1) + C(1)$$

$$2 = A - B + C$$

$$2 = 1 - B + 1$$

$$2 = 2 - B$$

$$-B = 2 - 2$$

$$-B = 0$$

$$\boxed{B = 0}$$

From $\textcircled{1}$,

$$\frac{F(z)}{z} = \frac{1}{z+1} + \frac{1}{(z-1)^2}$$

$$F(z) = \frac{z}{z+1} + \frac{z}{(z-1)^2}$$

$$z y(n) = \frac{z}{z+1} + \frac{z}{(z-1)^2}$$

$$y(n) = z^{-1} \left[\frac{z}{z+1} \right] + z^{-1} \left[\frac{z}{(z-1)^2} \right]$$

$$= z^{-1} \left[\frac{z}{z-(-1)} \right] + z^{-1} \left[\frac{z}{(z-1)^2} \right]$$

$$y(n) = (-1)^n + n$$

③ Find $z^{-1} \left[\frac{z}{z^2+4z+3} \right]$

Soln:

$$\text{Let } F(z) = \frac{z}{z^2+4z+3}$$

$$\frac{F(z)}{z} = \frac{1}{z^2+4z+3} = \frac{1}{(z+1)(z+3)}$$

$$\frac{F(z)}{z} = \frac{1}{(z+1)(z+3)} = \frac{A}{z+1} + \frac{B}{z+3} \rightarrow \text{①}$$

$$\frac{1}{(z+1)(z+3)} = \frac{A(z+3) + B(z+1)}{(z+1)(z+3)}$$

$$1 = A(z+3) + B(z+1)$$

put $z = -1$

$$1 = A(-1+3) + B(0)$$

$$1 = A(2)$$

$$\boxed{A = \frac{1}{2}}$$

put $z = -3$

$$1 = A(0) + B(-3+1)$$

$$1 = B(-2)$$

$$\boxed{B = -\frac{1}{2}}$$

From ①, $\frac{F(z)}{z} = \frac{1}{2} \left(\frac{1}{z+1} \right) - \frac{1}{2} \left(\frac{1}{z+3} \right)$

$$F(z) = \frac{1}{2} \left(\frac{z}{z+1} \right) - \frac{1}{2} \left(\frac{z}{z+3} \right)$$

$$z Y(z) = \frac{1}{2} \left(\frac{z}{z+1} \right) - \frac{1}{2} \left(\frac{z}{z+3} \right)$$

$$Y(z) = \frac{1}{2} z^{-1} \left(\frac{z}{z+1} \right) - \frac{1}{2} z^{-1} \left(\frac{z}{z+3} \right)$$

$$Y(z) = \frac{1}{2} (-1)^n - \frac{1}{2} (-3)^n$$

Inverse z-Transform using Cauchy's Residue Theorem.

① Find $z^{-1} \left[\frac{10z}{(z-1)(z-2)} \right]$

Soln:-

$$\text{Let } F(z) = \frac{10z}{(z-1)(z-2)}$$

$$\times z^{n-1}$$

$$F(z) z^{n-1} = \frac{10z}{(z-1)(z-2)} z^{n-1}$$

$$= \frac{10z^n}{(z-1)(z-2)}$$

$$z-1=0$$

$z=1$ is a simple pole.

$$z-2=0$$

$z=2$ is a simple pole

Let the contour $|z| > 2$.

$$\text{Res}_{z=1} F(z) z^{n-1} = \lim_{z \rightarrow 1} (z-1) \frac{10z^n}{(z-1)(z-2)}$$

$$= \lim_{z \rightarrow 1} \frac{10z^n}{z-2}$$

$$= \frac{10(1)^n}{1-2} = -10,$$

$$\begin{aligned} \text{Res}_{z=2} F(z) z^{n-1} &= \lim_{z \rightarrow 2} (z-2) \frac{10z^n}{(z-1)(z-2)} \\ &= \lim_{z \rightarrow 2} \frac{10z^n}{z-1} = \frac{10(2^n)}{2-1} = 10 \times 2^n \end{aligned}$$

1) State initial value theorem in Z-transform
 Soln:.

If $Z[f(t)] = F(z)$, then

$$f(0) = \lim_{z \rightarrow \infty} F(z).$$

2) State Final value theorem in Z-transform.
 Soln:.

If $Z[f(t)] = F(z)$, then

$$\lim_{t \rightarrow \infty} f(t) = \lim_{z \rightarrow 1} (z-1) F(z).$$

3) Define Unit Step Sequence

The unit step sequence $u(n)$ has values

$$u(n) = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

4) Define Unit impulse Sequence.

The unit impulse sequence $\delta(n)$ is defined as the sequence with values.

$$\delta(n) = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{for } n \neq 0 \end{cases}$$

5) Define Convolution of Sequence

① The convolution of two sequence $\{x(n) \& y(n)\}$ is defined as

$$i) \{x(n) * y(n)\} = \sum_{k=-\infty}^{\infty} f(k) g(n-k) \text{ if the}$$

sequences are non-causal

$$ii) \{x(n) * y(n)\} = \sum_{k=0}^{\infty} f(k) g(n-k) \text{ if the}$$

sequences are causal.

② The convolution of two functions $f(t) \& g(t)$ is defined as $f(t) * g(t) = \sum_{k=0}^n f(kT) g(n-k)T$.

period.

⑥ Find $Z[e^{-iat}]$ using Z-Transform

Soln:-

$$Z[e^{-iat}] = Z[1]_{z \rightarrow ze^{iaT}}$$

$$= \left[\frac{z}{z-1} \right]_{z \rightarrow ze^{iaT}}$$

$$= \left[\frac{ze^{iaT}}{ze^{iaT}-1} \right]$$

$$= \frac{ze^{iaT}}{e^{iaT} \left[z - \frac{1}{e^{iaT}} \right]}$$

$$= \frac{z}{z - e^{-iaT}}$$

$$= \frac{z}{z - e^{-iaT}}$$

$$\therefore Z(1) = \frac{z}{z-1}$$

⑦ Find $Z\left[\frac{a^n}{n!}\right]$ in Z-transform.

Soln:-

We know that $Z[a^n f(n)] = F\left[\frac{z}{a}\right]$

$$Z\left[a^n \frac{1}{n!}\right] = \left[Z\left[\frac{1}{n!}\right] \right]_{z \rightarrow \frac{z}{a}}$$

$$= \left[e^{\frac{1}{z}} \right]_{z \rightarrow \frac{z}{a}}$$

$$= e^{\frac{1}{(z/a)}}$$

$$= e^{\frac{a}{z}}$$

$$\therefore Z\left[\frac{1}{n!}\right] = e^{\frac{1}{z}}$$

or $Z\left[\frac{1}{n!}\right] = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{1}{z}\right)^n$
 $= 1 + \frac{1}{z} + \frac{1}{2!} \left(\frac{1}{z}\right)^2 + \dots$

⑧ State and prove initial value theorem in Z-transform
Statement:-

Initial value theorem

If $Z[f(n)] = F(z)$, then $f(0) = \lim_{z \rightarrow \infty} F(z)$.

Proof:-

$$Z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n}$$

$$F(z) = f(0) + \frac{f(1)}{z} + \frac{f(2)}{z^2} + \dots$$

$$= f(0) + \frac{f(1)}{z} + \frac{f(2)}{z^2} + \dots$$

$$\lim_{z \rightarrow \infty} F(z) = \lim_{z \rightarrow \infty} \left[f(0) + \frac{f(1)}{z} + \frac{f(2)}{z^2} + \dots \right]$$

$$= f(0) + 0 + 0 + \dots$$

$$= f(0)$$

$$\therefore \lim_{z \rightarrow \infty} \frac{1}{z} = 0$$

$$\therefore \lim_{z \rightarrow \infty} F(z) = f(0)$$

9) Find the z-transform of $(n+1)(n+2)$

Soln:

$$Z[(n+1)(n+2)] = Z[n^2 + 2n + n + 2]$$

$$= Z[n^2 + 3n + 2]$$

$$= Z(n^2) + 3Z(n) + 2Z(1)$$

$$= \frac{Z(Z+1)}{(Z-1)^3} + 3 \frac{Z}{(Z-1)^2} + 2 \frac{Z}{Z-1}$$

10) Find the z-transform of $(n+2)$.

Soln:

$$Z(n+2) = Z(n) + Z(2)$$

$$= Z(n) + 2Z(1)$$

$$= \frac{Z}{(Z-1)^2} + 2\left(\frac{Z}{Z-1}\right)$$

$$Z(1) = \frac{Z}{Z-1}$$

$$Z(n) = \frac{Z}{(Z-1)^2}$$

11) Find $Z\left[\frac{1}{(n+1)!}\right]$

Soln:

$$Z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n}$$

$$Z\left[\frac{1}{(n+1)!}\right] = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \left(\frac{1}{z}\right)^n$$

$$Z\left[\frac{1}{(n+1)!}\right] = \frac{1}{1!} \left(\frac{1}{z}\right)^0 + \frac{1}{2!} \left(\frac{1}{z}\right)^1 + \frac{1}{3!} \left(\frac{1}{z}\right)^2 + \dots$$

$$\frac{1}{(n+1)!} = Z\left[1 + \frac{\left(\frac{1}{z}\right)}{2!} + \frac{\left(\frac{1}{z}\right)^2}{3!} + \dots\right]$$

$$= Z\left[e^{\frac{1}{z}} - 1\right]$$

$$\frac{1}{(n+1)!} = ze^{\frac{1}{z}} - Z$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots$$

$$e^{x-1} = \frac{x}{1!} + \frac{x^2}{2!} + \dots$$

(12) Find $Z \left[\frac{1}{n(n+1)} \right]$

Soln:

$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1} \rightarrow (1)$$

$$\frac{1}{n(n+1)} = \frac{A(n+1) + Bn}{n(n+1)}$$

$$1 = A(n+1) + Bn$$

Put $n=0$

$$1 = A$$



Put $n=-1$

$$1 = A(0) - B$$

$$B = -1$$

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

$$Z \left[\frac{1}{n(n+1)} \right] = Z \left(\frac{1}{n} \right) - Z \left(\frac{1}{n+1} \right)$$

$$= \log \left(\frac{Z}{Z-1} \right) - Z \log \left(\frac{Z}{Z-1} \right)$$

$$= \log \left(\frac{Z}{Z-1} \right) (1-Z)$$

$$\therefore Z \left(\frac{1}{n} \right) = \log \left(\frac{Z}{Z-1} \right)$$

$$Z \left(\frac{1}{n+1} \right) = Z \log \left(\frac{Z}{Z-1} \right)$$

(13) Find $Z \left[\cos \left(\frac{n\pi}{2} \right) \right]$

Soln:

$$Z \left[\cos n \frac{\pi}{2} \right] = \frac{Z [Z - \cos \frac{\pi}{2}]}{Z^2 - 2Z \cos \frac{\pi}{2} + 1}$$

$$= \frac{Z(Z-0)}{Z^2 - 2Z(0) + 1}$$

$$= \frac{Z^2}{Z^2 + 1} ; |Z| > 1$$

$$(i) Z[F(z)] = \sum_{n=0}^{\infty} F(n) Z^{-n}$$

$$(ii) Z(\cos n\theta) = \frac{Z(Z - \cos \theta)}{Z^2 - 2Z \cos \theta + 1}$$

$$\theta = \frac{\pi}{2}$$

(14) Define First Shifting theorem

If $Z[f(t)] = F(z)$, then

$$Z\{e^{-at} f(t)\} = Z[f(t)]_{z \rightarrow e^{-at}}$$

15) Define Second Shifting theorem

If $z \{f(t)\} = F(z)$, then $z \{f(t+T)\} = zF(z) - zf(0)$.

① Derive the difference equation from

$$u_n = A 2^n + Bn$$

Soln:

Let $u_n = A 2^n + Bn \rightarrow \textcircled{1}$

~~Let~~ $u_{n+1} = A 2^{n+1} + B(n+1) = A \cdot 2^n \cdot 2^1 + B(n+1) \rightarrow \textcircled{2}$

$u_{n+2} = A 2^{n+2} + B(n+2) = A \cdot 2^n \cdot 2^2 + B(n+2) \rightarrow \textcircled{3}$

$$\begin{vmatrix} u_n & 1 & n \\ u_{n+1} & 2 & n+1 \\ u_{n+2} & 4 & n+2 \end{vmatrix} = 0$$

$$\begin{aligned} & \left(u_n [2(n+2) - 4(n+1)] - u_{n+1} [n+2 - 4n] \right) \\ & \quad + u_{n+2} [n+1 - 2n] = 0 \end{aligned}$$

$$\begin{aligned} & 2u_n(n+2) - 4u_n(n+1) - u_{n+1}(n+2) + u_{n+2}(n+1) \\ & \quad + 4n u_{n+1} - 2n u_{n+2} = 0 \end{aligned}$$

$$\begin{aligned} & [2u_n(n+2) - 2n] + u_{n+1}(-n-2+4n) \\ & \quad + u_{n+2}(n+1-2n) = 0 \end{aligned}$$

$$-2n u_n + u_{n+1}(3n-2) + u_{n+2}(1-n) = 0$$

(or)

$$u_n [2(n+2) - 4(n+1)] - u_{n+1} (n+2 - 4n) + u_{n+2} (n+1 - 2n) = 0$$

$$u_n(-2n) - u_{n+1}(-3n+2) + u_{n+2}(-n+1) = 0$$

$$-2n u_n + u_{n+1}(-3n+2) + u_{n+2}(1-n) = 0$$

② Derive the difference equation from

$$y_n = (A + Bn) 2^n$$

Soln

Let $y_n = (A + Bn) 2^n \rightarrow \textcircled{1}$

$$y_{n+1} = A 2^{n+1} + B(n+1) 2^{n+1}$$

$$y_{n+1} = A \cdot 2^n \cdot (2^1) + B(n+1) 2^n \cdot 2 \rightarrow \textcircled{2}$$

$$y_{n+2} = A 2^{n+2} + B(n+2) 2^{n+2}$$

$$= 2 \cdot 2^n \cdot (2^2) + B(n+2) 2^n \cdot 2^2 \rightarrow \textcircled{3}$$

$$\begin{vmatrix} y_n & 1 & n \\ y_{n+1} & 2 & 2(n+1) \\ y_{n+2} & 4 & 4(n+2) \end{vmatrix} = 0$$

$$\begin{vmatrix} + & + & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

$$y_n [8(n+2) - 8(n+1)] - y_{n+1} [4(n+2) - 4n] + y_{n+2} [2(n+1) - 2n] = 0$$

$$y_n [8n + 16 - 8n - 8] - y_{n+1} [4n + 8 - 4n] + y_{n+2} [2n + 2 - 2n] = 0$$

$$y_n (8) - y_{n+1} (8) + y_{n+2} (2) = 0$$

③

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{ist} (G_1(s)) dt$$

$$= G_1(s) \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{ist} dt \right]$$

$$= G_1(s) F(s)$$

$$= F(s) G_1(s)$$

By shifting property

$$F[f(x-a)] = e^{ias} F(s)$$

$$a = t$$

Parseval's Identity :

If $F(s)$ is the Fourier transform of $f(x)$, then

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(s)|^2 ds$$

Proof :

By convolution theorem,

$$F[f(x) * g(x)] = F(s) \cdot G(s)$$

$$f(x) * g(x) = F^{-1}[F(s) \cdot G(s)]$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) g(x-t) dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) G(s) e^{-isx} ds$$

$$\rightarrow \textcircled{1} e^0 = 1$$

Put $x=0$,

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) g(-t) dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) G(s) ds$$

$$\int_{-\infty}^{\infty} f(t) \overline{f(t)} dt = \int_{-\infty}^{\infty} F(s) \overline{F(s)} ds$$

$\because g(-t) = \overline{f(t)}$, therefore it follows that $G(s) = \overline{F(s)}$

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} |F(s)|^2 ds$$

Put $t=x$

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(s)|^2 ds$$

① Show that the Fourier Transform of $e^{-\frac{x^2}{2}}$ is $e^{-\frac{s^2}{2}}$.

(or)
Show that $e^{-\frac{x^2}{2}}$ is self-reciprocal w.r.to Fourier Transform.

Soln ::

Given $f(x) = e^{-\frac{x^2}{2}}$

$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$

$F[e^{-\frac{x^2}{2}}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{x^2}{2}\right)} e^{isx} dx$

$(a-b)^2 = a^2 + b^2 - 2ab$
 $(a-b)^2 - b^2 = a^2 - 2ab$
 $a = \frac{x}{\sqrt{2}}; 2ab = isx$
 $2 \frac{x}{\sqrt{2}} b = isx$
 $\frac{\sqrt{2}x}{2} b = isx$
 $b = \frac{is}{\sqrt{2}}$

$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{x^2}{2}\right) + isx} dx$

$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left[\left(\frac{x}{\sqrt{2}}\right)^2 - isx\right]} dx$

$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left[\left(\frac{x}{\sqrt{2}} - \frac{is}{\sqrt{2}}\right)^2 - \left(\frac{is}{\sqrt{2}}\right)^2\right]} dx$
 $= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{x}{\sqrt{2}} - \frac{is}{\sqrt{2}}\right)^2} e^{\frac{i^2 s^2}{2}} dx$

$i^2 = -1$
 Put $y = \frac{x}{\sqrt{2}} - \frac{is}{\sqrt{2}}$

$\frac{dy}{dx} = \frac{1}{\sqrt{2}}$

$dy = \frac{1}{\sqrt{2}} dx$

$x \rightarrow -\infty \Rightarrow y \rightarrow -\infty$
 $x \rightarrow \infty \Rightarrow y \rightarrow \infty$

$\int_{-\infty}^{\infty} e^{-y^2} dy = \sqrt{\pi}$

$= \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{2}} \int_{-\infty}^{\infty} e^{-\left(\frac{x}{\sqrt{2}} - \frac{is}{\sqrt{2}}\right)^2} dx$

$= \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{2}} \int_{-\infty}^{\infty} e^{-y^2} \sqrt{2} dy$

$= \frac{1}{\sqrt{2\pi}} \sqrt{2} e^{-\frac{s^2}{2}} \int_{-\infty}^{\infty} e^{-y^2} dy$

$= \frac{1}{\sqrt{\pi}} e^{-\frac{s^2}{2}} \sqrt{\pi}$

$F[e^{-\frac{x^2}{2}}] = e^{-\frac{s^2}{2}}$

Hence the proof.

② Find the Fourier transform of $e^{-a^2 x^2}$, $a > 0$,
 Hence, Show that $e^{-\frac{x^2}{2}}$ is self reciprocal
 under Fourier transform.

Soln:

Given $f(x) = e^{-a^2 x^2}$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$F[e^{-\frac{x^2}{2}}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2 x^2} e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2 x^2 + isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-[(ax)^2 - isx]} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-[(ax - \frac{is}{2a})^2 - (\frac{is}{2a})^2]} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(ax - \frac{is}{2a})^2} e^{\frac{(is)^2}{4a^2}} dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{4a^2}} \int_{-\infty}^{\infty} e^{-(ax - \frac{is}{2a})^2} dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{4a^2}} \int_{-\infty}^{\infty} e^{-y^2} \frac{1}{a} dy$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{a}\right) e^{-\frac{s^2}{4a^2}} \int_{-\infty}^{\infty} e^{-y^2} dy$$

$$= \frac{1}{a\sqrt{2}} \sqrt{\pi} e^{-\frac{s^2}{4a^2}}$$

$$= \frac{1}{a\sqrt{2}} e^{-\frac{s^2}{4a^2}}$$

$$= \left(\frac{a}{a}\right) e^{-\frac{s^2}{2(1/2)}}$$

$$F[e^{-\frac{x^2}{2}}] = e^{-\frac{s^2}{2}}$$

$$\begin{aligned} (a-b)^2 &= a^2 + b^2 - 2ab \\ (a-b)^2 - b^2 &= a^2 - 2ab \\ A &= ax \\ 2AB &= isx \\ 2a \times B &= isx \\ B &= \frac{is}{2a} \end{aligned}$$

$$i^2 = -1$$

$$\begin{aligned} y &= ax - \frac{is}{2a} \\ \frac{dy}{dx} &= a \\ \frac{dy}{a} &= dx \\ \int_{-\infty}^{\infty} e^{-y^2} \frac{1}{a} dy &= \int_{-\infty}^{\infty} e^{-y^2} dy = \sqrt{\pi} \end{aligned}$$

$$\begin{aligned} \text{put } a &= \frac{1}{\sqrt{2}} \\ a^2 &= \frac{1}{2} \end{aligned}$$

$$\int_{-\infty}^{\infty} e^{-y^2} dy = \sqrt{\pi}$$

Find the Fourier Transform of $f(x) = \begin{cases} a-|x|, & \text{for } |x| < a \\ 0 & \text{for } |x| > a > 0 \end{cases}$ hence

deduce that $\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$

Soln:

WKT $F(s) = F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-a}^a [a-|x|] e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^a [a-|x|] [\cos sx + i \sin sx] dx$$

→ Even → odd = 0

$$= \frac{1}{\sqrt{2\pi}} \cdot 2 \int_0^a (a-x) \cos sx dx + 0$$

$$= \sqrt{\frac{2}{\pi}} \int_0^a (a-x) \cos sx dx$$

$$= \sqrt{\frac{2}{\pi}} \left[(a-x) \frac{\sin sx}{s} - (-1) \left(-\frac{\cos sx}{s^2} \right) \right]_0^a$$

(a-a=0) → 0

$$= \sqrt{\frac{2}{\pi}} \left[\left(0 - \frac{\cos as}{s^2} \right) - \left(0 - \frac{1}{s^2} \right) \right]$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{1}{s^2} [1 - \cos as]$$

$$F(s) = \sqrt{\frac{2}{\pi}} \cdot \frac{1}{s^2} \left[2 \sin^2 \frac{as}{2} \right]$$

Formula

By inversion formula,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{-isx} ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} 2 \sqrt{\frac{2}{\pi}} \sin^2 \frac{as}{2} e^{-isx} ds$$

$$= \frac{2}{\sqrt{2\pi}} \times \frac{\sqrt{2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} \sin^2 \frac{as}{2} e^{-isx} ds$$

$$2 = \sqrt{2} \sqrt{2}$$

$$\frac{2}{\sqrt{2\pi}} = \frac{\sqrt{2} \sqrt{2}}{\sqrt{2} \sqrt{\pi}} = \frac{\sqrt{2}}{\sqrt{\pi}}$$

Diff	Integr
$u = a-x$	$v = \cos sx$
$u' = -1$	$v_1 = \frac{\sin sx}{s}$
$u'' = 0$	$v_2 = -\frac{\cos sx}{s^2}$
$\int u dv = uv_1 - u'v_2 + u''v_3 - \dots$	

$$\sqrt{\pi} \sqrt{\pi} = \pi$$

$$f(x) = \frac{2}{\pi} \int_{-\infty}^{\infty} \left(\frac{1}{s^2} \sin^2 \frac{as}{2} \right) e^{-isx} ds$$

$$f(x) = \frac{2}{\pi} \int_{-\infty}^{\infty} \left[\frac{\sin^2 \frac{as}{2}}{s} \right] e^{-isx} ds$$

$$x=0 \Rightarrow a=2$$

$$e^0 = 1$$

$$f(0) = \frac{2}{\pi} \int_{-\infty}^{\infty} \sin^2 \left(\frac{xs}{x} \right) \frac{e^{-is(0)}}{1} ds$$

$$= \frac{2}{\pi} \int_{-\infty}^{\infty} \sin^2 s ds$$

$$= \frac{2}{\pi} \left(2 \int_0^{\infty} \left(\frac{\sin^2 s}{s^2} \right) ds \right)$$

$$f(0) = \left(\frac{4}{\pi} \right) \int_0^{\infty} \frac{\sin^2 s}{s^2} ds$$

$$\int_0^{\infty} \frac{\sin^2 s}{s^2} ds = \frac{\pi}{4} f(0)$$

$$= \frac{\pi}{4} \times 2$$

$$= \frac{\pi}{2}$$

$$\int_0^2 \left(\frac{\sin t}{t} \right)^2 dt = \frac{\pi}{2}$$

Hence the proof

$$\text{even function} \left\{ = \frac{\sin^2 s}{s^2} \right\}$$

$$f(x) = \begin{cases} a-|x|, & |x| < a \\ 0, & \text{otherwise} \end{cases}$$

$$f(0) = a; a=2$$

s is a Dummy variable

$$s=t$$

Unit - IV
Two marks

- ① State the convolution theorem for Fourier transforms.

The Fourier transforms of the convolution of $f(x)$ & $g(x)$ is the product of their Fourier transforms.

$$F[f(x) * g(x)] = F(s) G(s) = F[f(x)] F[g(x)].$$

- ② State Parseval's identity for Fourier Transform.

If $F(s)$ is the Fourier transform of $f(x)$, then $\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(s)|^2 ds$.

- ③ State the Fourier integral theorem.

The Fourier integral theorem of $f(x)$ in the interval $(-L, L)$ is

$$f(x) = \frac{1}{\pi} \int_0^{\infty} \int_{-\infty}^{\infty} f(x) \cos \lambda(x-\alpha) d\alpha d\lambda.$$

- ④ Define Fourier Transform Pair (or) Define Fourier Transform and its inverse transform.

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{-isx} ds$$

- ⑤ Find the Fourier (cosine) transform of e^{-3x} .

W.K.T $F_c[e^{-ax}] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-ax} \cos sx dx$

$$F_c[e^{-3x}] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-3x} \cos sx dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{3}{s^2 + 3^2} \right]$$

$$\because \int_0^{\infty} e^{-ax} \cos bx dx = \frac{a}{a^2 + b^2}$$

6) Find ^{the} Fourier Cosine transform of e^{-x} .

WKT $F_c[e^{-ax}] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-ax} \cos sx \, dx$

$$F_c[e^{-x}] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-x} \cos sx \, dx$$

$$\therefore \int_0^{\infty} e^{-ax} \cos bx \, dx = \frac{a}{a^2 + b^2}$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{1}{1+s^2} \right]$$

7) Find the Fourier Sine transform of e^{-3x} .

WKT $F_s[e^{-ax}] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-ax} \sin sx \, dx$

$$F_s[e^{-3x}] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-3x} \sin sx \, dx$$

~~WKT~~ $F_s[e^{-ax}]$

$$\int_0^{\infty} e^{-ax} \sin bx \, dx = \frac{b}{a^2 + b^2}$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{s}{s^2 + 3^2} \right]$$

8) Find the Fourier Sine transform of e^{ax} .

$$F_s(f(x)) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin sx \, dx$$

WKT $F_s(e^{ax}) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{ax} \sin sx \, dx$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{ax} \sin sx \, dx = \left(\frac{s}{s^2 + a^2} \right)$$

9) Find the Fourier Sine transform of $\frac{1}{x}$.

WKT $F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin sx \, dx$

$$\therefore \int_0^{\infty} \frac{\sin ax}{x} \, dx = \frac{\pi}{2} \quad ; a > 0$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{1}{x} \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{\pi}{2} \right)$$

$$\pi = \sqrt{\pi} \sqrt{\pi} \\ 2 = \sqrt{2} \sqrt{2}$$

$$= \frac{\sqrt{2}}{\sqrt{\pi}} \frac{\sqrt{\pi} \sqrt{\pi}}{\sqrt{2} \sqrt{2}} \\ = \frac{\sqrt{\pi}}{\sqrt{2}}$$

(10) State the modulation theorem in Fourier Transform.

If $F(s)$ is the Fourier transform of $f(x)$, then $F[f(x) \cos ax] = \frac{1}{2} [F(s+a) + F(s-a)]$

(11) ^{2m or 8m} Shifting property

(i) $F[f(x-a)] = e^{ias}$

(ii) $F[e^{iax} f(x)] = F(s+a)$

Proof:

(i) WKT $F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$

$$F[f(x-a)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-a) e^{isx} dx$$

Put $t = x - a$

$$\frac{dt}{dx} = 1$$

$$dt = dx$$

$$x \rightarrow -\infty \Rightarrow t \rightarrow -\infty$$

$$x \rightarrow \infty \Rightarrow t \rightarrow \infty$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{is(t+a)} dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{ist} \cdot e^{isa} dt$$

$$= e^{isa} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{ist} dt$$

$$= e^{ias} F[f(t)]$$

$$= e^{isa} F(s)$$

(ii) WKT, $F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$

$$F[e^{iax} f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iax} f(x) e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i(s+a)x} dx$$

$$= F(s+a)$$

(12) Give a function which is self reciprocal under Fourier sine and cosine transforms

soln $\frac{1}{\sqrt{x}}$.

(13) State the Fourier transform of the derivatives of a function.

The Fourier transform of $F'(x)$, the derivative of $F(x)$ is $if(s)$, where $f(s)$ is the Fourier transform of $F(x)$.

$$F[F'(x)] = if(s).$$

① Solve for $f(x)$ from the integral equation
 $\int_0^\infty f(x) \cos \lambda x \, dx = e^{-\lambda}$.

Soln:

$$\text{Let } \int_0^\infty f(x) \cos \lambda x \, dx = e^{-\lambda}$$

$$\times \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos \lambda x \, dx = \sqrt{\frac{2}{\pi}} e^{-\lambda}$$

Formula

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}} e^{-\lambda}$$

Inversion formula

$$\int_0^\infty e^{-ax} \cos bx \, dx = \frac{a}{a^2 + b^2} \quad a > 0, b \in \mathbb{R}$$

By Fourier Inversion formula,

$$\begin{aligned} f(x) &= \sqrt{\frac{2}{\pi}} \int_0^\infty \overset{\text{cos series}}{F_c[f(x)]} \cos \lambda x \, d\lambda \\ &= \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} e^{-\lambda} \cos \lambda x \, d\lambda \\ &= \frac{2}{\pi} \int_0^\infty e^{-\lambda} \cos \lambda x \, d\lambda \\ &= \frac{2}{\pi} \left[\frac{1}{1+x^2} \right] \end{aligned}$$

② Find the Fourier Sine transform of e^{-ax} , $a > 0$ and hence deduce the inversion formula.

Soln: let $f(x) = e^{-ax}$

$$\text{WKT } F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$$

$$F_s[e^{-ax}] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin sx \, dx$$

$$F_s[e^{-ax}] = \sqrt{\frac{2}{\pi}} \left[\frac{s}{a^2 + s^2} \right]$$

By Fourier Inversion formula,

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \overset{\text{Sine series}}{F_s[f(x)]} \sin sx \, ds$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \left[\frac{s}{a^2 + s^2} \right] \sin sx \, ds$$

$$f(x) = \frac{2}{\pi} \int_0^\infty \frac{s}{a^2 + s^2} \sin sx \, ds$$

$$\therefore \int_0^\infty e^{-ax} \sin bx \, dx = \frac{b}{a^2 + b^2}$$

$$\therefore \int_0^{\infty} \frac{s}{a^2 + s^2} \sin sx \, ds = \frac{\pi}{2} f(x) \\ = \frac{\pi}{2} e^{-ax} ; a > 0$$

③ If $F_c(f(x)) = F_c(s)$, prove that $F_c(F_c(x)) = f(s)$.
(In the above F_c stands for the Fourier Cosine transform).

Soln:

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \underline{f(x)} \cos sx \, dx = F_c(s)$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_c(s) \cos sx \, ds$$

$$\therefore F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_c(x) \cos sx \, dx \\ = f(s)$$

① Find the Fourier Transform of

$$f(x) = \begin{cases} 1-x & \text{if } |x| < 1 \\ 0 & \text{if } |x| > 1 \end{cases} \quad \text{Hence deduce}$$

$$\text{that } \int_0^\infty \left(\frac{\sin t}{t} \right)^2 dt = \pi/2, \quad \int_0^\infty \left(\frac{\sin t}{t} \right) dt = \pi/3$$

Soln

Using (3) of Sum-over Huygens

or

(ii) using Parseval's identity.

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(s)|^2 ds \quad \rightarrow (1)$$

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-1}^1 [1-|x|]^2 dx$$

$$= 2 \int_0^1 (1-x)^2 dx$$

$$= 2 \int_0^1 (1-x)^2 dx \quad \left[\because |x| = x \text{ in } (0,1) \right]$$

$$= 2 \left[\frac{(1-x)^3}{3(-1)} \right]_0^1$$

$$= -\frac{2}{3} \left[(1-x)^3 \right]_0^1$$

$$= -\frac{2}{3} (0-1) = 2/3 \quad \rightarrow (2)$$

$$|F(s)|^2 = \frac{2}{\pi} \left[\frac{1-\cos s}{s^2} \right]$$

$$= \frac{2}{\pi} \left[\frac{2 \sin^2(s/2)}{s^2} \right]^2 = \frac{8}{\pi} \frac{\sin^4(s/2)}{s^4}$$

$$\int_{-\infty}^{\infty} |F(s)|^2 ds = \frac{8}{\pi} \int_{-\infty}^{\infty} \frac{\sin^4(s/2)}{s^4} ds$$

$$= \frac{16}{\pi} \int_0^{\infty} \frac{\sinh^4(s/2)}{s^4} ds$$

$$= \frac{16}{\pi} \int_0^{\infty} \frac{\sin^4 t}{(2t)^4} 2dt$$

$$= \frac{2}{\pi} \int_0^{\infty} \frac{\sin^4 t}{t^4} dt \rightarrow (3)$$

$$t = s/2$$

$$2t = s \Rightarrow 2dt = ds$$

$$s \rightarrow 0 ; t \rightarrow 0, s \rightarrow \infty \\ \Rightarrow t \rightarrow \infty$$

From (1) $\frac{2}{\pi} \int_0^{\infty} \frac{\sin^4 t}{t^4} dt = 2/3$ by (2) & (3).

$$\int_0^{\infty} \left(\frac{\sin t}{t} \right)^4 dt = \pi/3$$

Hence the proof.