

Unit-I

Random variable

Types of random variable:

There are two types of random variables.

- 1) Discrete random variable
- 2) Continuous " "

Formula :- Discrete random variable

$$1) \sum_i P(x_i) = 1$$

$$2) \text{Var}(ax+by) = a^2 \text{Var}(x) + b^2 \text{Var}(y)$$

$$3) \text{Mean} = E(x) = \sum x_i P(x_i)$$

μ

$$4) E(x^2) = \sum x_i^2 P(x_i)$$

$$5) \text{Variance} = \text{Var}(x) = E(x^2) - (E(x))^2$$

$$6) E(ax+by) = a E(x) + b E(y)$$

$$7) \text{Probability mass function } P(x) = P(X=x)$$

$$8) \text{Standard deviation} = \sqrt{\text{Var}(x)}$$

S.D. σ^2

Problems :-

APR-14

① X & Y are independent random variables with variance 2 & 3. Find the variance of $3x+4y$.

Soln:-

$$\text{Given, } \text{Var}(x) = 2$$

$$\text{Var}(y) = 3$$

$$\text{Var}(ax+by) = a^2 \text{Var}(x) + b^2 \text{Var}(y)$$

$$\text{Var}(3x+4y) = 3^2 \text{Var}(x) + 4^2 \text{Var}(y)$$

$$= 9(2) + 16(3)$$

$$= 18 + 48$$

$$= 66$$

- ② If the range of x is the Set $(0, 1, 2, 3, 4)$ and $P(X=x) = 0.2$ determine the mean & Variance of the random variable.

Soln:

x	0	1	2	3	4
$P(X=x)$	0.2	0.2	0.2	0.2	0.2

$$\text{Mean} = E(X) = \sum x_i P(x_i)$$

$$= 0(0.2) + 1(0.2) + 2(0.2) + 3(0.2)$$

$$+ 4(0.2)$$

$$= 0 + 0.2 + 0.4 + 0.6 + 0.8$$

$$E(X) = 2$$

$$E(X^2) = \sum x_i^2 P(x_i)$$

$$= 0^2(0.2) + 1^2(0.2) + 2^2(0.2) + 3^2(0.2) + 4^2(0.2)$$

$$= 0 + 0.2$$

$$E(X^2) = 6$$

$$\text{Var}(X) = E(X^2) - (E(X))^2$$

$$= 6 - (2)^2$$

$$= 6 - 4$$

$$\text{Var}(X) = 2$$

- ③ A discrete Probability d

x	0
$P(X=x)$	a

(i) Find x_1

(ii) Find P

(iii) Find t

Soln:

(i) Given $\sum_{i=1}^8$

$a +$

(ii) $P(X < 3)$

or

$$= 1 - P(X \geq 3)$$

$$= 1 - \frac{8}{9}$$

$$= \frac{9-8}{9}$$

$$= \frac{1}{9}$$

- ③ A r.v 'x' has the following Probability function $P(0 < x < 3)$

x	0	1	2	3	4
$P(x)$	k	$3k$	$5k$	$7k$	$9k$

Find the values of k .

Soln:

$$\sum p(x_i) = 1$$

$$P(0) + P(1) + P(2) + P(3) + P(4) = 1$$

$$k + 3k + 5k + 7k + 9k = 1$$

$$25k = 1$$

$$k = \frac{1}{25}$$

$P(X \geq 3)$

Nov-11

Q A discrete random variable X has the following probability distribution.

x	0	1	2	3	4	5	6	7	8
$P(X=x)$	a	$3a$	$5a$	$7a$	$9a$	$11a$	$13a$	$15a$	$17a$

(i) Find the value of a .

(ii) Find $P(X < 3)$, $P(0 < X < 3)$, $P(X \geq 3)$

(iii) Find the distribution function of X .

Soln:

(i) Given $\sum_{i=0}^8 P(x_i) = 1$

$$a + 3a + 5a + 7a + 9a + 11a + 13a + 15a + 17a = 1$$

$$81a = 1$$

$$a = \frac{1}{81}$$

x	0	1	2	3
$P(x)$	$\frac{1}{81}$	$\frac{3}{81}$	$\frac{5}{81}$	$\frac{7}{81}$

(ii) $P(X < 3) = P(X=0) + P(X=1) + P(X=2)$

or

$$\begin{aligned}
 &= 1 - P(X \geq 3) \\
 &= 1 - \frac{8}{9} \\
 &= \frac{9-8}{9} \\
 &= \frac{1}{9}
 \end{aligned}$$

$$= a + 3a + 5a$$

$$= 9a$$

$$= 9 \left(\frac{1}{81} \right)$$

$$= \frac{1}{9}$$

4	5	6	7	8
$\frac{9}{81}$	$\frac{11}{81}$	$\frac{13}{81}$	$\frac{15}{81}$	$\frac{17}{81}$

b) $P(0 < X < 3) = P(X=1) + P(X=2)$

$$= 3a + 5a$$

$$= 8a$$

$$= \frac{8}{81}$$

c) $P(X \geq 3) = 1 - P(X < 3)$

$$= 1 - \frac{1}{9}$$

$$= \frac{9-1}{9} = \frac{8}{9}$$

or

$$\begin{aligned}
 &= P(3) + P(4) + P(5) + P(6) + P(7) + P(8) \\
 &= \frac{7}{81} + \frac{9}{81} + \frac{11}{81} + \frac{13}{81} + \frac{15}{81} + \frac{17}{81} \\
 &= \frac{72}{81} = \frac{8}{9}
 \end{aligned}$$

(iii) Distribution Function :

x_i	$P(x_i)$	$F(x_i)$
0	$P(0) = \frac{1}{81}$	$F(0) = P(0) = \frac{1}{81}$
1	$P(1) = \frac{3}{81}$	$F(1) = F(0) + P(1) = \frac{1}{81} + \frac{3}{81} = \frac{4}{81}$
2	$P(2) = \frac{5}{81}$	$F(2) = F(1) + P(2) = \frac{4}{81} + \frac{5}{81} = \frac{9}{81}$
3	$P(3) = \frac{7}{81}$	$F(3) = F(2) + P(3) = \frac{9}{81} + \frac{7}{81} = \frac{16}{81}$
4	$P(4) = \frac{9}{81}$	$F(4) = F(3) + P(4) = \frac{16}{81} + \frac{9}{81} = \frac{25}{81}$
5	$P(5) = \frac{11}{81}$	$F(5) = F(4) + P(5) = \frac{25}{81} + \frac{11}{81} = \frac{36}{81}$
6	$P(6) = \frac{13}{81}$	$F(6) = F(5) + P(6) = \frac{36}{81} + \frac{13}{81} = \frac{49}{81}$
7	$P(7) = \frac{15}{81}$	$F(7) = F(6) + P(7) = \frac{49}{81} + \frac{15}{81} = \frac{64}{81}$
8	$P(8) = \frac{17}{81}$	$F(8) = F(7) + P(8) = \frac{64}{81} + \frac{17}{81} = \frac{81}{81} = 1$

⑤ A r.v. x has the following Probability function

$x = x_i$	0	1	2	3	4	5	6	7
$P(x = x_i)$	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2 + k$

a) Find k

b) Evaluate $P[x < 6]$, $P[x \geq 6]$

c) If $P[x \leq c] > \frac{1}{9}$, then find the minimum value of c .

d) Find $P[x < 2]$, $P[x > 3]$, $P[1 < x < 5]$

$$\textcircled{a} \quad \sum_i P(x_i) = 1$$

$$0 + k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 1$$

$$(10k^2 + 9k - 1 = 0)$$

$$k = -1, \quad k = \frac{1}{10}$$



-10
 $\wedge$
 10 -1 $\Rightarrow$ (9)
 $\frac{10}{10}, -\frac{1}{10}$
 opposite sign
 $-1, -\frac{1}{10}$

X	0	1	2	3	4	5	6
P(X)	0	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$(\frac{1}{10})^2 = \frac{1}{100}$	$\frac{2}{100}$

$$\textcircled{b} \text{ (i) } P[X \geq 6] = P[X=6] + P[X=7]$$

$$= \frac{2}{100} + \frac{17}{100}$$

$$= \frac{19}{100}$$

$$\begin{aligned} & \frac{7}{100} + \frac{1}{100} \\ &= \frac{7+1}{100} \\ &= \frac{8}{100} \end{aligned}$$

$$\textcircled{ii} \quad P[X < 6] = 1 - P[X \geq 6]$$

$$= 1 - \frac{19}{100}$$

$$= \frac{100-19}{100}$$

$$= \frac{81}{100}$$

$= 1, -2, -3, \dots, -50, 1$

or

$$P[X < 6] = P(0) + P(1) + P(2) + P(3) + P(4) + P(5)$$

$$= 0 + \frac{1}{10} + \frac{2}{10} + \frac{2}{10} + \frac{3}{10} + \frac{1}{100}$$

$$= \frac{8}{10} + \frac{1}{100}$$

$$= \frac{80+1}{100}$$

$$= \frac{81}{100}$$

c

x_i	$p(x_i)$	$F(x_i)$
0	0	$F(0) = P(0) = 0$
1	$\frac{1}{10}$	$F(1) = F(0) + P(1) = 0 + \frac{1}{10} = \frac{1}{10} < \frac{1}{2}$
2	$\frac{2}{10}$	$F(2) = F(1) + P(2) = \frac{1}{10} + \frac{2}{10} = \frac{3}{10} < \frac{1}{2}$
3	$\frac{2}{10}$	$F(3) = F(2) + P(3) = \frac{3}{10} + \frac{2}{10} = \frac{5}{10} = \frac{1}{2}$
4	$\frac{3}{10}$	$F(4) = F(3) + P(4) = \frac{5}{10} + \frac{3}{10} = \frac{8}{10} > \frac{1}{2}$
5	$\frac{1}{100}$	$F(5) = F(4) + P(5) = \frac{8}{10} + \frac{1}{100} = \frac{81}{100} > \frac{1}{2}$
6	$\frac{2}{100}$	$F(6) = F(5) + P(6) = \frac{81}{100} + \frac{2}{100} = \frac{83}{100} > \frac{1}{2}$
7	$\frac{17}{100}$	$F(7) = F(6) + P(7) = \frac{83}{100} + \frac{17}{100} = \frac{100}{100} = 1 > \frac{1}{2}$

∴ The minimum value of $c = 4$

$$\because P[X \leq c] > \frac{1}{2}$$

(i) $P(X < 2) = P(0) + P(1)$
 $= 0 + \frac{1}{10}$
 $= \frac{1}{10}$

(ii) $P(X > 3) = 1 - P(X \leq 3)$
 $= 1 - (P(0) + P(1) + P(2) + P(3))$
 $= \frac{3}{10} + \frac{1}{100} + \frac{2}{100} + \frac{17}{100}$
 $= \frac{3}{10} + \frac{20}{100}$
 $= \frac{30+20}{100}$
 $= \frac{50}{100}$
 $= \frac{1}{2}$

(iii) $P(1 < X < 5) = P(2) + P(3) + P(4)$
 $= \frac{2}{10} + \frac{2}{10} + \frac{3}{10}$
 $= \frac{7}{10}$

Discrete	
x_i	p_i
1	
2	
3	
4	

(6) If the r.v. X takes the values 1, 2, 3 & 4 such that $2P(X=1) = 3P(X=2) = P(X=3) = 5P(X=4)$. Find the probability distribution and cumulative distribution function of X .

Soln

X is a d.r.v

Given $2P(X=1) = 3P(X=2) = P(X=3) = 5P(X=4) = K$

X	1	2	3	4
$P(X)$	$\frac{K}{2}$	$\frac{K}{3}$	K	$\frac{K}{5}$

$$\sum_{i=1}^4 P(x_i) = 1$$

$$\frac{K}{2} + \frac{K}{3} + K + \frac{K}{5} = 1$$

$$\frac{15K + 10K + 30K + 6K}{30} = 1$$

$$\frac{61K}{30} = 1$$

$$61K = 30$$

$$K = \frac{30}{61}$$

$$2P(1) = K$$

$$P(1) = \frac{K}{2}$$

$$3P(2) = K$$

$$P(2) = \frac{K}{3}$$

$$1P(3) = K$$

$$P(3) = \frac{K}{1}$$

$$5P(4) = K$$

$$P(4) = \frac{K}{5}$$

Distribution function

x_i	$P(x_i)$	$F(x_i)$
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1	$\frac{15}{61}$	$F(1) = P(1) = \frac{15}{61}$
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2	$\frac{10}{61}$	$F(2) = F(1) + P(2) = \frac{15}{61} + \frac{10}{61} = \frac{25}{61}$
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3	$\frac{30}{61}$	$F(3) = F(2) + P(3) = \frac{25}{61} + \frac{30}{61} = \frac{55}{61}$
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4	$\frac{6}{61}$	$F(4) = F(3) + P(4) = \frac{55}{61} + \frac{6}{61} = \frac{61}{61} = 1$
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1.5) Continuous random variables

Defn:

A r.v X which takes all possible values in a given interval is called a c.r.v.
Eg:

Age, height, weight

Formula:-

$$\textcircled{1} \int_{-\infty}^{\infty} f(x) dx = 1$$

$$\textcircled{2} \text{ Mean} = \mu = E(X) = \int_{-\infty}^{\infty} x f(x) dx$$

$$\textcircled{3} E(X^2) = \int_{-\infty}^{\infty} x^2 f(x) dx$$

$$\textcircled{4} \text{ Variance} = \text{Var}(X) = E(X^2) - (E(X))^2$$

$$\textcircled{5} P[a \leq x \leq b] = F(b) - F(a)$$

$$b) 0 \leq F(x) \leq 1$$

$\textcircled{1}$ A c.r.v X that can assume any value between $x=2$ & $x=5$ has a density function given by $f(x) = k(1+x)$. Find $P[X < 4]$.

Soln:-

we know that $\int_{-\infty}^{\infty} f(x) dx = 1$ Given $f(x) = k(1+x)$

$$\int_{-\infty}^{\infty} f(x) dx = 1$$

$$\int_2^5 k(1+x) dx = 1$$

$$k \int_2^5 (1+x) dx = 1$$

$$k \left[x + \frac{x^2}{2} \right]_2^5 = 1$$

$$k \left[\left(\frac{5}{1} + \frac{25}{2} \right) - \left(\frac{2}{1} + \frac{4}{2} \right) \right] = 1$$

$$k \left[\left(\frac{10+25}{2} \right) - \left(\frac{4+4}{2} \right) \right] = 1$$

$$P[X < 4] = \int_2^4 f(x) dx$$

$$= \int_2^4 k(1+x) dx$$

$$= \int_2^4 \left(\frac{2}{27} \right) (1+x) dx$$

$$= \frac{2}{27} \int_2^4 (1+x) dx$$

$$= \frac{2}{27} \left[x + \frac{x^2}{2} \right]_2^4$$

$$k \left[\frac{35}{2} - \frac{8}{2} \right] = 1$$

$$k \left[\frac{27}{2} \right] = 1$$

$$\boxed{k = \frac{2}{27}}$$

$$\begin{aligned} &= \frac{2}{27} \left[\left(4 + \frac{16}{2} \right) - \left(2 + \frac{2^2}{2} \right) \right] \\ &= \frac{2}{27} [(4+8) - (2+2)] \\ &= \frac{2}{27} [12-4] \\ &= \frac{2}{27} (8) \\ &= \frac{16}{27} \end{aligned}$$

(2) Find the value of c given the pdf of a r.v

$$X \text{ as } f(x) = \begin{cases} c/x^3, & \text{if } 1 < x < \infty \\ 0, & \text{otherwise} \end{cases}$$

Soln:

$$\text{WKT } \int_{-\infty}^{\infty} f(x) dx = 1$$

$$\text{if } x^n = n x^{n-1}$$

$$\int x^n dx = \frac{x^{n+1}}{n+1}$$

$$\int_1^{\infty} \frac{c}{x^3} dx = 1$$

$$c \int_1^{\infty} \left(\frac{1}{x^3} \right) dx = 1$$

$$c \int_1^{\infty} x^{-3} dx = 1$$

$$c \left[\frac{x^{-3+1}}{-3+1} \right]_1^{\infty} = 1$$

$$c \left[\frac{x^{-2}}{-2} \right]_1^{\infty} = 1$$

$$\frac{c}{-2} \left[(x)^{-2} - (1)^{-2} \right]$$

$$\frac{c}{-2} \left[\frac{1}{x^2} \right]_1^{\infty} = 1$$

$$\frac{c}{-2} \left[\frac{1}{\infty} - \frac{1}{1^2} \right] = 1$$

$$\frac{c}{-2} (0-1) = 1$$

$$\frac{c}{-2} \times -1 = 1 \Rightarrow \boxed{c = 2}$$

probability density function

Nov-17

③ Let X be a C.R.V. with the P.d.f
 $f(x) = \frac{1}{4} ; 2 \leq x \leq 6$. Find the expected value and variance of X .

④ Show that it is a P.d.

Soln: ∴

$$\text{WRT } \int_{-\infty}^{\infty} f(x) dx$$

Soln: ∴

Given $f(x) = \frac{1}{4} ; 2 \leq x \leq 6$

$$\begin{aligned} E(X) &= \int_2^6 x f(x) dx \\ &= \int_2^6 x \left(\frac{1}{4}\right) dx \\ &= \frac{1}{4} \int_2^6 x dx \\ &= \frac{1}{4} \left[\frac{x^2}{2} \right]_2^6 \\ &= \frac{1}{8} (36 - 4) \\ &= \frac{1}{8} (32) \end{aligned}$$

$$\boxed{E(X) = 4}$$

$$\begin{aligned} E(X^2) &= \int_2^6 x^2 f(x) dx \\ &= \int_2^6 x^2 \left(\frac{1}{4}\right) dx \\ &= \frac{1}{4} \int_2^6 x^2 dx \\ &= \frac{1}{4} \left[\frac{x^3}{3} \right]_2^6 \\ &= \frac{1}{12} [216 - 8] \\ &= \frac{1}{12} [208] \end{aligned}$$

$$\begin{aligned} 6^3 &= 6 \times 6 \times 6 \\ 3 &= \frac{36 \times 6}{216} \end{aligned}$$

$$\boxed{E(X^2) = \frac{52}{3}}$$

May-201

⑤ If X is then find (

Soln: ∴

$$\begin{aligned} \textcircled{1} \quad & \int_{-1}^1 \dots \\ & \int_{-1}^1 k \\ & k \left[\frac{x}{1} \right] \\ & \frac{k}{3} \\ & \frac{k}{3} \end{aligned}$$

$$E(X^2) =$$

$$\begin{aligned} &= \int_{-1}^1 \\ &= k \end{aligned}$$

$$\begin{aligned} \text{Var}(X) &= E(X^2) - [E(X)]^2 \\ &= \frac{52}{3} - (4)^2 \\ &= \frac{52}{3} - \frac{16}{1} \\ &= \frac{52 - 48}{3} \end{aligned}$$

$$\boxed{\text{Var}(X) = \frac{4}{3}}$$

4) Show that the function $f(x) = \begin{cases} \frac{x^2}{3}, & -1 < x < 2 \\ 0, & \text{otherwise} \end{cases}$ is a p.d.f

Soln:.

$$\text{WKT } \int_{-\infty}^{\infty} f(x) dx = \int_{-1}^2 \frac{x^2}{3} dx$$

$$= \frac{1}{3} \left[\frac{x^3}{3} \right]_{-1}^2 = \frac{1}{9} [8 - (-1)]$$

$$= \frac{1}{9} (8+1) = \frac{9}{9} = 1$$

May-2011

5) If X is a c.r.v with p.d.f $f(x) = \begin{cases} Kx^2, & -1 < x < 1 \\ 0, & \text{elsewhere} \end{cases}$ then find

- ① The value of K
- ② The mean & variance of X
- ③ $P(\frac{1}{3} \leq X < 4)$

Soln:.

$$\textcircled{1} \int_{-1}^1 f(x) dx = 1$$

$$\int_{-1}^1 Kx^2 dx = 1$$

$$K \left[\frac{x^3}{3} \right]_{-1}^1 = 1$$

$$\frac{K}{3} [1 - (-1)] = 1$$

$$\frac{K}{3} (2) = 1$$

$$\boxed{K = \frac{3}{2}}$$

$$\textcircled{2} \text{ Mean } = E(X) = \int_{-1}^1 x f(x) dx$$

$$= \int_{-1}^1 x Kx^2 dx$$

$$= K \int_{-1}^1 x^3 dx$$

$$= K \left[\frac{x^4}{4} \right]_{-1}^1$$

$$= K/4 (1-1)$$

$$\boxed{E(X) = 0}$$

$$E(X^2) = \int_{-1}^1 x^2 f(x) dx$$

$$= \int_{-1}^1 x^2 Kx^2 dx$$

$$= K \int_{-1}^1 x^4 dx$$

$$= K \left[\frac{x^5}{5} \right]_{-1}^1$$

$$= \frac{K}{5} (1 - (-1))$$

$$E(X^2) = \frac{K}{5} (2)$$

$$= \frac{2K}{5}$$

$$= \frac{2}{5} \times \frac{3}{2}$$

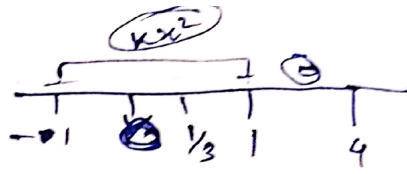
$$\boxed{E(X^2) = \frac{3}{5}}$$

$$\text{Var}(X) = E(X^2) - [E(X)]^2$$

$$= \frac{3}{5} - 0$$

$$\boxed{\text{Var}(X) = \frac{3}{5}}$$

$$(3) P\left(\frac{1}{3} \leq x < 4\right)$$



$$= \int_{1/3}^1 f(x) dx + \int_1^4 f(x) dx$$

$$= \int_{1/3}^1 kx^2 dx + 0$$

$$= k \left[\frac{x^3}{3} \right]_{1/3}^1$$

$$= \frac{k}{3} \left[1 - \left(\frac{1}{3}\right)^3 \right]$$

$$= \frac{k}{3} \left(1 - \frac{1}{27} \right) = \frac{k}{3} \left(\frac{27-1}{27} \right)$$

$$= \left(\frac{3}{27} \times \frac{1}{27} \right) \left(\frac{26}{27} \right)$$

$$= \frac{13}{27}$$

1.6) Moments - Moment Generating Functions

Binomial Expansions:-

$$(1-x)^{-1} = 1+x+x^2+x^3+\dots$$

$$(1+x)^{-1} = 1-x+x^2-x^3+\dots$$

$$(1-x)^{-2} = 1+2x+3x^2+4x^3+\dots$$

$$(1+x)^{-2} = 1-2x+3x^2-4x^3+\dots$$

M.G.F ∴

$$M_X(t) = E(e^{tx})$$

$$= \begin{cases} \int_{-\infty}^{\infty} e^{tx} f(x) dx, & \text{for a c.p.d} \\ \sum_{x=-\infty}^{\infty} e^{tx} P(x), & \text{for a d.p.d} \end{cases}$$

Problems

① Find the M.G.F for the distribution where

$$f(x) = \begin{cases} \frac{2}{3} & \text{at } x=1 \\ \frac{1}{3} & \text{at } x=2 \\ 0 & \text{otherwise} \end{cases}$$

Soln ∴

$$M_X(t) = E(e^{tx}) = \sum_{x=0}^{\infty} e^{tx} P(x)$$

$$= 0 + e^t \left(\frac{2}{3}\right) + e^{2t} \left(\frac{1}{3}\right) + 0 + \dots$$

$$= \frac{2}{3} e^t + \frac{1}{3} e^{2t}$$

$$= \frac{e^t}{3} [2 + e^t]$$

② If a R.V X has the MGF, $M_X(t) = \frac{3}{3-t}$ compute $E(X^2)$ & S.D of X .

Soln:

$$M_X(t) = \frac{3}{3-t} = \frac{3^1}{3(1-\frac{t}{3})} = \frac{1}{(1-\frac{t}{3})} = (1-\frac{t}{3})^{-1}$$

$$(1-x)^{-1} = 1+x+x^2+x^3+\dots$$

$$= 1 + \left(\frac{t}{3}\right) + \left(\frac{t}{3}\right)^2 + \dots$$

$$= 1 + \frac{t}{1!} \left(\frac{1}{3}\right) + \frac{t^2}{2!} \left(\frac{2}{9}\right) + \dots$$

$$E(X) = \frac{1}{3}$$

$$E(X^2) = \text{coefficient of } \frac{t^2}{2!}$$

$$= \frac{2}{9}$$

$$\text{Var}(X) = E(X^2) - (E(X))^2$$

$$= \frac{2}{9} - \frac{1}{9} = \frac{1}{9}$$

$$\text{S.D} = \sqrt{\text{Var}(X)} = \sqrt{\frac{1}{9}} = \frac{1}{3}$$

③ If the R.V X has the MGF, $M_X(t) = \frac{2}{2-t}$ find the variance of X .

Soln:

$$M_X(t) = \frac{2}{2-t} = \frac{2^1}{2(1-\frac{t}{2})} = \frac{1}{(1-\frac{t}{2})} = (1-\frac{t}{2})^{-1}$$

$$(1-x)^{-1} = 1+x+x^2+\dots$$

$$= 1 + \left(\frac{t}{2}\right) + \left(\frac{t}{2}\right)^2 + \dots$$

$$= 1 + \frac{t}{1!} \left(\frac{1}{2}\right) + \frac{t^2}{2!} \left(\frac{2}{4}\right) + \dots$$

$$E(X) = \frac{1}{2}$$

$$E(X^2) = \frac{2}{4} = \frac{1}{2}$$

$$\text{Var}(X) = E(X^2) - (E(X))^2$$

$$= \frac{1}{2} - \left(\frac{1}{2}\right)^2$$

$$= \frac{1}{2} - \frac{1}{4}$$

$$= \frac{4-2}{8} = \frac{2}{8} = \frac{1}{4}$$

③ Find the M.O.F of a r.v X having the Probability p.d.f $f(x) = \begin{cases} \frac{x}{4} e^{-x/2} & ; x > 0 \\ 0 & , \text{elsewhere} \end{cases}$

Also deduce the first four Moments about the origin.

$$\text{Given } f(x) = \frac{x}{4} e^{-x/2} ; x > 0$$

$$M_X(t) = E(e^{tx})$$

$$= \int_0^{\infty} e^{tx} f(x) dx$$

$$= \int_0^{\infty} e^{tx} \frac{x}{4} e^{-x/2} dx$$

$$= \frac{1}{4} \int_0^{\infty} x e^{tx - x/2} dx$$

$$= \frac{1}{4} \int_0^{\infty} x e^{-(\frac{1}{2} - t)x} dx$$

$$= \frac{1}{4} \frac{1!}{(\frac{1}{2} - t)^2}$$

$$\therefore \int_0^{\infty} x^n e^{-ax} dx = \frac{n!}{a^{n+1}}$$

$$= \frac{1}{4} \frac{1!}{\left(\frac{1-2t}{2}\right)^2}$$

$$= \frac{1}{4} \frac{4}{(1-2t)^2}$$

$$= \frac{1}{(1-2t)^2}$$

$$= (1-2t)^{-2}$$

$$\therefore (1-x)^{-2} = 1 + 2x + 3x^2 + \dots$$

$$= 1 + 2(2t) + 3(2t)^2 + 4(2t)^3 + 5(2t)^4 + \dots$$

$$= 1 + 4t + 12t^2 + 32t^3 + 80t^4 + \dots$$

$$= 1 + \frac{t}{1!}(4) + \frac{t^2}{2!}(24) + \frac{t^3}{3!}(192) + \frac{t^4}{4!}(1920) + \dots$$

$$\therefore \mu'_1 = 4; \mu'_2 = 24, \mu'_3 = 192, \mu'_4 = 1920$$

④ A r.v. x has pdf

$$f_X(x) = \begin{cases} kx^2 e^{-x} & ; x > 0 \\ 0 & ; \text{otherwise} \end{cases}$$

Find the r th moment of x about origin.

Hence, find the mean & variance.

Soln:

$$\text{wkt } \int_{-\infty}^{\infty} f(x) dx = 1$$

$$\therefore \int_0^{\infty} x^n e^{-ax} dx = \frac{n!}{a^{n+1}}$$

$$\int_0^{\infty} kx^2 e^{-x} dx = 1$$

$$\frac{k(2!)}{(1)^3} = 1$$

$$2k = 1 \Rightarrow k = \frac{1}{2}$$

$$\begin{aligned}
 \mu'_r &= E(x^r) = \int_0^{\infty} x^r f(x) dx \\
 &= \int_0^{\infty} x^r \frac{1}{2} x^2 e^{-x} dx \\
 &= \frac{1}{2} \int_0^{\infty} x^{r+2} e^{-x} dx
 \end{aligned}$$

$$\mu'_r = \frac{1}{2} (r+2)!$$

$$\because \int_0^{\infty} x^n e^{-x} dx = \frac{n!}{a^{n+1}}$$

Put $r=1 \Rightarrow \mu'_1 = \frac{1}{2} (1+2)! = \frac{1}{2} (3!) = \frac{1}{2} (1 \times 2 \times 3)$

$$\text{Mean, } \mu'_1 = 3$$

Put $r=2 \Rightarrow \mu'_2 = \frac{1}{2} (2+2)! = \frac{1}{2} (4!) = \frac{1}{2} (1 \times 2 \times 3 \times 4)$
 $= 12$

$$\text{variance} = \mu'_2 - (\mu'_1)^2$$

$$= 12 - (3)^2$$

$$= 12 - 9$$

$$\text{variance} = 3$$

Discrete Distributions

- * Binomial distribution
- * Poisson
- * Geometric

③ If a f.c.
 $P(X \leq 1)$,
of heads

Soln:
Given

$P(X$

⑦ Binomial Distributions

$$P(X=x) = {}^n C_x p^x q^{n-x} ; x=0,1,2,\dots,n, \quad P[X \leq 1] \\ p+q=1$$

Binomial			
MGF	Mean	Variance	S.D
$q + pe^t$	np	npq	$\sqrt{npq}$

④ The n
distrib
distrib

Problems

① Check whether the following data follow a binomial distribution or not. Mean = 3, Variance

Soln:

Soln:

Given Mean = $np = 3 \rightarrow$ ①

Variance = $npq = 4 \rightarrow$ ②

$$\frac{②}{①} \Rightarrow \frac{npq}{np} = \frac{4}{3} \Rightarrow 1 \cdot \frac{1}{3}$$

$$q = 1/3 > 1$$

$\therefore q > 1$ which is not possible $0 < q < 1$

② Under what conditions binomial distribution tends to Poisson distribution?

Soln:

$n \rightarrow \infty, p \rightarrow 0, \lambda = np$ is finite

③
①

③ If a fair coin is tossed twice, find $P(X \leq 1)$, where X denotes the number of heads in each experiment

Soln:

$$P = \frac{2}{4} = \frac{1}{2} \rightarrow \text{single time} \\ \text{Given } P = \frac{1}{2}; \quad q = \frac{1}{2}, \quad n = 2$$

$$p + q = 1 \\ q = 1 - p \\ q = 1 - \frac{1}{2} \\ q = \frac{1}{2}$$

$$P(X=x) = {}^n C_x p^x q^{n-x}$$

$$P[X \leq 1] = P[X=0] + P[X=1]$$

$$= \frac{{}^2 C_0}{\textcircled{1}} \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^2 + \frac{{}^2 C_1}{\textcircled{1}} \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^1$$

$$\begin{aligned} {}^2 C_0 &= 1 \\ {}^2 C_1 &= \frac{2}{1} = 2 \end{aligned}$$

$$= \frac{1}{4} + 2 \times \frac{1}{2} \times \frac{1}{2}$$

$$= \frac{1}{4} + \frac{1}{2} = \frac{2+4}{8} = \frac{6}{8} = \frac{3}{4}$$

④ The Mean & Standard deviation of binomial distribution are 5 & 2. Determine the distribution.

Soln:

$$\text{Mean} = np = 5 \rightarrow \textcircled{1}$$

$$\text{S.D} = \sqrt{npq} = 2 \rightarrow \textcircled{2}$$

$$(npq)^{\frac{1}{2} \times 2} = 2^2$$

$$npq = 4 \rightarrow \textcircled{3}$$

$$\frac{\textcircled{3}}{\textcircled{1}} \Rightarrow \frac{npq}{np} = \frac{4}{5}$$

$$q = \frac{4}{5}$$

$$p = 1 - q \\ = 1 - \frac{4}{5}$$

$$= \frac{5-4}{5}$$

$$p = \frac{1}{5}$$

$$\sqrt{\quad} = \frac{1}{2}$$

Sub ①,

$$np = 5$$

$$n\left(\frac{1}{5}\right) = 5$$

$$n = 25$$

$$P(X=x) = {}^n C_x p^x q^{n-x} \\ = {}^{25} C_x \left(\frac{1}{5}\right)^x \left(\frac{4}{5}\right)^{25-x}$$

⑤ In 256 sets of twelve tosses of a coin how many cases may one expect eight heads and four tails?

Soln:.

$$N = 256 \text{ (sets)}$$

$$n = 12 \text{ (tosses)}$$

Head	Tails	Total
8	4	12

$$X = 8, x = 4$$

$$P = \frac{1}{2} \begin{matrix} \rightarrow \text{single time} \\ \rightarrow \text{Head tail} \end{matrix}$$

$$q = 1 - P$$

$$= 1 - \frac{1}{2}$$

$$q = \frac{1}{2}$$

$$P(X=x) = nC_x p^x q^{n-x}$$

$$P(x=8) = N \cdot [nC_x p^x q^{n-x}]$$

$$= 256 \left[{}^{12}C_8 \left(\frac{1}{2}\right)^8 \left(\frac{1}{2}\right)^{12-8} \right]$$

$$256 \left[495 \left(\frac{1}{256} \right) \right] \frac{1}{16}$$

$$= \frac{495}{16}$$

$$= 30.937 \text{ times}$$

$$P(x=8) \approx 31 \text{ times}$$

$${}^{12}C_8 = \frac{12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5}{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8}$$

$$= 11 \times 5 \times 9$$

$$= 495$$

$$256 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$$

$$2^8 = 256$$

$$2^8 = 256$$

6) 6 dice are thrown 729 times. How many times do you expect atleast three dice to show 5 or 6?

$$P = \frac{2}{6} = \frac{1}{3} \quad \left| \quad \begin{aligned} q &= 1 - P \\ &= 1 - \frac{1}{3} \\ q &= \frac{2}{3} \end{aligned} \right| \quad n = 6$$

$$P(X=x) = {}^n C_x p^x q^{n-x} \quad ; x=0,1,2,\dots,6$$

$$P(X=x) = {}^n C_x p^x q^{n-x}$$

$$P(X \geq 3) = P(X=3) + P(X=4) + P(X=5) + P(X=6)$$

$$= 729 \left[{}^6 C_3 \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^3 + {}^6 C_4 \left(\frac{1}{3}\right)^4 \left(\frac{2}{3}\right)^2 \right.$$

$$+ {}^6 C_5 \left(\frac{1}{3}\right)^5 \left(\frac{2}{3}\right) + {}^6 C_6 \left(\frac{1}{3}\right)^6 \left(\frac{2}{3}\right)^0 \left. \right]$$

$$= 729 \left[\frac{160}{729} + \frac{60}{729} + \frac{12}{729} + \frac{1}{729} \right]$$

$$P(X \geq 3) = \frac{233}{729} \times 729 = 233 \text{ times}$$

In 729 times, when 6 dice are thrown

$$= \frac{729 \times 233}{729}$$

$$= 233 \text{ times}$$

$${}^6 C_3 = \frac{6 \times 5 \times 4}{1 \times 2 \times 3} = 5 \times 4 = 20$$

$${}^6 C_4 = \frac{6 \times 5 \times 4 \times 3}{1 \times 2 \times 3 \times 4} = 5 \times 3 = 15$$

$${}^6 C_5 = \frac{6 \times 5 \times 4 \times 3 \times 2}{1 \times 2 \times 3 \times 4 \times 5} = 6$$

$${}^6 C_6 = \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{1 \times 2 \times 3 \times 4 \times 5 \times 6} = 1$$

⑦ 10 coins are thrown simultaneously. Find the probability of getting atleast 7 heads.

Soln

Given $p = \frac{1}{2}$; $q = \frac{1}{2}$; $n = 10$

$$P(X=x) = {}^n C_x p^x q^{n-x}$$

$$P(X \geq 7) = P(X=7) + P(X=8) + P(X=9) + P(X=10)$$

$$= {}^{10}C_7 \left(\frac{1}{2}\right)^7 \left(\frac{1}{2}\right)^3 + {}^{10}C_8 \left(\frac{1}{2}\right)^8 \left(\frac{1}{2}\right)^2$$

$$+ {}^{10}C_9 \left(\frac{1}{2}\right)^9 \left(\frac{1}{2}\right) + {}^{10}C_{10} \left(\frac{1}{2}\right)^{10}$$

$$= \left(\frac{1}{2}\right)^{10} [{}^{10}C_7 + {}^{10}C_8 + {}^{10}C_9 + {}^{10}C_{10}]$$

$$= \frac{1}{2^{10}} [120 + 45 + 10 + 1]$$

$$= \frac{176}{1024}$$

$$= 0.1718$$

$${}^{10}C_7 = \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4}{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7}$$

$$= 10 \times 3 \times 4 = 120$$

$${}^{10}C_8 = \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3}{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8}$$

$$= 5 \times 9 = 45$$

$${}^{10}C_{10} = \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10}$$

$$\boxed{{}^{10}C_{10} = 1}$$

$${}^{10}C_9 = \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9}$$

$$= 10$$

Poisson Distribution

$$P(X=x) = \frac{e^{-\lambda} \lambda^x}{x!}; \quad x=0,1,2,\dots,n; \quad \lambda=np$$

Poisson			
M.G.F	Mean	Variance	S.D
$e^{\lambda(e^t-1)}$	λ	λ	$\sqrt{\lambda}$

- ① If 3% of the electric bulbs manufactured by a company are defective, find the probability that in a sample of 100 bulbs exactly 5 bulbs are defective ($e^{-3} = 0.0498$).

Soln:-

Given $p = 3\% = \frac{3}{100}$; $n=100$; $\lambda=np$

$\lambda = 3$

$\lambda = 3$

$$P(X=x) = \frac{e^{-\lambda} \lambda^x}{x!}$$

$$P(X=5) = \frac{e^{-3} 3^5}{5!} = \frac{0.0498 \times 243}{120} = 0.1008$$

$$3^5 = \frac{3 \times 3 \times 3 \times 3 \times 3}{1 \times 2 \times 3 \times 4 \times 5} = 120$$

- ② If X is Poisson variate $P(X=2) = 9P(X=4) + 90P(X=6)$ find Mean & Variance of X .

Soln:-

$$P(X=x) = \frac{e^{-\lambda} \lambda^x}{x!}; \quad x=0,1,2,\dots,n$$

$$P(X=2) = 9P(X=4) + 90P(X=6)$$

$$\frac{e^{-\lambda} \lambda^2}{2!} = 9 \frac{e^{-\lambda} \lambda^4}{4!} + 90 \frac{e^{-\lambda} \lambda^6}{6!}$$

$$\frac{e^{-\lambda} \lambda^2}{2!} = e^{-\lambda} \lambda^2 \left(\frac{9 \lambda^2}{4!} + \frac{90 \lambda^4}{6!} \right)$$

$$\frac{1}{2} = \frac{3 \lambda^2}{1 \times 2 \times 3 \times 4} + \frac{90 \lambda^4}{1 \times 2 \times 3 \times 4 \times 5 \times 6}$$

$$4! = 1 \times 2 \times 3 \times 4$$

$$\frac{1}{2} = \frac{3\lambda^2}{8} + \frac{\lambda^4}{8}$$

$$\frac{1}{2} = \frac{3\lambda^2 + \lambda^4}{8}$$

$$\lambda^4 + 3\lambda^2 = 4$$

$$\lambda^4 + 3\lambda^2 - 4 = 0$$

$$\lambda = 1, -1, \pm 2i$$

$$\text{Mean} = \text{Variance} = \lambda = 1$$

$$S.D = \sqrt{\lambda} = \sqrt{1} = 1$$

$$\begin{array}{r|rrrrr} 1 & 1 & 0 & 3 & 0 & -4 \\ & 0 & 1 & 1 & 4 & 4 \\ \hline -1 & 1 & 1 & 4 & 4 & 0 \\ & 0 & -1 & 0 & -4 & \\ \hline -4 & 1 & 0 & 4 & 0 & \\ & 0 & -4 & & & \end{array}$$

$$\lambda = 1, -1 \quad \lambda^2 = -4$$

$$\lambda = \pm 2i$$

③ The moment generating function of a r.v X is given by $M_X(t) = e^3(e^t - 1)$. Find $P(X=1)$

Soln: Given $M_X(t) = e^3(e^t - 1)$

M.G.F

$$M_X(t) = e^\lambda(e^t - 1)$$

$$\boxed{\lambda = 3}$$

$$P(X=x) = \frac{e^{-\lambda} \lambda^x}{x!}$$

$$P(X=1) = \frac{e^{-3} 3^1}{1!} = 0.0498 \times 3 = 0.1494$$

④ In a book of 520 pages, 390 typographical errors occur. Assuming Poisson's law for the number of errors per page, find the probability that a random sample of 5 pages will contain no errors.

Soln:

$$\lambda = \frac{\text{Errors}}{\text{Pages}} = \frac{390}{520} = 0.75$$

$$\boxed{\lambda = 0.75}$$

$$P(X=x) = \frac{e^{-\lambda} \lambda^x}{x!}$$

$$\begin{aligned} \therefore P[5 \text{ pages contain no errors}] &= P[X=0]^5 \\ &= \left[\frac{e^{-0.75} (0.75)^0}{0!} \right]^5 = (e^{-0.75})^5 \\ &= e^{-3.75} \\ &= 0.0235 \end{aligned}$$

⑤ If 2% electric bulbs produced by a company are defective, find the probability that in a sample of 200 bulbs exactly 4 are defective.

Soln

$$P = 2\% = \frac{2}{100}; \quad X = 4; \quad n = 200, \quad \lambda = np$$

$$P(X=x) = \frac{e^{-\lambda} \lambda^x}{x!}$$

$$= 200 \times \frac{2}{100}$$

$$\lambda = 4$$

$$P(X=4) = \frac{e^{-4} 4^4}{4!} = \frac{e^{-4} \times 4 \times 4 \times 4 \times 4}{1 \times 2 \times 3 \times 4}$$

$$e^{-4} = 0.0183$$

$$= \frac{32}{3} e^{-4} = 0.1952$$

APR-15
MAY-16
APR-18
MAY-17

⑥ messages arrive at a switch board in a Poisson manner at an average rate of six per hour. Find the probability for each of the following events.

① exactly two messages arrive with in one hour

② No message arrives with in one hour.

③ at least three messages arrives with one hour.

Soln:

$$\text{Mean } \lambda = 6$$

$$P(X=1) = e^{-6} 6^1$$

$$= 6 \cdot e^{-6}$$

$$= 0.0149$$

$$P(X=x) = \frac{e^{-\lambda} \lambda^x}{x!}$$

$$\textcircled{1} P(X=2) = \frac{e^{-6} 6^2}{2!} = \frac{e^{-6} \times 6 \times 6}{1 \times 2} = 18 e^{-6} = (0.0450)$$

$$\textcircled{2} P(X=0) = \frac{e^{-6} 6^0}{0!} = e^{-6} = 0.0025$$

$$\boxed{0! = 1}$$

$$6^0 = 1$$

$$\textcircled{3} P(X \geq 3) = 1 - P(X < 3)$$

$$= 1 - (0.0025 + 0.0149 + 0.0450)$$

$$= 1 - 25(0.0025) = 1 - [P(X=0) + P(X=1) + P(X=2)]$$

$$= 1 - 0.0625 = 1 - \left[\frac{e^{-6} 6^0}{0!} + \frac{e^{-6} 6^1}{1!} + \frac{e^{-6} 6^2}{2!} \right]$$

$$= 0.9375$$

$$= 1 - e^{-6} (1 + 6 + 18)$$

$$= 1 - 25 e^{-6}$$

Geometric Distribution

Table - I		Table - II	
$P(X=x) = q^{x-1} p ; x=1, 2, \dots$		$P(X=x) = q^x p ; x=0, 1, 2, \dots$	
$M_X(t)$	$\frac{pe^t}{(1-qe^t)}$	$M_X(t)$	$\frac{p}{(1-qe^t)}$
Mean	$\frac{1}{p}$	Mean	$\frac{q}{p}$
Variance	$\frac{q}{p^2}$	Variance	$\frac{q}{p^2}$

APR-17
APR-10
NOV-12
APR-15

(1) If the probability that an applicant for a driver's licence will pass the road test on any given trial is 0.8, what is the probability that he will finally pass the test

- on the fourth trial
- in fewer than 4 trials?

Soln

Given $p=0.8$ & $q=1-p=1-0.8=0.2$
 $P(X=r) = q^{r-1} p ; r=1, 2, 3, \dots \infty$

(a) $P(X=4) = (0.2)^{4-1} (0.8) = (0.2)^3 (0.8) = 0.0064$

(b) $P(X < 4) = P(X=1) + P(X=2) + P(X=3)$
 $= (0.2)^{1-1} (0.8) + (0.2)^{2-1} (0.8) + (0.2)^{3-1} (0.8)$
 $= (1 + 0.2 + (0.2)^2) (0.8)$
 $= 0.992$

APR-17

① X is a geometric variate taking values $1, 2, \dots, \infty$, then find $P(X \text{ is odd})$.

$$P(X=r) = q^{r-1} p$$

$$P(X \text{ is odd}) = P(X=1) + P(X=3) + P(X=5) + \dots$$

$$\begin{aligned}
 (1-x)^{-1} &= 1+x+x^2+x^3+\dots \\
 (1-q^2)^{-1} &= 1+q^2+(q^2)^2+(q^2)^3+\dots \\
 &= 1+q^2+q^4+q^6+\dots
 \end{aligned}$$

$$= p + q^2 p + q^4 p + \dots$$

$$= p(1+q^2+q^4+\dots)$$

$$= p(1-q^2)^{-1}$$

$$= p \left(\frac{1}{1-q^2} \right) = p \frac{1}{(1+q)(1-q)} = p \frac{1}{p(1-q)}$$

$$\begin{aligned}
 p+q &= 1 \\
 p &= 1-q
 \end{aligned}$$

$$\begin{aligned}
 a^2-b^2 &= (a+b)(a-b) \\
 &= \frac{1}{1+q}
 \end{aligned}$$

APR-13
NOV-17

② If the probability that a target is destroyed on any one shot is 0.5, what is the probability that it would be destroyed on 6th attempt?

Soln

$$\text{Given } p = 0.5 \text{ or } \frac{1}{2}; \quad p+q=1$$

$$q = 1 - 0.5$$

$$P(X=r) = q^{r-1} p$$

$$q = 0.5 \text{ or } \frac{1}{2}$$

$$P(X=6) = q^5 p$$

$$= \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)$$

$$= \frac{1}{2^6} = \frac{1}{64} = 0.156$$

NOV-16
APR-17

④ Balls are tossed at random into 50 boxes. Find the expected number of tosses required to get the first ball in the fourth box.

$$P = \frac{1}{50}$$

Mean:

$$E(Y) = \frac{1}{p} = \frac{1}{50}$$

⑤ Let $P(X=x) = \left(\frac{3}{4}\right) \left(\frac{1}{4}\right)^{x-1}$; $x=1, 2, 3, \dots$ be the

$$\text{① } P(X>4) \quad \text{② } P(X>4|X>2)$$

$$\text{③ } E(X) \quad \text{④ } \text{var}(X)$$

$$\text{⑤ } E(X) \quad \text{⑥ } \text{var}(X)$$

$$① P(X > k) = q^k = \left(\frac{1}{4}\right)^k$$

$$② P(X > 4) = \left(\frac{1}{4}\right)^4 = \frac{1}{256}$$

$$③ P[X > m+n | X > m] = \frac{q^{m+n}}{q^m} = q^n$$

$$P[X > 4 | X > 2] = q^2$$

$$= \left(\frac{1}{4}\right)^2$$

$$= \frac{1}{16}$$

$$④ \overset{\text{mean}}{E(X)} = \frac{1}{p} = \frac{1}{3/4} = \frac{4}{3}$$

$$⑤ \text{var}(X) = \frac{q}{p^2} = \frac{1}{4} \times \left(\frac{4}{3}\right)^2$$

$$= \frac{1}{4} \times \frac{16}{9}$$

$$= \frac{4}{9}$$

APR-10
APR-11
NOV-15
NOV-18
APR-6

⑥ Establish the memoryless Property of geometric distributions.

Statement:

If X has a G.D, then for any two positive integers m, n .

$$P[X > m+n | X > m] = P[X > n].$$

Proof:

$$P[X > m+n | X > m] = \frac{P[X > m+n \cap X > m]}{P[X > m]} = \frac{P[X > m+n]}{P[X > m]}$$

$$P(X=r) = q^{r-1} p; \quad r=1, 2, 3, \dots$$

$$P(X > k) = \sum_{r=k+1}^{\infty} q^{r-1} p$$

$$= q^{k+1-1} p + q^{k+2-1} p + q^{k+3-1} p + \dots$$

$$= q^k p + q^{k+1} p + q^{k+2} p + \dots$$

$$= q^k p [1 + q + q^2 + \dots]$$

$$= q^k p (1-q)^{-1}$$

$$= q^k p \frac{1}{(1-q)}$$

$$\boxed{p+q=1}$$

$$\boxed{p=1-q}$$

$$= q^k \left(\frac{p}{p} \right)$$

$$\boxed{P(X > k) = q^k}$$

$$\therefore P[X > m+n] = q^{m+n}$$

$$P[X > m] = q^m$$

$$\therefore P[X > m+n / X > m] = \frac{q^{m+n}}{q^m} = \frac{q^m \cdot q^n}{q^m} = q^n$$

$$\therefore P[X > m+n / X > m] = P[X > n]$$

Probability & Random Variables

Defn.: probability

Probability is a mathematical measurement of uncertainty.

Eg.:

* Tossing a coin $S = \{H, T\}$; $n(S) = 2$

* Throwing a dice $S = \{1, 2, 3, 4, 5, 6\}$; $n(S) = 6$

* Tossing 2 coins $S = \{HH, HT, TH, TT\}$; $n(S) = 4$

* Tossing 2 dice $S = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6),$
 $(2,1), (2,2), (2,3), (2,4), (2,5), (2,6),$
 $\dots$
 $(6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\}$
 $n(S) = 36.$

② Using the axiom

$$P(A^c) = 1 - P(A)$$

Soln.

$$\text{Let } A \cup A^c =$$

$$P(A \cup A^c)$$

$$P(A) + P(A^c)$$

$$P(A^c)$$

③ Prove that Zero.

WKT

1.1) Axioms of Probability

① What are the axioms of probability.

$S \rightarrow$ Sample space

$E \rightarrow$ Event in a random experiment

Axiom : 1 $0 \leq P(E) \leq 1$

Axiom : 2 $P(S) = 1$

Axiom : 3 For any sequence of mutually exclusive events $E_1, E_2, E_3, \dots$

((i.e) events for which $E_i \cap E_j = \emptyset$ when $i \neq j$)

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$

$P(E) \rightarrow$ The Probability of the event E .

Formula

$$1) P(A)$$

④ A lot of 10 goods of 10 goods Major defect from the lot at least

Soln.:

$$P(\text{at least 1})$$

$$10 + 4 + 2 = 16$$

② Using the axioms of probability, prove

$$P(A^c) = 1 - P(A)$$

Soln,

$$\text{Let } A \cup A^c = S$$

$$P(A \cup A^c) = P(S)$$

$$P(A) + P(A^c) = 1$$

$$P(A^c) = 1 - P(A)$$

③ prove that the probability of a null set is zero.

$$\text{WKT } S \cup \phi = S$$

$$P(S \cup \phi) = P(S)$$

$$P(S) + P(\phi) = P(S)$$

$$P(\phi) = P(S) - P(S)$$

$$P(\phi) = 0$$

Formula:

$$1) P(A) = \frac{n(A)}{n(S)}$$

④ A lot of integrated circuit chips consists of 10 good, 4 with minor defects and 2 with major defects. Two chips are randomly chosen from the lot. What is the probability that at least one chip is good?

Soln:

$$P(\text{at least one is good}) = \frac{n(A)}{n(S)} = \frac{10C_1 \times 6C_1 + 10C_2}{16C_2}$$

$$= \frac{10 \times 6 + 45}{120}$$

$$= \frac{60 + 45}{120} = \frac{105}{120} = \frac{7}{8}$$

$$10 + 4 + 2 = 16$$

NDV-06
 (5) Four persons are chosen at random from a group containing 3 men, 2 women & 4 children. Show that the chance that exactly two of them will be children is $\frac{10}{21}$.

Men	Women	Children	Total
3	2	4	9

4 persons out of 9 persons = ${}^9C_4 = 126 \rightarrow (1)$

$$\begin{aligned}
 \text{Total} &= \left. \begin{array}{l} \text{2 persons} \\ \text{2 children} \\ \text{out of 4 childrens} \end{array} \right\} \times \begin{array}{l} \text{2 persons} \\ \text{out of 5 persons} \end{array} \\
 &= {}^4C_2 \times {}^5C_2 \\
 &= 6 \times 10 \\
 &= 60 \rightarrow (2)
 \end{aligned}$$

$$\therefore \text{Probability} = \frac{(2)}{(1)} = \frac{60}{126} = \frac{10}{21}$$

(6) If A & B are events with $P(A) = \frac{3}{8}$; $P(B) = \frac{1}{2}$ & $P(A \cap B) = \frac{1}{4}$. find $P(A^c \cap B^c)$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{3}{8} + \frac{1}{2} - \frac{1}{4}$$

$$= \frac{3+4-2}{8}$$

$$P(A \cup B) = \frac{5}{8}$$

$$P(A^c \cap B^c) = 1 - P(A \cup B)$$

$$= 1 - \frac{5}{8}$$

$$= \frac{8-5}{8}$$

$$= \frac{3}{8}$$

⑦ If A & B are independent events with $P(A) = 0.4$ & $P(B) = 0.5$. Find $P(A \cup B)$.

$$\begin{aligned}
 P(A \cup B) &= P(A) + P(B) - \underbrace{P(A \cap B)}_{\downarrow} \\
 &= P(A) + P(B) - \underline{P(A) P(B)} \\
 &= 0.4 + 0.5 - 0.4 \times 0.5 \\
 &= 0.9 - 0.2 \\
 P(A \cup B) &= 0.7
 \end{aligned}$$

⑧ If you ^{you choose} flip a balanced coin, what is the probability of getting atleast one head? (B) 3/4

$$S = \{HH, H\underline{T}, \underline{T}H, \underline{T}\underline{T}\} = \left(\frac{3}{4}\right)$$

$$\begin{aligned}
 P(\text{atleast one head}) &= 1 - P(\text{no head}) \\
 &= 1 - \frac{1}{4} \\
 &= \frac{4-1}{4} \\
 &= \frac{3}{4}
 \end{aligned}$$

⑨ Let $P(A \cup B) = \frac{5}{6}$; $P(A \cap B) = \frac{1}{3}$ & $P(\bar{B}) = \frac{1}{2}$. Are the events A & B independent. Explain.

$$\begin{aligned}
 P(B) &= 1 - P(\bar{B}) \\
 &= 1 - \frac{1}{2}
 \end{aligned}$$

$$P(B) = \frac{1}{2}$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\frac{5}{6} = P(A) + \frac{1}{2} - \frac{1}{3}$$

$$P(A) = \frac{5}{6} - \frac{1}{2} + \frac{1}{3}$$

$$= \frac{5-3+2}{6}$$

$$= \frac{5}{6} + \frac{1}{3} - \frac{1}{2}$$

$$= \frac{15+6-9}{18} = \frac{12}{18} = \frac{2}{3}$$

$$= \frac{21}{18} - \frac{1}{2}$$

$$= \frac{42-18}{36}$$

$$= \frac{24}{36}$$

$$= \frac{2}{3}$$

$$P(A \cap B) = P(A) P(B)$$

$$\frac{1}{3} = \frac{1}{2} \cdot \frac{2}{3}$$

$$\frac{1}{3} = \frac{1}{3}$$

$\therefore A$ & B are independent.

① State and prove Bayes theorem

Statement:-

Let $B_1, B_2, \dots, B_n$ be an ^{partition} exhaustive & mutually exclusive random experiments and A be an event related to that B_i then

$$P(B_i/A) = \frac{P(B_i) P(A/B_i)}{\sum_{i=1}^n P(B_i) P(A/B_i)}$$

$i = 1, 2, \dots, n.$

Proof:

By Conditional probability,

$$P(B_i/A) = \frac{P(B_i \cap A)}{P(A)} \rightarrow \textcircled{1}$$

Using multiplication rule of Probability,

$$P(B_i \cap A) = P(B_i) P(A/B_i) \rightarrow \textcircled{2}$$

Using Total probability theorem,

$$P(A) = \sum_{i=1}^n P(B_i) P(A/B_i) \rightarrow \textcircled{3}$$

Sub $\textcircled{2}$ & $\textcircled{3}$ in $\textcircled{1}$,

$$P(B_i/A) = \frac{P(B_i) P(A/B_i)}{\sum_{i=1}^n P(B_i) P(A/B_i)}$$

Hence the proof.

NOV-23

101-0207W

① In a certain binary communication channel, owing to noise, the probability that a transmitted zero is received as a zero is 0.95 and the probability that transmitted one is received as one is 0.9 . If the probability that a zero is transmitted is 0.4 , find the probability that (i) a one is received (ii) a one was transmitted given that one was received.

Soln

$A \rightarrow$ event of transmitting one.

$\bar{A} \rightarrow$ " " " Zero.

$B \rightarrow$ " " receiving one.

$\bar{B} \rightarrow$ " " " Zero

Given $P(\bar{A}) = 0.4$; $P(B/A) = 0.9$; $P(\bar{B}/\bar{A}) = 0.95$

$$\begin{aligned} P(A) &= 1 - P(\bar{A}) \\ &= 1 - 0.4 \\ \boxed{P(A) &= 0.6} \end{aligned} \quad \begin{aligned} P(B/\bar{A}) &= 1 - P(\bar{B}/\bar{A}) \\ &= 1 - 0.95 \\ \boxed{P(B/\bar{A}) &= 0.05} \end{aligned}$$

By Total probability Theorem,

$$\begin{aligned} P(B) &= P(A) \times P(B/A) + P(\bar{A}) \times P(B/\bar{A}) \\ &= 0.6 \times 0.9 + 0.4 \times 0.05 \\ &= 0.56 \end{aligned}$$

By Baye's theorem

$$\begin{aligned} P(A/B) &= \frac{P(A) \times P(B/A)}{P(B)} \\ &= \frac{0.6 \times 0.9}{0.56} \\ &= \frac{0.54}{0.56} = \frac{27}{28} \end{aligned}$$

(2 marks) APR-11
APR-19

① Define discrete r.v

A r.v 'x' is said to be discrete if it has finite number of values.

Eg:

- * The number of children in a particular village
- * The number of doctors working in a particular district.

② Define continuous r.v APR-19

A r.v 'x' which takes all possible values in a given interval is called a c.r.v

Eg:

Age, height, weight, length, pressure, temperature, time, voltage.

③ Define r.v NOV-11

A function is a real number to each outcome in the sample space for random experiment

④ Test whether $f(x)$ NOV-14
APR-15
$$= \begin{cases} |x| & -1 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$
 can be the p.d.f of a c.r.v.

$$\int_{-\infty}^{\infty} f(x) dx = 1$$

$$\text{L.H.S} = \int_{-\infty}^{\infty} f(x) dx$$

$$f(x) = \begin{cases} -x & -1 \leq x \leq 0 \\ x & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$= \int_{-1}^1 |x| dx$$

$$= 2 \int_0^1 x dx$$

$$= 2 \left[\frac{x^2}{2} \right]_0^1 = \frac{2}{2} [1 - 0] = 1$$

(APR-19)
 (5) Show that for any events A & B in S ,
 $P(B) = P(B|A) P(A) + P(B|\bar{A}) P(\bar{A})$.

Soln

$$\begin{aligned}
 R.H.S &= P(B|A) P(A) + P(B|\bar{A}) P(\bar{A}) \\
 &= \frac{P(A \cap B)}{P(A)} P(A) + \frac{P(\bar{A} \cap B)}{P(\bar{A})} P(\bar{A}) \\
 &= P(A \cap B) + \frac{P(\bar{A} \cap B)}{\downarrow} \\
 &= P(A \cap B) + P(B) - P(A \cap B) \\
 &= P(B) \\
 &= L.H.S \\
 \therefore L.H.S &= R.H.S
 \end{aligned}$$

(NOV-19)
 (6) If A & B are mutually exclusive events
 $P(A) = 0.29$ & $P(B) = 0.43$. then find $P(\bar{A})$
 & $P(A \cup B)$.

Soln:

Given $P(A) = 0.29$

$P(B) = 0.43$

$P(\bar{A}) = 1 - P(A) = 1 - 0.29 = 0.71$

$P(A \cup B) = P(A) + P(B) = 0.29 + 0.43 = 0.72$.

(APR-19)
 (7) The p.d.f of the r.v X is given by

$$f(x) = \begin{cases} \frac{k(1-x^2)}{3} & ; 0 < x < 1 \\ 0 & ; \text{otherwise} \end{cases}$$

Find the value of k .

$$\int_{-\infty}^{\infty} f(x) dx = 1$$

$$\int_0^1 k(1-x^2) dx = 1$$

$$k \int_0^1 (1-x^2) dx = 1$$

$$k \left[x - \frac{x^3}{3} \right]_0^1 = 1$$

$$k \left(1 - \frac{1}{3} \right) = 1$$

$$k \left(\frac{3-1}{3} \right) = 1 \Rightarrow k \left(\frac{2}{3} \right) = 1 \Rightarrow k = \frac{3}{2}$$

APR-19

8) The M.G.F of r.v X is given $M(t) = e^3(e^t - 1)$
What is $P(X=0)$

$$M_X(t) = e^3(e^t - 1)$$
$$= e^{3e^t} e^{-3}$$

$$= \frac{1}{e^3} \left[1 + 3 \frac{e^t}{1!} + \frac{(3e^t)^2}{2!} + \dots \right]$$

$$M_X(t) = \frac{1}{e^3} + \frac{3}{e^3} \frac{e^t}{1!} + \frac{9}{2} e^{2t} + \dots$$

$P(0) = P(X=0) = \frac{1}{e^3}$

NOV-18

9) Find the mean of a normal distribution.
Soln:

$$\text{Mean} = E(X) = \mu$$

10) A r.v X is uniformly distributed between 3 & 15. Find the variance of X .

Soln:

$$\text{Given } a=3; b=15$$

$$\begin{aligned} \text{variance} &= \frac{1}{12} (b-a)^2 \\ &= \frac{1}{12} (15-3)^2 \\ &= \frac{1}{12} (12)^2 \end{aligned}$$

$\text{variance} = 12$

APR-19

11) What are the limitations of poisson distrib

* n is indefinitely large
(i.e) $n \rightarrow \infty$

* p is very small such that $p \rightarrow 0$

* $np = \lambda$ (a finite quantity)

$$p = \frac{\lambda}{n}$$

$$q = 1 - p$$

$$q = 1 - \frac{\lambda}{n}$$

(12) Show that the function $f(x) = \begin{cases} \frac{x^2}{3}; & -1 < x < 2 \\ 0; & \text{otherwise} \end{cases}$ is a P.d.f

Soln:

$$\begin{aligned} \int_{-\infty}^{\infty} f(x) dx &= \int_{-1}^2 \frac{x^2}{3} dx \\ &= \frac{1}{3} \left[\frac{x^3}{3} \right]_{-1}^2 \\ &= \frac{1}{9} [(2)^3 - (-1)^3] \\ &= \frac{1}{9} (8+1) \\ &= \frac{1}{9} \times 9 \\ &= 1 \end{aligned}$$

(13) A continuous r.v. x has a P.d.f $f(x) = k$, $0 \leq x \leq 1$. Find k .

$$\int_{-\infty}^{\infty} f(x) dx = 1$$

$$\int_0^1 k dx = 1$$

$$k [x]_0^1 = 1$$

$$k(1-0) = 1$$

$$\boxed{k=1}$$

(14) For a binomial distribution Mean is 2 & variance is $4/3$. Find the 1st term of the distribution.

$$\text{Mean} = np = 2 \rightarrow (1)$$

$$\text{variance } npq = 4/3 \rightarrow (2)$$

$$\frac{(2)}{(1)} = \frac{npq}{np} = \frac{4/3}{2}$$

$$q = \frac{4/3}{2} = \frac{2}{3}$$

$$p+q=1$$

$$p = 1-p = 1 - 2/3 = 1/3 \quad \boxed{p=1/3}$$

(14) An experiment succeeds twice as often as it fails. Find the chance that in the next 4 trials, there shall be at least one success. $n=4$; $P=2q=2(1-p)$
 $P=2-2p$

$$\begin{aligned} P(X \geq 1) &= 1 - P[X < 1] \\ &= 1 - P[X=0] \\ &= 1 - {}^4C_0 \left(\frac{2}{3}\right)^0 \left(\frac{1}{3}\right)^{4-0} \\ &= 1 - \frac{1}{81} \end{aligned}$$

$$= \frac{81-1}{81} = \frac{80}{81}$$

From (1)

$$np = 2$$

$$n(1/3) = 2$$

$$\boxed{n=6}$$

$$\begin{aligned} P(X=x) &= {}^nC_x p^x q^{n-x} \\ P(X=0) &= {}^6C_0 \left(\frac{1}{3}\right)^0 \left(\frac{2}{3}\right)^{6-0} \\ &= \frac{2^6}{3^6} = \frac{64}{729} \end{aligned}$$

- (16) Let $M_X(t) = \frac{1}{1-t}$, $|t| < 1$, M.G.F of r.v. X . Find $E(X)$ & $E(X^2)$. NOV-19

$$M_X(t) = \frac{1}{1-t} = (1-t)^{-1} = 1 + t + t^2 + \dots$$

$$= 1 + \frac{t}{1!} (1) + \frac{t^2}{2!} (2) + \dots$$

$$E(X) = 1$$

$$E(X^2) = 2.$$

- (17) Let A & B be two events such that $P(A) = \frac{1}{3}$, $P(B) = \frac{3}{4}$ & $P(A \cap B) = \frac{1}{4}$. Compute $P(A|B)$ & $P(\bar{A} \cap \bar{B})$. APR-19

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{(1/4)}{(3/4)} = \frac{1}{3}$$

$$P(\bar{A} \cap \bar{B}) = P(\overline{A \cup B})$$

$$= 1 - P(A \cup B)$$

$$= 1 - [P(A) + P(B) - P(A \cap B)]$$

$$= 1 - \frac{1}{3} - \frac{3}{4} + \frac{1}{4}$$

$$= \frac{3-1}{3} - \frac{3+1}{4}$$

$$= \frac{2}{3} - \frac{2}{4} = \frac{8-6}{12} = \frac{2}{12} = \frac{1}{6}$$

- (18) A bag contains 8 white & 4 black balls. If 5 balls are drawn at random, what is the probability that 3 are white & 2 are black? NOV-19

White	Black	Total
8	4	12

$$S = \{ 5 \text{ balls are taken out of } 12 \}$$

$$n(S) = {}^{12}C_5 = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} = 792$$

(3 are white & 2 are black)

$$n(A) = {}^8C_3 \times {}^4C_2$$

$$n(A) = 336$$

$$P(A) = \frac{n(A)}{n(S)} = \frac{336}{792} = 0.4242$$

NOV-15
APR-19

(19) Let X be the random variable which denote the number of heads in three tosses of a fair coin. Determine the P.m.f of X .

$$P[X=x] = {}^3C_x \left(\frac{1}{2}\right)^3 ; x=0, 1, 2, 3.$$

x	0	1	2
$P(X=x)$	${}^3C_0 \left(\frac{1}{2}\right)^3$ $= \left(\frac{1}{2}\right)^3$	${}^3C_1 \left(\frac{1}{2}\right)^3$ $= 3 \left(\frac{1}{2}\right)^3$	${}^3C_2 \left(\frac{1}{2}\right)^3$ $= \frac{3 \times 2}{1 \times 2} \left(\frac{1}{2}\right)^3$ $= 3 \left(\frac{1}{2}\right)^3$

$${}^3C_3 \left(\frac{1}{2}\right)^3$$

$$= \left(\frac{1}{2}\right)^3$$

	Binomial	Poisson	Geometric		Uniform	Exponential	Normal
$P(X=x)$	$n C x p^x q^{n-x}$ $x=0,1,2,\dots,n$	$\frac{e^{-\lambda} \lambda^x}{x!}$ $x=0,1,2,\dots,\infty$	$q^{x-1} p$ $x=1,2,\dots$	$q^x p$ $x=0,1,2,\dots$	$f(x) = \begin{cases} \frac{1}{b-a}, & a \leq x < b \\ 0, & \text{otherwise} \end{cases}$	$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$	$f(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2\sigma^2} \left(\frac{x-\mu}{\sigma}\right)^2}$
$P(X \leq t)$	$q + p e^t$	$e^\lambda (e^t - 1)$	$\frac{p e^t}{(1 - q e^t)}$	$\frac{p}{(1 - q e^t)}$	$\frac{e^{bt} - e^{at}}{(b-a)t}$	$\frac{\lambda}{\lambda - t} ; \lambda > t$	
n	np	λ	$\frac{1}{p}$	$\frac{q}{p}$	$\frac{a+b}{2}$	$\frac{1}{\lambda}$	μ
npq	npq	λ	$\frac{q}{p^2}$	$\frac{q}{p^2}$	$\frac{(b-a)^2}{12}$	$\frac{1}{\lambda^2}$	σ^2
$\sqrt{npq}$	$\sqrt{npq}$	$\sqrt{\lambda}$	Memory less property $P(X > s+t X > s) = P(X > t)$		$\sqrt{\frac{(b-a)^2}{12}}$	$\sqrt{\frac{1}{\lambda^2}}$	$\sqrt{\sigma^2}$
						Memory less property $P(X > t) = e^{-\lambda t}$	

Discrete r.v.s	Continuous r.v.s
$E(X)$	$E(X)$
$E(X^2)$	$E(X^2)$
$\text{Var}(X)$	$\text{Var}(X)$
S.D.	S.D.

	Discrete r.v.'s	Continuous r.v.'s
$E(x)$	$\sum x p(x)$	$\int_{-\infty}^{\infty} x f(x) dx$
$E(x^2)$	$\sum x^2 p(x)$	$\int_{-\infty}^{\infty} x^2 f(x) dx$
variance mean sq	$E(x^2) - (E(x))^2$	$E(x^2) - (E(x))^2$
S.D to	$\sqrt{\text{var}(x)}$	$\sqrt{\text{var}(x)}$

Uniform Distribution (or) Rectangular "

Formula:

$$f(x) = \begin{cases} \frac{1}{b-a}, & a < x < b \\ 0, & \text{otherwise} \end{cases}$$

	Uniform
$M_x(t)$	$\frac{e^{bt} - e^{at}}{(b-a)t}$
Mean	$\frac{a+b}{2}$
Variance	$\frac{(b-a)^2}{12}$
S.D	$\sqrt{\frac{(b-a)^2}{12}}$

① If x is a r.v with cdf as $F(x)$. Show that the r.v $y = F(x)$ is uniformly distributed in (0,1)
SOLN

Given $y = F(x)$ at (0,1)

$$g(y) = f(x) \left| \frac{dx}{dy} \right|$$

$$= f(x) \left| \frac{1}{\frac{dy}{dx}} \right|$$

$$= f(x) \left| \frac{1}{f(x)} \right|$$

$$= 1$$

$$g(y) = \begin{cases} 1, & 0 < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

Nov-06

② If x is uniformly distributed over (0,10)
Calculate the probability that (a) $x > 6$
(b) $3 < x < 8$.

SOLN: Given $f(x) = \frac{1}{b-a}, a < x < b$
 $= \frac{1}{10-0}, 0 < x < 10$

$$f(x) = \frac{1}{10}$$

$$\begin{aligned}
 &= \int_a^b f(x) dx \\
 \textcircled{a} \quad P[X > 6] &= \frac{1}{10} \int_6^{10} dx = \frac{1}{10} [x]_6^{10} \\
 &= \frac{1}{10} [10 - 6] \\
 &= \frac{4}{10} = \frac{2}{5}
 \end{aligned}$$

$$\begin{aligned}
 \textcircled{b} \quad P[3 < X < 8] &= \frac{1}{10} \int_3^8 dx = \frac{1}{10} [x]_3^8 \\
 &= \frac{1}{10} [8 - 3] \\
 &= \frac{5}{10} = \frac{1}{2}
 \end{aligned}$$

$\textcircled{3}$ X is APR-10
APR-11 uniformly distributed with mean 1 and variance $\frac{4}{3}$, find $P(X < 0)$.

Soln:

Given Mean = 1 $\Rightarrow \frac{b+a}{2} = 1$

$b+a = 2 \rightarrow \textcircled{1}$

variance = $\frac{4}{3} \Rightarrow \frac{(b-a)^2}{12} = \frac{4}{3}$

$(b-a)^2 = \frac{4 \times 12}{3} = 16$

$\sqrt{(b-a)^2} = \sqrt{16 \times 1} = 4$

$b-a = 4 \rightarrow \textcircled{2}$

From $\textcircled{1}$ & $\textcircled{2}$

$b+a = 2$

$b-a = 4$

$2b = 6$

$b = \frac{6}{2} = 3$

$b = 3$

Sub $b=3$ in $\textcircled{1}$

$b+a = 2$

$a = 2 - b$

$a = 2 - 3$

$a = -1$

$f(x) = \frac{1}{b-a} = \frac{1}{3-(-1)} = \frac{1}{3+1} = \frac{1}{4}$

$$\begin{aligned}
 P(X < 0) &= \int_{-1}^0 f(x) dx = \frac{1}{4} [x]_{-1}^0 \\
 &= \int_{-1}^0 \frac{1}{4} dx = \frac{1}{4} (0 - (-1)) \\
 &= \frac{1}{4} \int_{-1}^0 dx = \frac{1}{4}
 \end{aligned}$$

④ Show that ^(Nov-06) for the uniform distribution
 $f(x) = \frac{1}{2a}$, $-a < x < a$ the M.G.F
 about origin is $\frac{\sinh at}{at}$
Soln:

Given $f(x) = \frac{1}{2a}$, $-a < x < a$

$$\therefore M_X(t) = \int_a^b e^{tx} f(x) dx$$

$$= \int_{-a}^a e^{tx} \frac{1}{2a} dx$$

$$= \frac{1}{2a} \int_{-a}^a e^{tx} dx$$

$$\begin{aligned} \sinh at &= \frac{e^{at} - e^{-at}}{2} \\ &= \frac{1}{2a} \left[\frac{e^{tx}}{t} \right]_{-a}^a \\ &= \frac{1}{2at} \left[e^{ta} - e^{-ta} \right] \\ &= \frac{1}{at} \left[\frac{e^{at} - e^{-at}}{2} \right] \end{aligned}$$

$$\boxed{M_X(t) = \frac{1}{at} \sinh at}$$

Exponential Distribution

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & , x \geq 0, \lambda > 0 \\ 0 & , \text{otherwise} \end{cases}$$

Exponential			
M.g.f	Mean	variance	S.D
$\frac{\lambda}{\lambda - t}; \lambda > t$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\sqrt{\frac{1}{\lambda^2}}$

① Establish the memoryless property of Exponential Distribution.

Statement:

If X is E.D, then

$$P(X > s + t | X > s) = P(X > t), \forall s, t > 0.$$

Proof:

Unit-II

Correlation Analysis

Karl Pearson's Co-efficient of Correlation

$$r(x,y) = \frac{\text{Cov}(x,y)}{\sigma_x \sigma_y}$$

(or)
 r_{xy}

$$\sigma_x = \sqrt{\text{Var}(x)}$$

$$\sigma_y = \sqrt{\text{Var}(y)}$$

$$\bar{x} = \frac{\sum x}{n} ; \bar{y} = \frac{\sum y}{n}$$

$$\sigma_x = \sqrt{\frac{1}{n} \sum x^2 - \bar{x}^2}$$

$$\sigma_y = \sqrt{\frac{1}{n} \sum y^2 - \bar{y}^2}$$

$$\text{Cov}(x,y) = \frac{1}{n} \sum xy - \bar{x}\bar{y}$$

Problems

- ① Calculate the correlation Co-efficient for the following heights (in inches) of fathers x & their sons y .

x	65	66	67	67	68	69	70	72
y	67	68	65	68	72	72	69	71

Soln :

x	y	x^2	y^2	xy
65	67	4225	4489	4355
66	68	4356	4624	4488
67	65	4489	4225	4355
67	68	4489	4624	4556
68	72	4624	5184	4896
69	72	4761	5184	4968
70	69	4900	4761	4830
72	71	5184	5041	5112
$\sum x = 574$	$\sum y = 552$	$\sum x^2 = 37028$	$\sum y^2 = 38132$	$\sum xy = 37560$

$$\bar{x} = \frac{\sum x}{n} = \frac{544}{8} = 68$$

$$\boxed{\bar{x} = 68}$$

$$\bar{y} = \frac{\sum y}{n} = \frac{552}{8} = 69$$

$$\boxed{\bar{y} = 69}$$

$$\bar{x}\bar{y} = 68 \times 69 = 4692$$

$$\boxed{\bar{x}\bar{y} = 4692}$$

$$\begin{aligned}\sigma_x &= \sqrt{\frac{1}{n} \sum x^2 - \bar{x}^2} \\ &= \sqrt{\frac{1}{8} (37028) - 68^2} \\ &= \sqrt{4628.5 - 4624}\end{aligned}$$

$$\boxed{\sigma_x = 2.121}$$

$$\begin{aligned}\sigma_y &= \sqrt{\frac{1}{n} \sum y^2 - \bar{y}^2} \\ &= \sqrt{\frac{1}{8} (38132) - 69^2} \\ &= \sqrt{4766.5 - 4761}\end{aligned}$$

$$\boxed{\sigma_y = 2.345}$$

$$\text{Cov}(x, y) = \frac{1}{n} \sum xy - \bar{x}\bar{y}$$

$$= \frac{1}{8} (37560) - 68 \times 69$$

$$= 4695 - 4692$$

$$\boxed{\text{Cov}(x, y) = 3}$$

$$r(x, y) = \frac{\text{Cov}(x, y)}{\sigma_x \sigma_y}$$

$$= \frac{3}{(2.121)(2.345)}$$

$$= \frac{3}{4.973}$$

$$\boxed{r(x, y) = 0.6032}$$

② Find the correlation coefficient for the following data

x	10	14	18	22	26	30
y	18	12	24	6	30	36

Soln:-

x	y	x ²	y ²	xy
10	18	100	324	180
14	12	196	144	168
18	24	324	576	432
22	6	484	36	132
26	30	676	900	780
30	36	900	1296	1080
120	126	2680	3276	2772

H.W
① Compute the
x & y using the

x	1	=
y	8	.

② Find the c
industrial
the follow

x	55
y	35

$$\bar{x} = \frac{\sum x}{n} = \frac{120}{6} = 20$$

$$\boxed{\bar{x} = 20}$$

$$\bar{y} = \frac{\sum y}{n} = \frac{126}{6} = 21$$

$$\boxed{\bar{y} = 21}$$

$$\begin{aligned} \text{Cov}(x, y) &= \frac{\sum xy}{n} - \bar{x}\bar{y} \\ &= \frac{2772}{6} - 20 \times 21 \\ &= 462 - 420 \end{aligned}$$

$$\boxed{\text{Cov}(x, y) = 42}$$

$$\begin{aligned} r(x, y) &= \frac{\text{Cov}(x, y)}{\sigma_x \sigma_y} \\ &= \frac{42}{46.667 \times 10.25} \\ &= 0.5998 \end{aligned}$$

$$\boxed{r(x, y) = 0.60}$$

$$\begin{aligned} \sigma_x &= \sqrt{\frac{\sum x^2}{n} - \bar{x}^2} \\ &= \sqrt{\frac{2680}{6} - (20)^2} \end{aligned}$$

$$= \sqrt{\frac{1340}{3} - 400}$$

$$\boxed{\sigma_x = 46.667}$$

$$\sigma_y = \sqrt{\frac{\sum y^2}{n} - \bar{y}^2}$$

$$= \sqrt{\frac{3276}{6} - (21)^2}$$

$$= \sqrt{546 - 441}$$

$$= \sqrt{105}$$

$$\boxed{\sigma_y = 10.25}$$

H.W

① Compute the coefficient of correlation between X & Y using the following data.

X	1	3	5	7	8	10
Y	8	12	15	17	18	20

Ans

0.9878

② Find the coefficient of correlation between industrial production and export using the following data.

X	55	56	58	59	60	60	62
Y	35	38	37	39	44	43	44

Ans

0.8972

Rank correlation!!

$$r = 1 - \frac{6 \sum d_i^2}{n(n^2-1)} ;$$

$$d_i = x_i - y_i$$

n = no. of items.

① Find the rank correlation coefficient from the following data.

X	1	2	3	4	5	6	7
Y	4	3	1	2	6	5	7

Soln:

X	Y	$d_i = x_i - y_i$	d_i^2
1	4	-3	9
2	3	-1	1
3	1	2	4
4	2	2	4
5	6	-1	1
6	5	1	1
7	7	0	0
		0	20

$$n=7$$

$$r = 1 - \frac{6 \sum d_i^2}{n(n^2-1)}$$

$$= 1 - \frac{6 \times 20}{7(49-1)}$$

$$r = 0.6429$$

Regression
coefficient of correlation:-

$$r = 0 \text{ to } 1$$

$$r = \sqrt{b_{yx} \times b_{xy}}$$

$$\bar{x} = \frac{\sum x}{n} \quad ; \quad \bar{y} = \frac{\sum y}{n}$$

$$b_{yx} = \frac{\sum (x - \bar{x})(y - \bar{y})}{\sum (x - \bar{x})^2}$$

$$b_{xy} = \frac{\sum (x - \bar{x})(y - \bar{y})}{\sum (y - \bar{y})^2}$$

Line x on y :-

$$x - \bar{x} = b_{xy}(y - \bar{y})$$

Line y on x :-

$$y - \bar{y} = b_{yx}(x - \bar{x})$$

Problems

- ① obtain the equation of the lines of reg from the following data.

x	1	2	3	4	5	6	7
y	9	8	10	12	11	13	14

Soln:-

x	y	$x - \bar{x}$ $x - 4$	$y - \bar{y}$ $y - 11$	$(x - \bar{x})^2$	$(y - \bar{y})^2$	$(x - \bar{x})(y - \bar{y})$
1	9	-3	-2	9	4	6
2	8	-2	-3	4	9	6
3	10	-1	-1	1	1	1
4	12	0	1	0	1	0
5	11	1	0	1	0	0
6	13	2	2	4	4	4
7	14	3	3	9	9	9
28	77	0	0	28	28	26

$$\bar{x} = \frac{\sum x}{n} = \frac{28}{7} = 4 \quad \bar{y} = \frac{\sum y}{n} = \frac{77}{7} = 11$$

$$\boxed{\bar{x} = 4}$$

$$\boxed{\bar{y} = 11}$$

$$b_{yx} = \frac{\sum (x - \bar{x})(y - \bar{y})}{\sum (x - \bar{x})^2} = \frac{26}{28} = 0.928$$

$$\boxed{b_{yx} = 0.928}$$

$$b_{xy} = \frac{\sum (x - \bar{x})(y - \bar{y})}{\sum (y - \bar{y})^2} = \frac{26}{28} = 0.928$$

$$\boxed{b_{xy} = 0.928}$$

$$r = \sqrt{b_{yx} \times b_{xy}}$$

$$= \sqrt{0.928 \times 0.928}$$

$$\boxed{r = 0.8939}$$

Line x on y :-

$$x - \bar{x} = b_{xy} (y - \bar{y})$$

$$x - 4 = 0.928 (y - 11)$$

$$x - 4 = 0.928y - 10.208$$

$$x - 4 - 0.928y + 10.208 = 0$$

$$\boxed{x - 0.928y + 6.208 = 0}$$

Line y on x :-

$$y - \bar{y} = b_{yx} (x - \bar{x})$$

$$y - 11 = 0.928 (x - 4)$$

$$y - 11 = 0.928x - 3.712$$

$$y - 11 - 0.928x + 3.712 = 0$$

$$y - 0.928x - 7.288 = 0$$

② Find the equation of the regression line y on x from the following data

X	3	5	6	8	9	11
Y	2	3	4	6	5	8

Soln:

x	y	$x - \bar{x}$	$y - \bar{y}$	$(x - \bar{x})^2$	$(y - \bar{y})^2$	$(x - \bar{x})(y - \bar{y})$
3	2	-4	-3	16	9	12
5	3	-2	-2	4	4	4
6	4	-1	-1	1	1	1
8	6	1	1	1	1	1
9	5	2	0	4	0	0
11	8	4	3	16	9	12
42	28	0	0	42	24	30

3) The two lines
 $8x - 1$
 $40x$
 The vari
 (i) The
 (ii) Cor

Soln: (i) $\frac{\sum xy}{\sum x^2}$
 Given $\frac{30}{40}$

① $\times 5 \Rightarrow$
 ② $\times 1 \Rightarrow$

$$\bar{x} = \frac{\sum x}{n} = \frac{42}{6} = 7 \quad \left| \quad \bar{y} = \frac{\sum y}{n} = \frac{28}{6} = 4.6 \approx 5$$

$\bar{x} = 7$

$\bar{y} = 5$

$$b_{yx} = \frac{\sum (x - \bar{x})(y - \bar{y})}{\sum (x - \bar{x})^2}$$

$$= \frac{30}{42}$$

$$= \frac{5}{7}$$

$b_{yx} = 0.7$

Line y on x:

$$y - \bar{y} = b_{yx} (x - \bar{x})$$

$$y - 5 = 0.7 (x - 7)$$

$$y - 5 = 0.7x - 4.9$$

$$y - 5 - 0.7x + 4.9 = 0$$

$$y - 0.7x - 0.1 = 0$$

Mean

(ii)

① $\Rightarrow 8x = 10$
 $x = \frac{10}{8}$

$b_{xy} =$

② $\Rightarrow 10y =$
 $y =$

b_{yx}

③ The two lines of regression are

$$8x - 10y + 66 = 0 \rightarrow (A)$$

$$40x - 18y - 214 = 0 \rightarrow (B)$$

The variance of x is 9. Find

(i) The mean values of x & y .

(ii) Correlation coefficient between x & y .

Soln: (i) Mean...

Given $8\bar{x} - 10\bar{y} = -66 \rightarrow (1)$

$$40\bar{x} - 18\bar{y} = 214 \rightarrow (2)$$

$$(1) \times 5 \Rightarrow 40\bar{x} - 50\bar{y} = -330$$

$$(2) \times 1 \Rightarrow \begin{array}{r} 40\bar{x} - 18\bar{y} = 214 \\ \hline (-) \quad (+) \quad (-) \end{array}$$

$$-32\bar{y} = -544$$

$$\bar{y} = \frac{-544}{-32}$$

$$\boxed{\bar{y} = 17}$$

$$\bar{y} = 17 \text{ in (1)}$$

$$8\bar{x} - 10\bar{y} = -66$$

$$8\bar{x} - 10(17) = -66$$

$$8\bar{x} = -66 + 170$$

$$8\bar{x} = 104$$

$$\bar{x} = \frac{104}{8} = 13$$

$$\boxed{\bar{x} = 13}$$

Mean

$$\bar{x} = 13, \bar{y} = 17.$$

(ii)

1st x & y
2nd y & x

$$(A) \Rightarrow 8x = 10y - 66$$

$$x = \frac{10}{8}y - \frac{66}{8}$$

$$\boxed{b_{xy} = \frac{10}{8}}$$

$$(B) \Rightarrow 18y = 40x - 214$$

$$y = \frac{40}{18}x - \frac{214}{18}$$

$$\boxed{b_{yx} = \frac{40}{18}}$$

$$r = \sqrt{b_{xy} b_{yx}}$$

$$= \sqrt{\left(\frac{10}{8}\right)\left(\frac{40}{18}\right)}$$

$$r = 1.66 \neq 1$$

$$(A) \Rightarrow 10y = 8x + 66$$

$$y = \frac{8}{10}x + \frac{66}{10}$$

$$\boxed{b_{yx} = \frac{8}{10}}$$

$$(B) \Rightarrow 40x = 18y + 214$$

$$x = \frac{18}{40}y + \frac{214}{40}$$

$$\boxed{b_{xy} = \frac{18}{40}}$$

$$r = \sqrt{b_{xy} b_{yx}}$$

$$= \sqrt{\frac{18}{10} \times \frac{8}{10}}$$

$$r = \pm 0.6$$

$$\therefore r = 0.6$$

2.1) Joint probability - marginal & Conditional distributions

16 Marks

Discrete Random Variables

- ① The joint probability mass function of (x, y) is given by $p(x, y) = K(2x + 3y)$, $x = 0, 1, 2$; $y = 1, 2, 3$. Find all the marginal and Conditional Probability distribution. Also, find the probability distribution of $(x+y)$ & $P(x+y > 3)$.

Soln:

Marginal Distribution

$x \backslash y$	1	2	3	Total	$K = \frac{1}{72}$
0	$3K$	$6K$	$9K$	$18K$	$\frac{18}{72}$
1	$5K$	$8K$	$11K$	$24K$	$\frac{24}{72}$
2	$7K$	$10K$	$13K$	$30K$	$\frac{30}{72}$
Total	$15K$	$24K$	$33K$	$72K$	$\frac{72}{72} = 1$
$K = \frac{1}{72}$	$\frac{15}{72}$	$\frac{24}{72}$	$\frac{33}{72}$	$\frac{72}{72} = 1$	

$$p(x, y) = K(2x + 3y)$$

$$p(0, 1) = K(0 + 3) = 3K = \frac{3}{72}$$

$$p(0, 2) = K(0 + 6) = 6K = \frac{6}{72}$$

$$p(0, 3) = K(0 + 9) = 9K = \frac{9}{72}$$

$$p(1, 1) = K(2 + 3) = 5K = \frac{5}{72}$$

$$p(1, 2) = K(2 + 6) = 8K = \frac{8}{72}$$

$$p(1, 3) = K(2 + 9) = 11K = \frac{11}{72}$$

$$p(2, 1) = K(4 + 3) = 7K = \frac{7}{72}$$

$$p(2, 2) = K(4 + 6) = 10K = \frac{10}{72}$$

$$p(2, 3) = K(4 + 9) = 13K = \frac{13}{72}$$

$$\text{Total Probability} = 1$$

$$72K = 1$$

$$K = \frac{1}{72}$$

Marginal Distribution x & y

$$P(X=0) = \frac{18}{72}, P(X=1) = \frac{24}{72}; P(X=2) = \frac{30}{72}$$

$$P(Y=1) = \frac{15}{72}; P(Y=2) = \frac{24}{72}; P(Y=3) = \frac{33}{72}$$

Conditional Distribution:-

$$\underline{X=0}$$

$$\frac{P(0, 1)}{P(X=0)} = \frac{3}{72} \times \frac{72}{18} = \frac{1}{6}$$

$$\frac{P(0, 2)}{P(X=0)} = \frac{6}{72} \times \frac{72}{18} = \frac{1}{3}$$

$$\underline{Y=1}$$

$$\frac{P(0, 1)}{P(Y=1)} = \frac{3}{72} \times \frac{72}{15} = \frac{1}{5}$$

$$\frac{P(1, 1)}{P(Y=1)} = \frac{5}{72} \times \frac{72}{15} = \frac{5}{15} = \frac{1}{3}$$

$$\frac{P(0,3)}{P(X=0)} = \frac{9}{72} \times \frac{72}{180} = \frac{1}{2}$$

when $X=1$

$$\frac{P(1,1)}{P(X=1)} = \frac{5}{72} \times \frac{72}{24} = \frac{5}{24}$$

$$\frac{P(1,2)}{P(X=1)} = \frac{8}{72} \times \frac{72}{24} = \frac{1}{3}$$

$$\frac{P(1,3)}{P(X=2)} = \frac{11}{72} \times \frac{72}{24} = \frac{11}{24}$$

$X=2$

$$\frac{P(2,1)}{P(X=2)} = \frac{7}{72} \times \frac{72}{30} = \frac{7}{30}$$

$$\frac{P(2,2)}{P(X=2)} = \frac{10}{72} \times \frac{72}{30} = \frac{1}{3}$$

$$\frac{P(2,3)}{P(X=2)} = \frac{13}{72} \times \frac{72}{30} = \frac{13}{30}$$

$$\frac{P(2,1)}{P(Y=1)} = \frac{7}{72} \times \frac{72}{15} = \frac{7}{15}$$

$Y=2$

$$\frac{P(0,2)}{P(Y=2)} = \frac{6}{72} \times \frac{72}{24} = \frac{1}{4}$$

$$\frac{P(1,2)}{P(Y=2)} = \frac{8}{72} \times \frac{72}{24} = \frac{1}{3}$$

$$\frac{P(2,2)}{P(Y=2)} = \frac{10}{72} \times \frac{72}{24} = \frac{5}{12}$$

$Y=3$

$$\frac{P(0,3)}{P(Y=3)} = \frac{9}{72} \times \frac{72}{33} = \frac{3}{11}$$

$$\frac{P(1,3)}{P(Y=3)} = \frac{11}{72} \times \frac{72}{33} = \frac{1}{3}$$

$$\frac{P(2,3)}{P(Y=3)} = \frac{13}{72} \times \frac{72}{33} = \frac{13}{33}$$

Probability Distribution :-

$X+Y$	P
1 $P(0,1)$	$\frac{3}{72}$
2 $P(0,2) + P(1,1)$	$\frac{6}{72} + \frac{5}{72} = \frac{11}{72}$
3 $P(0,3) + P(1,2) + P(2,1)$	$\frac{9}{72} + \frac{8}{72} + \frac{7}{72} = \frac{24}{72}$
4 $P(1,3) + P(2,2)$	$\frac{11}{72} + \frac{10}{72} = \frac{21}{72}$
5 $P(2,3)$	$\frac{13}{72}$
Total	1

$$P(X+Y \leq 3) = P(1) + P(2) + P(3) = \frac{3}{72} + \frac{11}{72} + \frac{24}{72} = \frac{38}{72}$$

(Q) The joint distribution of X & Y is given by
 $f(x,y) = \frac{x+y}{21}$; $x=1,2,3$, $y=1,2$. Find the
 marginal distribution & conditional distribution.

Soln:-

Marginal distribution:

$X \backslash Y$	1	2	3	Total
1	$\frac{2}{21}$	$\frac{3}{21}$	$\frac{4}{21}$	$\frac{9}{21}$
2	$\frac{3}{21}$	$\frac{4}{21}$	$\frac{5}{21}$	$\frac{12}{21}$
total	$\frac{5}{21}$	$\frac{7}{21}$	$\frac{9}{21}$	1

$P(x,y)$

$$f(x,y) = \frac{x+y}{21}$$

$$f(1,1) = \frac{1+1}{21} = \frac{2}{21}$$

$$f(1,2) = \frac{1+2}{21} = \frac{3}{21}$$

$$f(2,1) = \frac{2+1}{21} = \frac{3}{21}$$

$$f(2,2) = \frac{2+2}{21} = \frac{4}{21}$$

$$f(3,1) = \frac{3+1}{21} = \frac{4}{21}$$

$$f(3,2) = \frac{3+2}{21} = \frac{5}{21}$$

Marginal distribution X & Y

$$P(X=1) = \frac{5}{21} ; P(X=2) = \frac{7}{21} , P(X=3) = \frac{9}{21}$$

$$P(Y=1) = \frac{9}{21} ; P(Y=2) = \frac{12}{21}$$

Conditional Distribution:

$$\frac{P(1,1)}{P(X=1)} = \frac{\frac{2}{21}}{\frac{5}{21}} = \frac{2}{5}$$

$$\frac{P(2,1)}{P(X=1)} = \frac{\frac{3}{21}}{\frac{5}{21}} = \frac{3}{5}$$

$$\frac{P(3,1)}{P(X=1)} =$$

$$\frac{P(1,2)}{P(X=2)} =$$

$$\frac{P(2,2)}{P(X=2)} =$$

$$\frac{P(3,2)}{P(X=2)} =$$

$X \backslash Y$	1	2	Total
1	$\frac{2}{21}$	$\frac{3}{21}$	$\frac{5}{21}$
2	$\frac{3}{21}$	$\frac{4}{21}$	$\frac{7}{21}$
3	$\frac{4}{21}$	$\frac{5}{21}$	$\frac{9}{21}$
Total	$\frac{9}{21}$	$\frac{12}{21}$	1

Conditional distribution

X	Y
<u>$X=1$</u> $\frac{P(1,1)}{P(X=1)} = \frac{2}{21} \times \frac{21}{5} = \frac{2}{5}$ $\frac{P(1,2)}{P(X=1)} = \frac{3}{21} \times \frac{21}{5} = \frac{3}{5}$ <u>$X=2$</u> $\frac{P(2,1)}{P(X=2)} = \frac{3}{21} \times \frac{21}{7} = \frac{3}{7}$ $\frac{P(2,2)}{P(X=2)} = \frac{4}{21} \times \frac{21}{7} = \frac{4}{7}$ <u>$X=3$</u> $\frac{P(3,1)}{P(X=3)} = \frac{4}{21} \times \frac{21}{9} = \frac{4}{9}$ $\frac{P(3,2)}{P(X=3)} = \frac{5}{21} \times \frac{21}{9} = \frac{5}{9}$	<u>$Y=1$</u> $\frac{P(1,1)}{P(Y=1)} = \frac{2}{21} \times \frac{21}{9} = \frac{2}{9}$ $\frac{P(2,1)}{P(Y=1)} = \frac{3}{21} \times \frac{21}{9} = \frac{1}{3}$ $\frac{P(3,1)}{P(Y=1)} = \frac{4}{21} \times \frac{21}{9} = \frac{4}{9}$ <u>$Y=2$</u> $\frac{P(1,2)}{P(Y=2)} = \frac{3}{21} \times \frac{21}{12} = \frac{1}{4}$ $\frac{P(2,2)}{P(Y=2)} = \frac{4}{21} \times \frac{21}{12} = \frac{1}{3}$ $\frac{P(3,2)}{P(Y=2)} = \frac{5}{21} \times \frac{21}{12} = \frac{5}{12}$

Continuous Random Variables

- ① The joint probability density function of a two-dimensional random variable (x, y) is given by $f(x, y) = xy^2 + \frac{x^2}{8}$; $0 \leq x \leq 2$, $0 \leq y \leq 1$. Compute
- (1) $P(X > 1)$
 - (2) $P(Y < \frac{1}{2})$
 - (3) $P[X > 1 / Y < \frac{1}{2}]$

Soln:

Given $f(x, y) = xy^2 + \frac{x^2}{8}$; $0 \leq x \leq 2$, $0 \leq y$.

Marginal Density of X:

$$\begin{aligned} f(x) &= \int_{-\infty}^{\infty} f(x, y) dy = \int_0^1 \left(xy^2 + \frac{x^2}{8} \right) dy \\ &= \left[\frac{xy^3}{3} + \frac{x^2}{8} y \right]_0^1 = \left(\frac{x}{3} + \frac{x^2}{8} - 0 \right) \end{aligned}$$

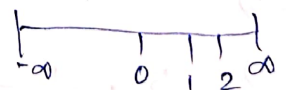
$$\boxed{f(x) = \frac{x}{3} + \frac{x^2}{8}}$$

marginal Density of Y:

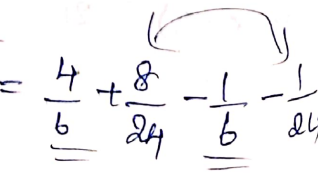
$$\begin{aligned} f(y) &= \int_{-\infty}^{\infty} f(x, y) dx = \int_0^2 \left(xy^2 + \frac{x^2}{8} \right) dx \\ &= \left[\frac{x^2}{2} y^2 + \frac{x^3}{8 \times 3} \right]_0^2 = \left[\frac{(2)^2}{2} y^2 + \frac{(2)^3}{8 \times 3} - 0 \right] \end{aligned}$$

$$\boxed{f(y) = 2y^2 + \frac{1}{3}}$$

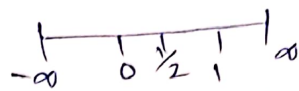
① $P(X > 1) = \int_1^2 f(x) dx$



$$\begin{aligned} &= \int_1^2 \left(\frac{x}{3} + \frac{x^2}{8} \right) dx = \left[\frac{x^2}{6} + \frac{x^3}{24} \right]_1^2 \\ &= \frac{3}{6} + \frac{7}{24} = \frac{12+7}{24} = \frac{19}{24} \end{aligned}$$



$$\textcircled{2} P\left[y < \frac{1}{2}\right] = \int_0^{\frac{1}{2}} f(y) dy$$



$$= \int_0^{\frac{1}{2}} \left(2y^2 + \frac{1}{3}\right) dy = \left[\frac{2y^3}{3} + \frac{1}{3}y\right]_0^{\frac{1}{2}}$$

$$= \left[\frac{2}{3} \left(\frac{1}{8}\right) + \frac{1}{3} \left(\frac{1}{2}\right) - 0\right]$$

$$= \frac{1}{12} + \frac{1}{6} = \frac{1+2}{12} = \frac{3}{12} = \frac{1}{4}$$

$$\textcircled{3} P[X > 1 \mid y < \frac{1}{2}] = \frac{\int_{x=1}^2 \int_{y=0}^{\frac{1}{2}} \left(xy^2 + \frac{x^2}{8}\right) dy dx}{\left(\frac{1}{4}\right)}$$

$$= 4 \int_1^2 \left[\frac{xy^3}{3} + \frac{x^2y}{8}\right]_0^{\frac{1}{2}} dx$$

$$= 4 \int_1^2 \left(x \left(\frac{1}{2}\right)^3 + \frac{x^2 \left(\frac{1}{2}\right)}{8}\right) dx$$

$$= 4 \int_1^2 \left(\frac{x}{24} + \frac{x^2}{16}\right) dx$$

$$= 4 \left[\frac{x^2}{48} + \frac{x^3}{48}\right]_1^2$$

$$= \frac{4}{48} (x^2 + x^3)_1^2$$

$$= \frac{1}{12} ((2^2 + 2^3) - (1^2 + 1^3))$$

$$= \frac{1}{12} (4 + 8 - 2)$$

$$= \frac{1}{12} (10)$$

$$= \frac{5}{6}$$

$$\textcircled{4} P[X < Y] = \int_{y=0}^1 \int_{x=0}^y \left(xy^2 + \frac{x^2}{8}\right) dx dy$$

$$= \int_0^1 \left[\frac{x^2y^2}{2} + \frac{x^3}{24}\right]_{x=0}^{x=y} dy$$

$$= \int_0^1 \left(\frac{y^2y^2}{2} + \frac{y^3}{24}\right) dy$$

$$= \left[\frac{y^5}{10} + \frac{y^4}{96}\right]_0^1 = \frac{1}{10} + \frac{1}{96} = \frac{96+10}{960} = \frac{106}{960}$$

$$= \frac{53}{480}$$

② Determine the value of c that makes the function $F(x,y) = c(x+y)$ a joint probability density function over the range $0 < x < 3$ & $x < y < x+2$. Also Determine the following

i) $P(X < 1, Y < 2)$

ii) $P(Y > 2)$

iii) $E(X)$.

Soln:

Given $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(x,y) dx dy = 1$

$$= x^2 + 2x + \frac{1}{2}(x^2 + 4 + 4x) - \frac{x^2}{2} - \frac{x^2}{2}$$

$$= 2x + \frac{1}{2}(4) + \frac{2}{2}x$$

$$= 4x + 2$$

$$c \int_0^3 \int_x^{x+2} (x+y) dy dx = 1$$

$$c \int_0^3 \left[xy + \frac{y^2}{2} \right]_x^{x+2} dx = 1$$

$$c \int_0^3 \left(x(x+2) + \frac{(x+2)^2}{2} - \left(x^2 + \frac{x^2}{2} \right) \right) dx = 1$$

$$c \int_0^3 (4x + 2) dx = 1$$

$$c \left(\frac{4x^2}{2} + 2x \right)_0^3 = 1$$

$$c \left(\frac{4 \times 9}{2} + 2 \times 3 \right) = 1$$

$$c (18 + 6) = 1$$

$$c (24) = 1$$

$$\boxed{c = \frac{1}{24}}$$

i) $P(X < 1; Y < 2) = \int_0^1 \int_x^{y=2} \frac{1}{24} (x+y) dy dx$

$$= \frac{1}{24} \int_0^1 \left(2x + \frac{2^2}{2} - \left(x^2 + \frac{x^2}{2} \right) \right) dx = \frac{1}{24} \int_0^1 \left(xy + \frac{y^2}{2} \right)_{y=x}^{y=2} dx$$

$$= \frac{1}{24} \int_0^1 \left(2x + 2 - \frac{3x^2}{2} \right) dx = \frac{1}{24} \int_0^1 \left(2x + \frac{4}{2} - \frac{3x^2}{2} \right) dx$$

$$= \frac{1}{24} \left(\frac{2x^2}{2} + 2x - \frac{3x^3}{6} \right)_0^1 = \frac{1}{24} \left(\frac{2x^2}{2} + 2x - \frac{x^3}{2} \right)_0^1$$

$$= \frac{1}{24} \left(1^2 + 2(1) - \frac{3(1)^3}{6} \right) = \frac{1}{24} \left(\frac{2x^2}{2} + \frac{x^3}{6} \right)_0^1$$

$$= \frac{1}{24} \left(3 - \frac{3}{6} \right) = \frac{1}{24} \left(1 + \frac{1}{6} \right)$$

$$= \frac{1}{24} \left(\frac{18+3}{6} \right) = \frac{1}{24} \left(\frac{7}{6} \right)$$

$$= \frac{1}{24} \left(\frac{15}{6} \right)$$

$$= \frac{5}{48}$$

③ The joint pdf of the random variables x, y is defined as

$$f(x, y) = \begin{cases} 25e^{-5y} & ; 0 < x < 0.2 ; y > 0 \\ 0 & ; \text{otherwise} \end{cases}$$

(i) Find the marginal pdf's of x & y .

(ii) What is the covariance of x & y ?

(iii) What is the correlation coefficient of x & y ?

Soln: (iv) Are x & y are independent.

(i) Marginal Density of x & y .

$$\begin{aligned} f(x) &= \int_0^{\infty} 25e^{-5y} dy \\ &= 25 \int_0^{\infty} e^{-5y} dy \\ &= 25 \left[\frac{e^{-5y}}{-5} \right]_0^{\infty} \\ &= -5 \left[e^{-5y} \right]_0^{\infty} \\ &= -5 (e^{\infty} - e^0) \\ &= -5 (0 - 1) \\ &= 5 \end{aligned}$$

$$\begin{cases} e^{\infty} = 0 \\ e^0 = 1 \end{cases}$$

$$\boxed{f(x) = 5}$$

$$\begin{aligned} f(y) &= \int_0^{0.2} 25e^{-5y} dx \\ &= 25e^{-5y} \int_0^{0.2} dx \\ &= 25e^{-5y} \left[x \right]_0^{0.2} \\ &= 25e^{-5y} (0.2 - 0) \\ &= 25e^{-5y} (0.2) \\ &= 5e^{-5y} \end{aligned}$$

$$\boxed{f(y) = 5e^{-5y}}$$

(ii) Covariance of x & y .

Formula:

$$\text{Cov}(x, y) = E(xy) - E(x)E(y)$$

$$\begin{aligned} E(x) &= \int_{-\infty}^{\infty} x f(x) dx \\ &= \int_0^{0.2} x (5) dx \\ &= 5 \int_0^{0.2} x dx \\ &= 5 \left[\frac{x^2}{2} \right]_0^{0.2} \\ &= \frac{5}{2} (0.2)^2 \\ &= \frac{5}{2} (0.04) \\ &= 0.1 \end{aligned}$$

$$\boxed{E(x) = 0.1}$$

$$\begin{aligned} E(y) &= \int_{-\infty}^{\infty} y f(y) dy \\ &= \int_0^{\infty} y (5e^{-5y}) dy \\ &= 5 \int_0^{\infty} y e^{-5y} dy \\ &= 5 \left[y \left(\frac{e^{-5y}}{-5} \right) - (1) \left(\frac{e^{-5y}}{25} \right) \right]_0^{\infty} \\ &= 5 \left[(0 - 0) - \left(0 - \frac{e^0}{25} \right) \right] \\ &= 5 \left(\frac{1}{25} \right) = \frac{1}{5} \end{aligned}$$

$$\boxed{E(y) = \frac{1}{5}}$$

$$\begin{aligned}
 E(xy) &= \int_0^\infty \int_0^y xy f(x,y) dx dy \\
 &= \int_0^\infty \int_0^y xy (25e^{-5y}) dx dy \\
 &= 25 \int_0^\infty \int_0^y xy e^{-5y} dx dy \\
 &= 25 \int_0^\infty y e^{-5y} dy \int_0^y x dx \\
 &= 25 \left(\frac{1}{25} \right) \left(\frac{x^2}{2} \right)_0^y \\
 &= \frac{0.2^2}{2} = \frac{0.04}{2} = 0.02
 \end{aligned}$$

$$\int_0^\infty y e^{-5y} dy = \frac{1}{25}$$

$$E(xy) = 0.02$$

$$\begin{aligned}
 \text{Cov}(x,y) &= E(xy) - E(x)E(y) \\
 &= 0.02 - (0.1)\left(\frac{1}{5}\right) \\
 &= 0.02 - 0.02
 \end{aligned}$$

$$\text{Cov}(x,y) = 0$$

(iii) Correlation coefficient of x & y

Formula

$$r(x,y) = \frac{\text{Cov}(x,y)}{\sigma_x \sigma_y} = 0$$

(iv) Independent: $f(x,y) = f(x)f(y) \Rightarrow 5 \times 5 e^{-5y} = 25 e^{-5y}$
 $25 e^{-5y} = 25 e^{-5y}$
 x & y are independent.

(4) Given the joint p.d.f of (x,y) as

$$f(x,y) = \begin{cases} 8xy, & 0 < x < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

Find the marginal & conditional p.d.f of x & y . Are x & y are independent?

Soln:

Given $f(x,y) = \begin{cases} 8xy, & 0 < x < y < 1 \\ 0, & \text{otherwise} \end{cases}$

Marginal Density function of x & y :

$$f(x) = \int_{-\infty}^{\infty} f(x,y) dy = \int_x^1 8xy dy = 4x \left[\frac{y^2}{2} \right]_x^1$$

$$\begin{aligned}
 f(y) &= \int_{-\infty}^{\infty} f(x,y) dx \\
 &= \int_0^y 8xy dx \\
 &= 4y \left[\frac{x^2}{2} \right]_0^y \\
 &= 4y(y^2)
 \end{aligned}$$

$$f(y) = 4y^3$$

Conditional distribution

$$f(x|y) = \frac{f(x,y)}{f(y)} = \frac{8xy}{4y^3} = \frac{2x}{y^2}$$

$$f(y|x) = \frac{f(x,y)}{f(x)} = \frac{8xy}{4x(1-x^2)} = \frac{2y}{1-x^2}$$

Independent

$$\begin{aligned}
 f(x) \cdot f(y) &= f(x,y) \rightarrow \text{independent} \\
 f(x) \cdot f(y) &\neq f(x,y) \rightarrow \text{not independent}
 \end{aligned}$$

$$\begin{aligned}
 f(x) f(y) &= 4x(1-x^2) 4y^3 \\
 &= 16xy^3(1-x^2)
 \end{aligned}$$

$$f(x) f(y) \neq f(x,y)$$

$\therefore x$ & y are not independent.

⑤ Find k if the joint p.d.f of a bivariate r.v. (x,y) is given by

$$f(x,y) = \begin{cases} k(1-x)(1-y) & \text{if } 0 < x < 1, 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Soln:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) dx dy = 1$$

$$\int_0^1 \int_0^1 k (1-x)(1-y) dx dy = 1$$

$$k \int_0^1 (1-x) dx \int_0^1 (1-y) dy = 1$$

$$k \left[x - \frac{x^2}{2} \right]_0^1 \left[y - \frac{y^2}{2} \right]_0^1 = 1$$

$$k \left(1 - \frac{1}{2} \right) \left(1 - \frac{1}{2} \right) = 1$$

$$k \left(\frac{1}{2} \right) \left(\frac{1}{2} \right) = 1$$

$$\boxed{k = 4}$$

Unit-II (2 marks)

- ① Find the acute angle between the two line of regression.

Soln ∴

$$\tan \theta = \left| \left(\frac{1-r^2}{r} \right) \left(\frac{\sigma_x \sigma_y}{\sigma_x^2 + \sigma_y^2} \right) \right|$$

- ② If X & Y are independent r.v's with variance 2 & 3, then find the variance of $3X+4Y$.

Soln

Given $\text{Var}(X) = 2$

$\text{Var}(Y) = 3$.

$$\text{Var}(3X+4Y) = a^2 \text{Var}(X) + b^2 \text{Var}(Y)$$

$$= 3^2 \times 2 + 4^2 \times 3$$

$$= 9 \times 2 + 16 \times 3$$

$$= 18 + 48$$

$$\boxed{\text{Var}(3X+4Y) = 66}$$

- ③ Define covariance & coefficient of correlation between two r.v's X & Y .

Soln ∴

Covariance ∴

$$\text{Cov}(X, Y) = E(XY) - E(X)E(Y)$$

Coefficient of correlation ∴

$$r(X, Y) = \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y}$$

Where $\sigma_X = \sqrt{\frac{1}{n} \sum x^2 - \bar{x}^2}$

$$\sigma_Y = \sqrt{\frac{1}{n} \sum y^2 - \bar{y}^2}$$

4) Let x & y be two independent R.V's with $\text{var}(x) = 9$ & $\text{var}(y) = 3$. Find $\text{var}(4x - 2y + 6)$

Soln:
Given $\text{var}(x) = 9$ &
 $\text{var}(y) = 3$.

$$\begin{aligned}\text{var}(4x - 2y + 6) &= 4^2 \text{var}(x) + (-2)^2 \text{var}(y) \\ &= 16 \times 9 + 4 \times 3 \\ &= 144 + 12\end{aligned}$$

$$\boxed{\text{var}(4x - 2y + 6) = 156}$$

5) State the Central limit theorem.
(or)
Lindberg-Levy's form

Statement:

Let $x_1, x_2, \dots, x_n$ be a sequence of independent identically distributed r.v's with $E(x_i) = \mu$ & $\text{var}(x_i) = \sigma^2$, $i = 1, 2, \dots, n$ & if $S_n = x_1 + x_2 + \dots + x_n$, then under certain general conditions S_n follows a normal distribution with mean $= n\mu$ & variance $= n\sigma^2$ as $n \rightarrow \infty$.

6) Find the value of k if $f(x, y) = kxy e^{-(x^2 + y^2)}$, $x \geq 0, y \geq 0$ is the joint Pdf.

Soln:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dy dx = 1$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} kxy e^{-(x^2 + y^2)} dy dx = 1$$

$$k \int_{-\infty}^{\infty} x e^{-x^2} dx \int_{-\infty}^{\infty} y e^{-y^2} dy = 1$$

$$k \int_0^{\infty} e^{-t} \frac{1}{2} dt \int_0^{\infty} e^{-t} \frac{1}{2} dt$$

Put $x^2 = t$	$x \rightarrow 0$
$2x dx = dt$	$t \rightarrow 0$
$x dx = \frac{1}{2} dt$	$x \rightarrow \infty$
	$t \rightarrow \infty$

$$k \cdot \frac{1}{2} \left(\frac{e^{-t}}{-1} \right)_0^\infty - \frac{1}{2} \left(\frac{e^{-t}}{-1} \right)_0^\infty = 1$$

$$e^\infty = 0$$

$$e^0 = 1$$

$$\frac{k}{4} (e^\infty - e^0) (e^\infty - e^0) = 1$$

$$\frac{k}{4} (0 - 1) (0 - 1) = 1$$

$$\frac{k}{4} = 1$$

$$\boxed{k = 4}$$

⑦ The joint pmf of two dimensional r.v x & y is given by $f(x,y) = \begin{cases} kxy; & x=1,2,3; y=1,2,3 \\ 0; & \text{otherwise} \end{cases}$
Find the value of k .

~~XXXX~~

$x \backslash y$	1	2	3	Total
1	k	$2k$	$3k$	$6k$
2	$2k$	$4k$	$6k$	$12k$
3	$3k$	$6k$	$9k$	$18k$
Total	$6k$	$12k$	$18k$	$\frac{36k}{36k}$

$f(x,y) = kxy$

$f(1,1) = k$
 $f(1,2) = k(1)(2) = 2k$
 $f(1,3) = k(1)(3) = 3k$
 $f(2,1) = k(2)(1) = 2k$
 $f(2,2) = k(2)(2) = 4k$
 $f(2,3) = k(2)(3) = 6k$
 $f(3,1) = k(3)(1) = 3k$
 $f(3,2) = k(3)(2) = 6k$
 $f(3,3) = k(3)(3) = 9k$

⑧ If the joint pdf of a bivariate r.v (x,y) is given by $f(x,y) = \begin{cases} k(1-x)(1-y) & \text{if } 0 < x < 1, 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$
Find the value of k .

Total probability: $36k = 1$
 $\boxed{k = \frac{1}{36}}$

Soln: $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) dx dy = 1$

$$\int_0^1 \int_0^1 k(1-x)(1-y) dx dy = 1$$

$$k \int_0^1 (1-x) dx \int_0^1 (1-y) dy = 1$$

$$k \left(x - \frac{x^2}{2} \right)_0^1 \left(y - \frac{y^2}{2} \right)_0^1 = 1$$

$$K(1 - \frac{1}{2} - 0)(1 - \frac{1}{2} - 0) = 1$$

$$K(\frac{1}{2})(\frac{1}{2}) = 1$$

$$\frac{K}{4} = 1$$

$$K = 4$$

9) State the equations of two regression lines & angle between them?

Regression line ∴

$$y - \bar{y} = r \frac{\sigma_y}{\sigma_x} (x - \bar{x})$$

$$x - \bar{x} = r \frac{\sigma_x}{\sigma_y} (y - \bar{y})$$

Angle

$$\tan \theta = \left| \frac{1-r^2}{r} \left(\frac{\sigma_x \sigma_y}{\sigma_x^2 + \sigma_y^2} \right) \right|$$

10) If the joint pdf of (x,y) is given by

$$f(x,y) = 6 \begin{cases} e^{-(x+y)} & ; x, y \geq 0 \\ 0 & ; \text{otherwise} \end{cases} \quad \text{Check}$$

whether x & y are independent.

Soln:

Marginal density of x & y:

$$\begin{aligned} f(x) &= \int_{-\infty}^{\infty} f(x,y) dy \\ &= \int_0^{\infty} e^{-(x+y)} dy \\ &= e^{-x} \int_0^{\infty} e^{-y} dy \\ &= e^{-x} [-e^{-y}]_0^{\infty} \\ &= -e^{-x} (e^{\infty} - e^0) \\ &= -e^{-x} (0 - 1) \\ &= e^{-x} \end{aligned}$$

$$\begin{matrix} \infty \\ e=0 \\ e^0=1 \end{matrix}$$

$$f(x) = e^{-x}$$

$$\begin{aligned} f(y) &= \int_{-\infty}^{\infty} f(x,y) dx \\ &= \int_0^{\infty} e^{-(x+y)} dx \\ &= e^{-y} \int_0^{\infty} e^{-x} dx \\ &= e^{-y} [e^{-x}]_0^{\infty} \\ &= -e^{-y} (e^{\infty} - e^0) \\ &= -e^{-y} (0 - 1) \\ &= e^{-y} \end{aligned}$$

$$f(y) = e^{-y}$$

Independent ∴

$$f(x) f(y) = f(x,y)$$

$$e^{-x} e^{-y} = e^{-(x+y)}$$

x & y are independent.

- (11) Let x & y be r.v.'s with joint density function
 $f_{xy}(x,y) = \begin{cases} 4xy & ; 0 \leq x \leq 1; 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$
 Find $E(xy)$.

Soln

$$\begin{aligned} E(xy) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f(x,y) dx dy \\ &= \int_0^1 \int_0^1 xy (4xy) dx dy \\ &= 4 \int_0^1 x^2 dx \int_0^1 y^2 dy \\ &= 4 \left[\frac{x^3}{3} \right]_0^1 \left[\frac{y^3}{3} \right]_0^1 \\ &= \frac{4}{9} (1-0) (1-0) \end{aligned}$$

$$\boxed{E(xy) = \frac{4}{9}}$$

- (12) Let (x,y) be a two dimensional r.v. Define covariance of (x,y) . If x & y are independent, what will be the covariance of (x,y) .

Soln:

$$\text{Cov}(x,y) = E(xy) - E(x)E(y)$$

If x & y are independent, then $\text{Cov}(x,y) = 0$

- (13) what do you mean by Correlation between two random variables?

Soln

Degree of relationship & nature "

- (14) The joint P.m.f of a two dimensional r.v. is given by $P(x,y) = K(2x+y)$; $x=1, 2$ & $y=1, 2$ where K is a Constant. Find the value of

$x \backslash y$	1	2	Total
1	$3K$	$4K$	$7K$
2	$5K$	$6K$	$11K$
Total	$8K$	$10K$	$\frac{18K}{18K}=1$

$$\begin{aligned} P(x,y) &= K(2x+y) \\ P(1,1) &= K(2(1)+1) \\ P(1,2) &= K(2(1)+2) \\ P(2,1) &= K(2(2)+1) \\ P(2,2) &= K(2(2)+2) \\ \text{Total probability } \sum_{x,y} P(x,y) &= 1 \end{aligned}$$

8 marks

Unit - III

Analytic Functions

Bilinear Transformation

The transformation $w = \frac{az+b}{cz+d}$ is called a bilinear transformation if $ad-bc \neq 0$.

Formula:-

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

① Find the bilinear transformation that maps the points $z = 0, -1, i$ into the points $w = i, 0, \infty$ respectively.

Soln:- Given $z_1 = 0, z_2 = -1, z_3 = i$
 $w_1 = i, w_2 = 0, w_3 = \infty$

Formula:-

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

omit $w_3 = \infty$

$$\frac{w-w_1}{w_2-w_1} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{w-i}{0-i} = \frac{(z-0)(-1-i)}{(z-i)(-1-0)}$$

$$\frac{w-i}{-i} = \frac{z}{z-i} (1+i)$$

$$w-i = \frac{z}{z-i} (1+i)(-i)$$

$$w-i = \frac{z}{z-i} (-i-i^2)$$

$$w-i = \frac{z}{z-i} (-i+1)$$

$$w = \frac{z}{z-i} (-i+1) + i$$

$$= \frac{-zi+z+i(z-i)}{z-i}$$

$$= \frac{-zi+z+iz-i^2}{z-i}$$

$$w = \frac{z+1}{z-i}$$

$$i^2 = -1$$

② Find the bilinear transformation that maps points $\infty, i, 0$ onto $0, i, \infty$ respectively.

Soln:

Given
$$\begin{array}{l} z_1 = \infty \\ z_2 = i \\ z_3 = 0 \end{array} \quad \begin{array}{l} w_1 = 0 \\ w_2 = i \\ w_3 = \infty \end{array}$$

Formula:

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

Omit $z_1 = \infty, w_3 = \infty$

$$\frac{w-w_1}{w_2-w_1} = \frac{z-z_3}{z_2-z_3}$$

$$\frac{w-0}{i-0} = \frac{z-0}{i-0}$$

$$\frac{w}{i} = \frac{z}{i}$$

$$w = \frac{z^2}{z}$$

$$w = -\frac{1}{z} \quad (\text{or}) \quad w = \frac{0z + (-1)}{1z + 0}$$

$$w = \frac{az+b}{cz+d}$$

③ Find the bilinear transformation which maps the points $1, i, -1$ onto the points $0, 1, \infty$ respectively.

Soln:

Given
$$\begin{array}{l} z_1 = 1, z_2 = i, z_3 = -1 \\ w_1 = 0, w_2 = 1, w_3 = \infty \end{array}$$

Formula:

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{w-w_1}{w_2-w_1} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{w-0}{1-0} = \frac{(z-1)(i+1)}{(z+1)(i-1)}$$

Omit $w_3 = \infty$

$$w = \frac{(z-1)(-i)}{z+1}$$

$$\frac{i+1}{i-1} = \frac{i+1}{i-1} \times \frac{i+1}{i+1}$$

$$= \frac{i^2 + i + i + 1}{i^2 - 1^2}$$

$$= \frac{2i}{-2} = -i$$

$$(a+bi)^2 = a^2 + b^2 + 2iab$$

$$w = \frac{-iz + i}{(1)z + 1}$$



Q) Find the bilinear transformation which maps $z = 1, i, -1$ respectively onto $w = i, 0, -i$. Hence find the fixed points.

Soln:

Given $z_1 = 1, z_2 = i, z_3 = -1$

$w_1 = i, w_2 = 0, w_3 = -i$

Formula:

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{w-i}{i-0} \cdot \frac{0+i}{-i-w} = \frac{z-1}{1-i} \cdot \frac{i+1}{-1-i}$$

$$\frac{w-i}{-w+i} = \left(\frac{z-1}{z+1} \right) \left(\frac{i+1}{i-1} \right)$$

$$\frac{w-i}{w+i} = - \left(\frac{z-1}{z+1} \right) (-i)$$

$$\frac{w-i}{w+i} = \frac{zi-i}{z+1}$$

$$\frac{(w-i) + (w+i)}{(w-i) - (w+i)} = \frac{(zi-i) + (z+1)}{(zi-i) - (z+1)}$$

by componendo & dividendo

$$\frac{2w}{-2i} = \frac{z(i+1) + (1-i)}{z(i-1) - (1+i)}$$

$$w = i \left(\frac{z(i+1) + (1-i)}{z(i-1) - (1+i)} \right)$$

$$\frac{A}{B} = \frac{C}{D} \Rightarrow \frac{A+B}{A-B} = \frac{C+D}{C-D}$$

⑤ Find the bilinear transformation which maps the points $z=0, z=1, z=\infty$ into the points $w=i, w=1$ & $w=-i$.

Soln:

Given $z_1=0, z_2=1, z_3=\infty$

$w_1=i, w_2=1, w_3=-i$

Formula:

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

omit $z_3=\infty$

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{z-z_1}{z_2-z_1}$$

$$\frac{(w-i)(1+i)}{(w+i)(1-i)} = \frac{z-0}{1-0}$$

$$\frac{w-i}{w+i} = z \left(\frac{1-i}{1+i} \right)$$

$$\frac{w-i}{w+i} = z(-i)$$

$$w-i = -zi(w+i)$$

$$w-i = -zwi - zi^2$$

$$w-i = -zwi + z$$

$$w + zwi = z + i$$

$$w(1+zi) = z+i$$

$$w = \frac{z+i}{1+zi}$$

$$\frac{1-i}{1+i} = \frac{1-i}{1+i} \times \frac{1-i}{1-i}$$

$$= \frac{1-i^2-2i}{1-i^2+2i}$$

$$= \frac{1-(-1)-2i}{1-(-1)+2i}$$

$$= \frac{1+1-2i}{1+1+2i}$$

$$= \frac{2-2i}{2+2i}$$

$$= \frac{2(1-i)}{2(1+i)}$$

$$= \frac{1-i}{1+i}$$

$$\frac{1-i}{1+i} = \frac{1-i}{1+i} \times \frac{1-i}{1-i}$$

$$= \frac{1-i^2-2i}{1-i^2+2i}$$

$$= \frac{1-(-1)-2i}{1-(-1)+2i}$$

$$= \frac{1+1-2i}{1+1+2i}$$

$$= \frac{2-2i}{2+2i}$$

$$= \frac{2(1-i)}{2(1+i)}$$

$$= \frac{1-i}{1+i}$$

⑥ Find the bilinear transformation that maps $z_1=0, z_2=1, z_3=\infty$ onto $w_1=-1, w_2=-i, w_3=1$.

Soln:

Given $z_1=0, z_2=1, z_3=\infty$
 $w_1=-1, w_2=-i, w_3=1$

Formula :-

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{z-z_1}{z_2-z_1}$$

omit $z_3=\infty$

$$\frac{(w+1)(-i-1)}{(w-1)(1-i)} = \frac{z-0}{1-0}$$

$$\frac{w+1}{w-1} = -z \left(\frac{1-i}{1+i} \right)$$

$$= -z(-i)$$

$$= zi$$

$$\frac{w+1}{w-1}$$

$$w+1 = zi(w-1)$$

$$= zwi - zi$$

$$w+1 \rightarrow w - zwi = -1 - zi$$

$$w(1-zi) = -1 - zi$$

$$w(1-zi) = -1 - zi$$

$$w = \frac{-1-zi}{1-zi}$$

$$\frac{1-i}{1+i} = -i$$

verification :-

$$\frac{z=0}{w = \frac{-1-0}{1-0}}$$

$$w = -1$$

(gm)

Image Method

Unit-III

- ① Find the image of the region $y > 1$ under the transformation $w = (1-i)z$.

Soln:

$$\text{Let } w = (1-i)z$$

$$u+iv = (1-i)(x+iy)$$

$$= x+iy - ix - i^2 y$$

$$= \underline{x+iy} - \underline{ix} + \underline{y}$$

$$u+iv = \underline{x+y} + i\underline{(y-x)}$$

$$\begin{aligned} v &= y-x \\ x &= y-v \end{aligned}$$

$$u = x+y$$

$$u = (y-v) + y$$

$$u = 2y - v$$

$$\boxed{u+v = 2y}$$

$$\boxed{y = \frac{u+v}{2}}$$

$$v = y-x$$

$$v = u - x - x$$

$$v = u - 2x$$

$$2x = u - v$$

$$\boxed{x = \frac{u-v}{2}}$$

$$\begin{aligned} u &= x+y \\ u-x &= y \end{aligned}$$

∴ Hence image region $y > 1$ is $\frac{u+v}{2} > 1$ in the w -plane.

- ② Find the image of the circle $|z| = 1$ under the transformation $w = 5z$

Soln:

$$\text{Let } w = 5z$$

$$u+iv = 5(x+iy)$$

$$u+iv = 5x + 5iy$$

$$u = 5x \Rightarrow x = \frac{u}{5}$$

$$v = 5y \Rightarrow y = \frac{v}{5}$$

Given $|z| = \lambda$

$$|x+iy| = \lambda$$

$$x^2 + y^2 = \lambda^2$$

$$\left(\frac{u}{5}\right)^2 + \left(\frac{v}{5}\right)^2 = \lambda^2$$

$$\frac{u^2}{25} + \frac{v^2}{25} = \lambda^2$$

$$\frac{u^2 + v^2}{25} = \lambda^2$$

$$u^2 + v^2 = 25\lambda^2$$

$$u^2 + v^2 = (5\lambda)^2$$

③ Find the image $2x+y-3=0$ under the transformation $w = z+2i$

Soln:

Let $w = z+2i$

$$u+iv = (x+iy)+2i$$

$$u+iv = x+i(y+2)$$

$$u = x ; v = y+2$$

$$x = u ; y = v-2$$

Given $2x+y-3=0$

$$2u + (v-2) - 3 = 0$$

$$2u + v - 5 = 0$$

(4) Find the image of $x=2$ under $w=\frac{1}{z}$.

Soln:

$$\text{let } w = \frac{1}{z}$$

$$z = \frac{1}{w}$$

$$x+iy = \frac{1}{u+iv}$$

$$= \frac{1}{u+iv} \times \frac{u-iv}{u-iv}$$

$$= \frac{u-iv}{u^2+v^2}$$

$$x+iy = \frac{u}{u^2+v^2} + i \left(\frac{-v}{u^2+v^2} \right)$$

$$x = \frac{u}{u^2+v^2}; \quad y = \frac{-v}{u^2+v^2} \rightarrow (2)$$

$\hookrightarrow (1)$

Given $x=2$

$$2 = \frac{u}{u^2+v^2} \text{ by (1)}$$

$$2(u^2+v^2) = u$$

$$2u^2+2v^2-u=0$$

$$\div 2 \quad u^2+v^2-\frac{u}{2}=0$$

$$\left(u-\frac{1}{4}\right)^2+v^2-\frac{1}{16}=0$$

$$\left(u-\frac{1}{4}\right)^2+v^2=\left(\frac{1}{4}\right)^2$$

∴ which is a circle whose centre is

$(\frac{1}{4}, 0)$ and radius $\frac{1}{4}$.

$x=2$ in the z plane is transformed into a circle.

$$(u - \frac{1}{4})^2 + v^2 = (\frac{1}{4})^2 \text{ in the } w \text{ plane.}$$

⑤ Find the image of $|z - 2i| = 2$ under the transformation $w = \frac{1}{z}$.

Soln:

let $w = \frac{1}{z}$

$$z = \frac{1}{w}$$

$$\begin{aligned} \therefore x + iy &= \frac{1}{u + iv} \\ &= \frac{1}{u + iv} \times \frac{u - iv}{u - iv} \\ &= \frac{u - iv}{u^2 + v^2} \end{aligned}$$

$$x + iy = \frac{u}{u^2 + v^2} + i \left(\frac{-v}{u^2 + v^2} \right)$$

$$x = \frac{u}{u^2 + v^2} \rightarrow \textcircled{1} ; y = \frac{-v}{u^2 + v^2} \rightarrow \textcircled{2}$$

Given $|z - 2i| = 2$

$$|x + iy - 2i| = 2$$

$$|x + i(y - 2)| = 2$$

$$x^2 + (y - 2)^2 = 4$$

$$x^2 + y^2 + 4 - 4y = 4$$

$$x^2 + y^2 + 4 - 4y - 4 = 0$$

$$x^2 + y^2 - 4y = 0 \rightarrow \textcircled{3}$$

Sub ① & ② in ③

$$\left(\frac{u}{u^2 + v^2} \right)^2 + \left(\frac{-v}{u^2 + v^2} \right)^2 - 4 \left(\frac{-v}{u^2 + v^2} \right) = 0$$

$$\frac{u^2}{(u^2 + v^2)^2} + \frac{v^2}{(u^2 + v^2)^2} + \frac{4v}{u^2 + v^2} = 0$$

$$\frac{u^2 + v^2 + 4v(u^2 + v^2)}{(u^2 + v^2)^2} = 0$$

$$u^2 + v^2 + 4v(u^2 + v^2) = 0$$

$$\frac{(u^2 + v^2)(1 + 4v)}{(u^2 + v^2)^2} = 0$$

$$\frac{1 + 4v}{u^2 + v^2} = 0$$

$$\boxed{1 + 4v = 0}$$

which is a straight line in w plane.

Unit-11
 ⑦ Show that the map $w = \frac{1}{z}$ maps the totality of circles and lines as circles or lines.

(or)
 Show that the transformation $w = \frac{1}{z}$ transform all circles and straight lines in the z -plane into circles or straight lines in the w -plane.

Soln:-

$$\text{Let } w = \frac{1}{z}$$

$$z = \frac{1}{w}$$

$$x+iy = \frac{1}{u+iv}$$

$$= \frac{1}{u+iv} \times \frac{u-iv}{u-iv}$$

$$= \frac{u-iv}{u^2+v^2}$$

$$x+iy = \frac{u}{u^2+v^2} - i \frac{v}{u^2+v^2}$$

$$x = \frac{u}{u^2+v^2} \rightarrow \textcircled{1}$$

$$y = \frac{-v}{u^2+v^2} \rightarrow \textcircled{2}$$

The general equation is



$$a(x^2+y^2) + 2gx + 2fy + c = 0$$

$$a \left[\frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2} \right] + 2g \left(\frac{u}{u^2+v^2} \right) + 2f \left(\frac{-v}{u^2+v^2} \right) + c = 0$$

$$a \left[\frac{u^2+v^2}{(u^2+v^2)^2} \right] + 2g \left(\frac{u}{u^2+v^2} \right) - 2f \frac{v}{u^2+v^2} + c = 0$$

$$- 2f \frac{v}{u^2+v^2} + c = 0$$

The transformed equation
 $C(u^2+v^2) + 2gu - 2fv + a = 0$

① If $w = \phi + i\psi$ represents the complex potential for an electric field and $\psi = x^2 - y^2 + \frac{x}{x^2 + y^2}$ determine the function ϕ .

Soln

By Milne Thomson Method,

$$f(z) = i \left(z^2 + \frac{1}{z} \right)$$

$$z = x + iy$$

$$\phi + i\psi = i \left[(x + iy)^2 + \frac{1}{x + iy} \right]$$

$$(a+b)^2 = a^2 + b^2 + 2ab$$

$$= i \left[x^2 + i^2 y^2 + 2ixy + \frac{1}{x+iy} \times \frac{x-iy}{x-iy} \right]$$

$$\begin{aligned} i^2 &= -1 \\ a^2 - b^2 &= (a+b)(a-b) \end{aligned}$$

$$= i \left[x^2 - y^2 + 2ixy + \frac{x-iy}{x^2 + y^2} \right]$$

$$= i \left[x^2 - y^2 + 2ixy + \frac{x}{x^2 + y^2} - \frac{iy}{x^2 + y^2} \right]$$

$$= i \left\{ x^2 - y^2 + 2ixy + \frac{x}{x^2 + y^2} - \frac{iy}{x^2 + y^2} \right\}$$

$$= \underline{i x^2 - i y^2 - 2xy + \frac{ix}{x^2 + y^2} + \frac{y}{x^2 + y^2}}$$

$$\phi + i\psi = i \left[\underbrace{x^2 - y^2 + \frac{ix}{x^2 + y^2}}_{\text{Imaginary}} - 2xy + \frac{y}{x^2 + y^2} \right]$$

Comparing Real & Imaginary Parts, Real

$$\phi = -2xy + \frac{y}{x^2 + y^2}$$

② Prove that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) |f(z)|^2 = 4|f'(z)|^2$, if $f(z)$ is analytic, (or)

If $f(z) = u + iv$ is a regular function of z in a domain D , then $\nabla^2 |f(z)|^2 = 4|f'(z)|^2$

Soln:

Let $f(z) = u + iv$

$$|f(z)| = |u + iv|$$

$$= \sqrt{u^2 + v^2}$$

$$|f(z)|^2 = u^2 + v^2$$

Consider $\frac{\partial^2}{\partial x^2} |f(z)|^2 = \frac{\partial^2}{\partial x^2} (u^2 + v^2)$

$$= \frac{\partial}{\partial x} \left[\frac{\partial}{\partial x} (u^2 + v^2) \right]$$

$$= \frac{\partial}{\partial x} \left[2u \frac{\partial u}{\partial x} + 2v \frac{\partial v}{\partial x} \right]$$

$$= 2 \frac{\partial}{\partial x} \left[u \frac{\partial u}{\partial x} + v \frac{\partial v}{\partial x} \right]$$

$$u \frac{\partial v}{\partial x} + v \frac{\partial u}{\partial x} = \frac{1}{2} \frac{\partial}{\partial x} (u^2 + v^2)$$

$$= 2 \left[u \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} \frac{\partial u}{\partial x} + v \frac{\partial^2 v}{\partial x^2} + \frac{\partial v}{\partial x} \frac{\partial v}{\partial x} \right]$$

||| $\frac{\partial^2}{\partial x^2} |f(z)|^2 = 2 \left[u \frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x} \right)^2 + v \frac{\partial^2 v}{\partial x^2} + \left(\frac{\partial v}{\partial x} \right)^2 \right]$

$\frac{\partial^2}{\partial y^2} |f(z)|^2 = 2 \left[u \frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial u}{\partial y} \right)^2 + v \frac{\partial^2 v}{\partial y^2} + \left(\frac{\partial v}{\partial y} \right)^2 \right]$ ①

From ① + ② $\Rightarrow$ ②

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} |f(z)|^2 = 2 \left[u \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + \left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 + v \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) + \left(\frac{\partial v}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2 \right]$$

$$f(z) = u + iv$$

$$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial y}$$

$$|f'(z)| = \sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2}$$

$$|f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2$$

find $f(z)$ & v if

$$u = \frac{\sin 2x}{\cos 2x + \cosh 2y}$$

u, v are harmonic

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

C.R. eqns

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}; \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$+ v \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) + \left(\frac{\partial v}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2$$

$$= 2 \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(-\frac{\partial v}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 \right]$$

$$= 2 \left[2 \left(\frac{\partial u}{\partial x} \right)^2 + 2 \left(\frac{\partial v}{\partial x} \right)^2 \right]$$

$$= 2 \times 2 \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial x} \right)^2 \right]$$

$$= 4 |f'(z)|^2$$

Hence the proof

(1) If $f(z) = u + iv$ is analytic

Soln:

Unit-1(11)

Analytic Functions

2 marks

Defn: Analytic Function
(or) Holomorphic (or) Regular function

A function $f(z)$ is said to be analytic in a region R if $f(z)$ is analytic at all points in the region R .

Eg:-

$$f(z) = z^2, e^z, \cos z.$$

Defn: Continuity

If $f(z)$ is said to be continuous at $z = z_0$ then

$$\lim_{z \rightarrow z_0} f(z) = f(z_0).$$

Defn: Entire Function (or) Integral function

A function which is analytic everywhere in the finite plane is called an entire function.

An entire function is analytic everywhere except at $z = \infty$.

Eg $e^z, \sin z, \cos z, \sinh z, \cosh z.$

Defn: Limit of a function

Let $f(z)$ be a single valued function defined at all points in some neighbourhood of point z_0 .

Then the limit of $f(z)$ as z approaches z_0 is w_0

$$\lim_{z \rightarrow z_0} f(z) = w_0$$

Properties of Analytic Function

- 1) An analytic function with constant real part is constant.
- 2) An analytic function with constant imaginary part is constant.
- 3) An analytic function with constant modulus is constant.
- 4) If $f(z)$ & $\overline{f(z)}$ are analytic then $f(z)$ is constant.

Polar form of C-R equations

In cartesian coordinates, any point z is $z = x + iy$.

In Polar coordinates,

$$it \text{ is } z = re^{i\theta}$$

Where $r \rightarrow$ Modulus

$\theta \rightarrow$ argument.

Cauchy-Riemann Equations

Necessary & Sufficient Conditions:-

A complex function

$f(z) = u(x, y) + iv(x, y)$ to be analytic in a region R are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

or $u_x = v_y \quad \& \quad v_x = -u_y$

Defn: Harmonic Functions

An expression of the form

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ is called}$$

the Laplace equation in two dimensions. (or)

A function $u(x, y)$ is called harmonic when $u_{xx} + u_{yy} = 0$.

Properties of Harmonic function

1) If $f(z) = u + iv$ is an analytic function, then u is a harmonic

2) If $f(z) = u + iv$ is an analytic then v is a harmonic

C.R. Equations

Sufficient Condition

$$u_x \neq v_y, u_y = -v_x$$

$\therefore$ C.R. equations are not satisfied.

$\therefore f(z)$ is not analytic

② Show that the function $f(z) = \bar{z}$ is nowhere differentiable (or) analytic.

Sol

C.R. equations

$$u_x = v_y, u_y = -v_x$$

$$f(z) = \bar{z}$$

$$= x - iy$$

$$(u + iv)$$

$$z = x + iy$$

$$\bar{z} = x - iy$$

$u = x$	$v = -y$
$u_x = 1$	$v_x = 0$
$u_y = 0$	$v_y = -1$

$$u_x \neq v_y, u_y = -v_x$$

$\therefore$ C.R. equations are not satisfied

$\therefore f(z)$ is not analytic (or) nowhere

① Test whether the function

$$f(z) = x^2 + iy^2 \text{ an analytic}$$

or not.

Sol:

C.R. Equation

$$u_x = v_y, u_y = -v_x$$

$$\text{Given } f(z) = x^2 + iy^2$$

$$(u + iv)$$

$u = x^2$	$v = y^2$
$u_x = 2x$	$v_x = 0$
$u_y = 0$	$v_y = 2y$

③ Test the analyticity of the function $w = \sin z$.

Soln

C.R. Equations

$$u_x = v_y, u_y = -v_x$$

$$w = f(z) = \sin z$$

$$= \sin(x + iy)$$

$$= \sin x \cosh y + i \cos x \sinh y$$

$$= \sin x \cosh y + i \cos x \sinh y$$

$$d(\cosh y)$$

$$= \sinh y$$

$$d(\sinh y)$$

$$= \cosh y$$

$u = \sin x \cosh y$	$v = \cos x \sinh y$
$u_x = \cos x \cosh y$	$v_x = -\sin x \sinh y$
$u_y = \sin x \sinh y$	$v_y = \cos x \cosh y$

$$u_x = v_y ; u_y = -v_x$$

$\therefore$ C.R equations are satisfied

$\therefore f(z)$ is an analytic

④ Determine whether the function $2xy + i(x^2 - y^2)$ is analytic or not.

⑤ Find the real & imaginary part of $f(z) = 2z^3 - 3z$.

Ans

$$u = 2x^3 - 6xy^2 - 3x \text{ (real)}$$

$$v = 6x^2y - 2y^3 - 3y \text{ (imaginary)}$$

⑥ P.T $f(z) = \sinh z$ is analytic

$$\sinh z = \sinh(x+iy)$$

$$\begin{aligned} i \sinh x &= \sin ix \\ \sinh x &= -i \sin ix \end{aligned}$$

$$= -i \sin i(x+iy)$$

$$= -i (\sin ix - y)$$

$$= -i [\sin ix \cosh y - \cosh ix \sinh y]$$

$$= -i [i \sinh x \cosh y - \cosh x \sinh y]$$

$$= \sinh x \cosh y + i \cosh x \sinh y$$

(3)

⑦ P.T $f(z) = \log z$ is analytic

$$f(z) = \log z = \log(x+iy)$$

$$= \frac{1}{2} \log(x^2+y^2) + i \tan^{-1}\left(\frac{y}{x}\right)$$

$$\log(a+ib) = \frac{1}{2} \log(a^2+b^2) + i \tan^{-1}\left(\frac{b}{a}\right)$$

$u = \frac{1}{2} \log(x^2+y^2)$	$v = \tan^{-1}(y/x)$
---------------------------------	----------------------

$$u_x = \frac{1}{2} \cdot \frac{1}{x^2+y^2} (2x)$$

$$= \frac{x}{x^2+y^2}$$

$$v_x = \frac{1}{1+(y/x)^2} \cdot \frac{x(0)-y(1)}{x^2}$$

$$= \frac{x^2}{x^2+y^2} \cdot \left(-\frac{y}{x^2}\right)$$

$$u_y = \frac{1}{2} \cdot \frac{1}{x^2+y^2} (2y)$$

$$= \frac{-y}{x^2+y^2}$$

$$u_y = \frac{y}{x^2+y^2}$$

$$v_y = \frac{1}{1+(y/x)^2} \cdot \frac{x(1)-y(0)}{x^2}$$

$$= \frac{x^2}{x^2+y^2} \cdot \frac{x}{x^2}$$

$$= \frac{x}{x^2+y^2}$$

$$\therefore u_x = v_y \text{ \& } u_y = -v_x$$

C.R equations are Satisfied.
 $f(z)$ is analytic

⑧ $f(z) = z^3$

$$= (x+iy)^3$$

$$= x^3 + 3x^2(iy) - 3xy^2 - iy^3$$

$$u = x^3 - 3xy^2 \quad v = 3x^2y - y^3$$

⑨ $w = z^2 = (x+iy)^2$

$$= x^2 + i2xy - y^2$$

$$u = x^2 - y^2 ; v = 2xy$$

⑩ $f(z) = |z|^2 = \overline{z}z = x^2 + y^2$
 $u = x^2 + y^2 ; v = 0$

$$= e^{x+iy} = e^x \cdot e^{iy}$$

$$= e^x [\cos y + i \sin y]$$

$$u = e^x \cos y; \quad v = e^x \sin y$$

$$(12) \quad f(z) = z^n = (r e^{i\theta})^n$$

$$= r^n e^{in\theta}$$

$$= r^n [\cos n\theta + i \sin n\theta]$$

$$u = r^n \cos n\theta \quad v = r^n \sin n\theta$$

$$\frac{\partial u}{\partial r} = n r^{n-1} \cos n\theta \quad \frac{\partial v}{\partial r} = n r^{n-1} \sin n\theta$$

$$\frac{\partial u}{\partial \theta} = -n r^n \sin n\theta \quad \frac{\partial v}{\partial \theta} = n r^n \cos n\theta$$

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}$$

$$\frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}$$

C.R eqns is Satisfied.

$f(z)$ is analytic

Harmonic Function

A function $u(x, y)$ is called harmonic when

$$u_{xx} + u_{yy} = 0$$

① verify whether the function

$$u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$$

is harmonic.

Soln:

$$u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$$

$$\begin{aligned} u_x &= 3x^2 - 3y^2 + 6x & u_y &= -6xy - 6y \\ u_{xx} &= 6x + 6 & u_{yy} &= -6x - 6 \\ \text{from } ① + ② &\Rightarrow & & \Rightarrow ② \\ u_{xx} + u_{yy} &= 6x + 6 - 6x - 6 = 0 \end{aligned}$$

u is a harmonic

② Show that $u = 2x - x^3 + 3xy^2$ is harmonic.

③ Prove that $u = e^x \cos y$ is a harmonic function.

④ $u = 3x^2y - y^3$

⑤ $y + e^x \cos y$

⑥ $v(x, y) = \frac{-y}{x^2 + y^2}$

⑦ $u = x^2 - y^2$

⑧ $v = \frac{-y}{x^2 + y^2}$

$$\frac{\partial v}{\partial x} = \frac{(x^2 + y^2)(0) - (-y)(2x)}{(x^2 + y^2)^2}$$

$$\begin{aligned} \frac{\partial^2 v}{\partial x^2} &= \frac{(x^2 + y^2)(2y) - (2xy) 2(x^2 + y^2)x}{(x^2 + y^2)^4} \\ &= \frac{(x^2 + y^2) [2y - 4x^2y]}{(x^2 + y^2)^4} \end{aligned}$$

$$= \frac{2y^2 - 4x^2y}{(x^2 + y^2)^3} \rightarrow ①$$

$$\frac{\partial v}{\partial y} = \frac{(x^2 + y^2)(-1) - (-y)(2y)}{(x^2 + y^2)^2}$$

$$= \frac{-x^2 - y^2 + 2y^2}{(x^2 + y^2)^2}$$

$$= \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

$$\frac{\partial^2 v}{\partial y^2} = \frac{(x^2 + y^2)^2(2y) - (y^2 - x^2)(2y)}{(x^2 + y^2)^4}$$

$$= \frac{(x^2 + y^2) [(x^2 + y^2)(2y) - 2y(y^2 - x^2)]}{(x^2 + y^2)^4}$$

$$= \frac{2x^2y + 2y^3 - 4y^3 + 4x^2y}{(x^2 + y^2)^3}$$

$$= \frac{-2y^3 + 6x^2y}{(x^2 + y^2)^3} \rightarrow ②$$

from ① + ②

$v_{xx} + v_{yy} = 0$
 $\therefore v$ is harmonic function

Properties of Analytic Function.

① An analytic function with constant modulus is constant.

Proof: C.R. equations

Let $f(z) = u + iv$ $\left. \begin{array}{l} u_x = v_y \\ u_y = -v_x \end{array} \right\}$ ①

$$|f(z)| = \sqrt{u^2 + v^2}$$

Modulus = Constant

$$\sqrt{u^2 + v^2} = c$$

$$u^2 + v^2 = c^2$$

Diff. w.r.to x

Diff. w.r.to y

$$2u u_x + 2v v_x = 0$$

$$2u u_y + 2v v_y = 0$$

$\div 2$

$$u u_x + v v_x = 0$$

$\div 2$

$$u u_y + v v_y = 0$$

$$-u v_x + v u_x = 0$$

by ①

$$\begin{vmatrix} u & v \\ v & -u \end{vmatrix} = -u^2 - v^2$$

$$= -(u^2 + v^2)$$

$$= -c^2 \neq 0.$$

$$\therefore u_x = 0 \text{ \& } v_x = 0$$

$$u = \text{constant \&}$$

$$v = \text{constant.}$$

$$\therefore f(z) = u + iv \text{ is constant}$$

② An analytic function whose real part is constant is constant. (or) An analytic function with constant real part is constant.

Proof:

C.R. equations

$$u_x = v_y ; u_y = -v_x$$

$$\text{Let } f(z) = u + iv \rightarrow \text{imaginary}$$

$$\text{Real} = \text{Constant}$$

$$u = c$$

$$u_x = 0 \mid u_y = 0 \rightarrow ②$$

$$f(z) = u + iv$$

$$f'(z) = u_x + i v_x$$

$$= u_x - i u_y \text{ by ①}$$

$$= 0 - i(0) \text{ by ②}$$

$$f'(z) = 0$$

$\therefore f'(z)$ is a constant.

③ An analytic function with constant imaginary part is constant.

Proof

By C.R. equations

$$u_x = v_y ; u_y = -v_x \rightarrow ①$$

$$\text{Let } f(z) = u + iv \rightarrow \text{imaginary}$$

Imaginary = Constant

$$v = c$$

$$v_x = 0 \mid v_y = 0 \rightarrow ②$$

$$f(z) = u + iv$$

$$f'(z) = u_x + i v_x$$

$$= v_y + i v_x \text{ by ①}$$

$$= 0 + i(0) \text{ by ②}$$

$$f'(z) = 0$$

$\therefore f'(z)$ is a constant

Q If $f(z) = u + iv$ is analytic.
 Prove that the curves $u = \text{constant}$
 & $v = \text{constant}$ are orthogonal.
 proof

C.R. equations

(1) $u_x = v_y$; $u_y = -v_x$ → (1)

Let $f(z) = u + iv$

$u(x,y) = \text{constant}$	$v(x,y) = \text{constant}$
Slope $m_1 = -\frac{u_x}{u_y}$	Slope $m_2 = -\frac{v_x}{v_y}$

$$m_1 m_2 = -\frac{u_x}{u_y} \times \frac{v_x}{v_y}$$

$$= -\frac{u_x}{u_y} \times \frac{u_y}{u_x} \text{ by (1)}$$

$\therefore m_1 m_2 = -1$ ✓

$\therefore$ The curves are orthogonal

Conformal mapping

- ① Define Critical point of a transformation

A point z_0 at which the mapping $w=f(z)$ is not conformal is called the critical point. The critical point of the transformation $w=f(z)$ are given by $\frac{dw}{dz} = 0$

- ② Find the critical points of the transformation $w = z + \frac{1}{z}$.

Soln:

Given $w = z + \frac{1}{z}$

$$\frac{dw}{dz} = 1 - \frac{1}{z^2} = 0$$

$$1 - \frac{1}{z^2} = 0$$

$$z^2 = 1$$

$$z = \pm 1$$

Critical points } $z = 1,$
 $z = -1$

- ③ Find the fixed points of the mapping $w = \frac{1}{z+2i}$ (or) Invariant points.

Soln:

$$w = \frac{1}{z+2i}$$

$$w = z$$

$$z = \frac{1}{z+2i}$$

$$z(z+2i) = 1$$

$$z^2 + 2zi - 1 = 0$$

$$a=1, b=2i, c=-1$$

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-2i \pm \sqrt{(2i)^2 - 4(1)(-1)}}{2(1)}$$

$$= \frac{-2i \pm \sqrt{4i^2 + 4}}{2}$$

$$z = \frac{-2i \pm \sqrt{-4+4}}{2}$$

$$= \frac{-2i}{2}$$

$$z = -i$$

- ④ Find the invariant points of the transformation

$$w = \frac{1+z}{1-z}$$

Soln

Given $z = \frac{1+z}{1-z}$

$$z(1-z) = 1+z$$

$$z - z^2 - 1 - z = 0$$

$$-z^2 - 1 = 0$$

$$z^2 + 1 = 0$$

$$z^2 = -1 = i^2$$

$$z = \pm i$$

Invariant points } $z = +i,$
 $z = -i$

- ⑤ Find the image of $x^2 + y^2 = 4$ under the transformation $w = 3z$.

Soln:

$$w = 3z$$

$$u+iv = 3(x+iy)$$

$$u+iv = 3x + i3y$$

$$u = 3x \quad i \quad v = 3y$$

$$x = \frac{u}{3} \quad y = \frac{v}{3}$$

$$x^2 + y^2 = 4$$

$$\left(\frac{u}{3}\right)^2 + \left(\frac{v}{3}\right)^2 = 4$$

$$\frac{u^2}{9} + \frac{v^2}{9} = 4$$

$$u^2 + v^2 = 4 \times 9$$

$$u^2 + v^2 = 36$$

① Find the critical point of the transformation $w = 1 + \frac{2}{z}$

$$w = 1 + \frac{2}{z}$$

$$\frac{dw}{dz} = 0 - \frac{2}{z^2}$$

$$\frac{dw}{dz} = 0$$

$$-\frac{2}{z^2} = 0$$

$$z = \infty$$

$$\frac{1}{z^1}$$

$$\frac{-1}{z^2}$$

$$\frac{-1}{z^3}$$

$$\frac{-2}{z^4}$$

$$\frac{-1}{z^5}$$

$$x^n = nx^{n-1}$$

① Find the bilinear transformation which maps the points $z = 0, -i, -1$ into $w = i, 1, 0$ respectively.

Soln.

$$\text{Given } z_1 = 0, z_2 = -i, z_3 = -1$$

$$w_1 = i, w_2 = 1, w_3 = 0$$

By the cross ratio of bilinear transformation

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-i)(1-0)}{(w-0)(1-i)} = \frac{(z-0)(-i+1)}{(z+1)(-i-0)}$$

$$\frac{w-i}{w-i} = \frac{z(-i+1)}{(z+1)(-i)}$$

$$\frac{w-i}{w-w_1} = \frac{-zi+z}{-zi-i}$$

$$(w-i)(-iz-i) = (w-w_1)(-iz+z)$$

$$-wiz - wi + zi^2 + i^2$$

$$= -wiz + wz + wi^2 z - wiz$$

$$-wiz - wi - z - 1 + 2wiz - wz + wz$$

$$wi^2 z - wi - z - 1 = 0$$

$$w(i^2 - i) - (z+1) = 0$$

$$w(i^2 - i) = z+1$$

$$w = \frac{z+1}{i^2 - i}$$

② Find the bilinear transformation which maps the points $z = -1, 0, i$ into $w = -1, i, 1$ respectively.

Soln.

$$\text{Given } z_1 = -1, w_1 = -1$$

$$z_2 = 0, w_2 = i$$

$$z_3 = i, w_3 = 1$$

Bilinear transformation is,

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$\frac{(w+1)(i-1)}{(-1-i)(1-w)} = \frac{(z+1)(0-i)}{(-1-0)(i-z)}$$

$$\frac{(w+1)(i-1)}{(-1-i)(1-w)} = \frac{i(z+1)}{(-1-0)(i-z)}$$

$$\frac{(w+1)(i-1)}{(-1-i)(1-w)} = \frac{i(z+1)}{(-1-0)(i-z)}$$

$$\frac{(w+1)(i-1)}{(-1-i)(1-w)} = \frac{i(z+1)}{(-1-0)(i-z)}$$

$$\frac{w+1}{w-1} = \frac{i^2 - i - i + 1}{i^2 - i - i + 1} = \frac{i(z+1)}{i-z}$$

$$\frac{w+1}{w-1} = \frac{-1-1}{-2} = \frac{-2}{-2} = 1$$

$$\frac{w+1}{w-1} = \frac{-2i+x}{-2} = \frac{i(z+1)}{i-z}$$

$$\frac{w+1}{w-1} = \frac{-2i}{-2} = \frac{i(z+1)}{i-z}$$

$$\frac{w+1}{w-1} = \frac{z+1}{i-z}$$

$$\frac{w+1+w-1}{w+1-w+1} = \frac{1+z+i-z}{1+z-i+z}$$

$$\frac{2w}{2} = \frac{1+i}{1-i+2z} \Rightarrow w = \frac{1+i}{1-i+2z}$$

$$\frac{w-w_1}{w_1-w_2} \times \frac{w_2-w_3}{w_3-w} = \frac{z-z_1}{z_1-z_2} \times \frac{z_2-z_3}{z_3-z}$$

H.W

① $z = -1, i, 1$ onto $w = 1, i, -1$

② $z = -1, 0, 1$ onto $w = 0, i, 3i$

③ $z = -i, 0, i$ onto $w = -1, i, 1$

④ $z = 1, i, -1$ onto $w = 0, 1, \infty$

Ans ① $w = -\frac{1}{z}$

② $w = \frac{3i(1+z)}{3-z}$

③ $w = i \cdot \left(\frac{i-z}{1+z} \right)$

④ $w = -\frac{1}{z}$

Define Conformal mapping

Let $f(z)$ be a single valued function defined in a region. Then the mapping $w = f(z)$ preserves angle between every pair of curves both in magnitude & sense of orientation is said to be Conformal mapping.

Unit - III

Formula:

① Bilinear transformation

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

② Harmonic Function

$$u_{xx} + u_{yy} = 0$$

③ Analytic Function

C.R Equation

$$u_x = v_y$$

$$u_y = -v_x$$

④ Critical points

$$\frac{dw}{dz} = 0$$

⑤ Fixed Point (or)

Invariant points

$z = w$ (X) Apply

⑥ Image

⑦ properties of Analytic function

Unit - ~~III~~ IV
Complex Integration

(2 marks) which is convergent at every point inside C

(1) State Cauchy's Integral theorem.

If a function $f(z)$ is analytic & its derivative $f'(z)$ is continuous at all points inside & on a simple closed curve C , then

$$\int_C f(z) dz = 0$$

(2) State Cauchy's Integral formula

Soln

If $f(z)$ is analytic inside and on a closed curve C of a simply connected region R and a is any point within C , then

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a} dz$$

the integration around C being taken in the +ve direction.

(3) State Taylor's theorem

A function $f(z)$, analytic inside a circle C with centre at a , can be expanded in the series

$$f(z) = f(a) + \frac{f'(a)}{1!}(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \frac{f'''(a)}{3!}(z-a)^3 + \dots + \infty$$

(4) classify the various types of singularities & give one example for each type.

Soln:

Defn Isolated Singularity

A point $z=z_0$ is said to be isolated singularity of $f(z)$ if

(i) $f(z)$ is not analytic at $z=z_0$

(ii) There exists a neighbourhood of $z=z_0$ containing no other singularity.

Eg: $f(z) = \frac{1}{z}$

Defn: Removable Singularity

A singular point $z=z_0$ is called a removable singularity of $f(z)$ if $\lim_{z \rightarrow z_0} f(z)$ exists finite

Eg:

$$f(z) = \frac{\sin z}{z}$$

Defn: Poles

If we can find a true integer n such that

$$\lim_{z \rightarrow z_0} (z - z_0)^n f(z) \neq 0$$

then $z = z_0$ is called pole of order n for $f(z)$.

Eg: $f(z) = \frac{1}{(z-4)^2 (z-3)^4}$

Here $z=4$ is a pole of order 2
 $z=3$ " " " " 4.

Defn: Essential Singularity

If the principal part contain an infinite number of terms, then $z = z_0$ is known as an essential singularity.

Eg: $f(z) = e^{\frac{1}{z}}$

Taylor's Series

① $z=a$

$$f(z) = f(a) + \frac{f'(a)}{1!} (z-a) + \frac{f''(a)}{2!} (z-a)^2 + \dots$$

② $z=0$

$a \neq 0$

$$f(z) = f(0) + \frac{f'(0)}{1!} z + \frac{f''(0)}{2!} z^2 + \dots$$

Laurent's Series

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n}$$

where

$$a_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(z)}{(z-a)^{n+1}} dz$$

$$b_n = \frac{1}{2\pi i} \int_{C_2} \frac{f(z)}{(z-a)^{-n+1}} dz$$

① Expand $f(z) = e^z$ in a Taylor's Series about $z=0$.
Soln:

Function	Value at $z=0$
$f(z) = e^z$	$f(0) = e^0 = 1$
$f' = e^z$	$f' = e^0 = 1$
$f'' = e^z$	$f'' = e^0 = 1$
$f''' = e^z$	$f''' = e^0 = 1$

$$f(z) = f(0) + \frac{z}{1!} f'(0) + \frac{z^2}{2!} f''(0) + \dots$$
$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \dots$$

② Expand $f(z) = \log(1+z)$ as a Taylor's Series about $z=0$ if $|z| < 1$.

Soln:

Function	Value at $z=0$
$f = \log(1+z)$	$f(0) = \log(1+0) = \log 1 = 0$
$f' = \frac{1}{1+z}$	$f' = \frac{1}{1+0} = 1$
$f'' = -\frac{1}{(1+z)^2}$	$f'' = \frac{-1}{(1+0)^2} = -1$
$f''' = \frac{2}{(1+z)^3}$	$f''' = \frac{2}{(1+0)^3} = 2$

$$f(z) = f(0) + \frac{z}{1!} f'(0) + \frac{z^2}{2!} f''(0) + \dots$$

$$= 0 + \frac{z}{1!} (1) + \frac{z^2}{2!} (-1) + \frac{z^3}{3!} (2) + \dots$$

$$= z - \frac{z^2}{2} + \frac{z^3}{3} - \dots$$

~~$$f(z) = f(0) + \frac{z}{1!} f'(0) + \dots$$~~

$$f(z) = \frac{1}{\sqrt{2}} + \frac{(z - \pi/4)}{1!} \left(\frac{1}{\sqrt{2}} \right) + \frac{(z - \pi/4)^2}{2!} \left(\frac{-1}{\sqrt{2}} \right) + \dots$$

③ Expand $\frac{1}{z-2}$ at $z=1$ in a Taylor's series.
Soln.

Function	Value at $z=1$
$f(z) = \frac{1}{z-2}$	$f(1) = \frac{1}{1-2} = -1$
$f' = \frac{-1}{(z-2)^2}$	$f' = \frac{-1}{(1-2)^2} = -1$
$f'' = \frac{2}{(z-2)^3}$	$f'' = \frac{2}{(1-2)^3} = -2$

$$f(z) = \frac{(z-a)^0}{0!} f'(a) + \frac{(z-a)^1}{1!} f'(a) + \frac{(z-a)^2}{2!} f''(a) + \dots$$

$$= -1 + \frac{(z-1)}{1!} (-1) + \frac{(z-1)^2}{2!} (-2) + \dots$$

$$= -1 - (z-1) - (z-1)^2 - \dots$$

④ Find the Taylor series for $f(z) = \sin z$ about $z = \pi/4$
Soln.

Function	values at $z = \pi/4$
$f(z) = \sin z$	$f(\pi/4) = \sin \pi/4 = \frac{1}{\sqrt{2}}$ <u>($\sin 45^\circ$)</u>
$f' = \cos z$	$f' = \cos \pi/4 = \frac{1}{\sqrt{2}}$
$f'' = -\sin z$	$f'' = -\sin \pi/4 = -\frac{1}{\sqrt{2}}$
$f''' = -\cos z$	$f''' = -\cos \pi/4 = -\frac{1}{\sqrt{2}}$

$$f(z) = f(a) + \frac{(z-a)}{1!} f'(a) + \frac{(z-a)^2}{2!} f''(a) + \dots$$

⑤ Expand $f(z) = \sin z$ in a Taylor's series about $z=0$
Soln.

Function	value at $z=0$
$f(z) = \sin z$	$f(0) = \sin 0 = 0$
$f'(z) = \cos z$	$f' = \cos 0 = 1$
$f'' = -\sin z$	$f'' = \sin 0 = 0$
$f''' = -\cos z$	$f''' = -\cos 0 = -1$

$$f(z) = f(0) + \frac{z}{1!} f'(0) + \frac{z^2}{2!} f''(0) + \dots$$

$$= 0 + \frac{z}{1!} (1) + \frac{z^2}{2!} (0) + \frac{z^3}{3!} (-1) + \dots$$

$$= \frac{z}{1!} - \frac{z^3}{3!} + \dots$$

Cauchy's Integral theorem

If $f(z)$ is analytic & $f'(z)$ is continuous on and inside a simple closed curve C , then $\int_C f(z) dz = 0$.

① Evaluate $\int_C \frac{dz}{z+4}$, where

C is the circle $|z|=2$.

Soln.

$$z+4=0$$

$$z = -4 \Rightarrow$$

C is $|z|=2$.

$$z = -4 \Rightarrow |z| = |-4| = 4 > 2$$

$\therefore z = -4$ lies outside C .

By Cauchy's Integral Theorem,

$$\int_C f(z) dz = 0$$

$$\int_C \frac{dz}{z+4} = 0$$

② Evaluate $\int_C \frac{dz}{2z-3}$ where

C is $|z|=1$.

Soln:

$$2z-3=0$$

$$2z=3$$

$$z = \frac{3}{2}$$

$$|z| = \left| \frac{3}{2} \right| = \frac{3}{2} > 1.$$

$z = \frac{3}{2}$ lies outside C .

By C.I.T, $\int_C \frac{dz}{2z-3} = 0$

Evaluate

③ $\int_C \frac{z dz}{(z-1)(z-2)}$ where C is $|z| = \frac{1}{2}$

Soln

$$z-1=0 \quad z-2=0$$

$$z=1 \quad ; \quad z=2 \quad \therefore C \text{ is } |z| = \frac{1}{2}$$

$$|z| = |1| = 1 > \frac{1}{2} \quad |z| = 2 > \frac{1}{2}$$

$\therefore z=1, 2$ lies outside C .

By C.I.T, $\int_C \frac{z dz}{(z-1)(z-2)} = 0$.

④ Evaluate $\int_C \frac{e^z}{z-1} dz$ if C is $|z|=2$.

Soln

$$z-1=0$$

$$z=1$$

$\therefore C$ is $|z|=2$

$$|z| = |1| = 1 < 2$$

$\therefore z=1$ lies inside C .

$\therefore$ By C.I.T, $\int_C f(z) dz = 0$

$$\int_C \frac{e^z}{z-1} dz = 0$$

⑤ Evaluate $\int_C e^{\frac{1}{2}z} dz$ where

C is $|z|=1$.

Soln:

$$\text{Let } f(z) = e^{\frac{1}{2}z}$$

$f(z)$ is analytic inside & on C .

By C.I.T,

$$\int_C e^{\frac{1}{2}z} dz = 0$$

Cauchy's Integral formula

Let $f(z)$ be an analytic function inside and on a simple closed contour C , taken in the +ve sense. If 'a' is any point interior to C , then

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z) dz}{z-a}$$

(or)

$$\int_C \frac{f(z) dz}{z-a} = 2\pi i f(a)$$

① Evaluate $\int_C \frac{z}{z-2} dz$

where C is $|z|=1$ & $|z|=3$

Soln

$$\text{Let } \int_C \frac{z}{z-2} dz$$

$$z-2=0 \Rightarrow z=2$$

$\therefore z$ lies outside C , $|z|=2$

$$\int_C \frac{z}{z-2} dz = 0$$

z lies outside C , $|z|=3$

$$\int_C \frac{z}{z-2} dz = 2\pi i f(2)$$

$$= 2\pi i (2)$$

$$\int_C z dz = 4\pi i$$

② Evaluate $\int_c \frac{e^z}{z-1} dz$ if

c is $|z|=2$.

Soln

$$z-1=0 \Rightarrow z=1 \Rightarrow |z|=1 < 2$$

$\therefore z$ lies inside $c, |z|=2$.

$$\int_c \frac{e^z}{z-1} dz = 2\pi i f(1)$$

$$= 2\pi i e$$

$$f(z) = e^z = e^1 = e$$

③ Evaluate $\int_c \frac{dz}{z^2-7z+12}$

where c is the circle $|z|=3.5$

Soln:

$$z^2-7z+12=0$$

$$(z-4)(z-3)=0$$

$$z-4=0$$

$$z=4$$

$$|z|=|4|=4 > 3.5$$

z lies outside c

$$z-3=0$$

$$z=3$$

$$|z|=|3|=3 < 3.5$$

z lies inside c .

$$\therefore \int_c \frac{dz}{(z-4)(z-3)} = \int_c \left(\frac{1}{z-4} \right) dz$$

$$f(z) = \frac{1}{z-4} \text{ is analytic inside } c.$$

By C.I. Formula,

$$\int_c \frac{f(z)}{z-a} dz = 2\pi i f(a)$$

$$= 2\pi i f(3)$$

$$f(3) = \frac{1}{3-4} = -1$$

$$\therefore \int_c \frac{dz}{(z-4)(z-3)} = 2\pi i (-1) = -2\pi i$$

Formula:

$$\int_c \frac{f(z)}{(z-a)^{n+1}} dz = \frac{2\pi i}{n!} f^{(n)}(a)$$

④ Evaluate $\int_c \frac{z+4}{z^2+2z+5} dz$

where c is the circle $|z+1+i|=2$ using Cauchy's integral formula

Soln:

$$z^2+2z+5=0$$

$$a=1, b=2, c=5$$

$$z = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{-2 \pm \sqrt{4-20}}{2}$$

$$= -1 \pm \sqrt{-16}$$

$$= -1 \pm 4i$$

$$z = -1 + 2i$$

c is circle

$$|z+1+i|=2$$

$$z = -1 + 2i$$

$$|-1+2i+1+i| = |3i| = 3 > 2$$

z lies outside c .

$$z = -1 - 2i$$

$$|-1-2i+1+i| = |-i| = 1 < 2$$

z lies inside c .

$$\int_c \frac{z+4}{z^2+2z+5} dz = \int_c \frac{z+4}{(z-(-1+2i))(z-(-1-2i))} dz$$

$$= \int_c \frac{z+4}{z-(-1+2i)} dz$$

$f(z) = \frac{z+4}{z-(-1+2i)}$ is analytic

By C.I. Formula

$$\int_c \frac{f(z)}{z-a} dz = 2\pi i f(a)$$

$$= 2\pi i f(-1+2i)$$

$$z = -1 - 2i$$

$$= 2\pi i \left[\frac{-1-2i+4}{-1-2i-(1+2i)} \right]$$

$$= 2\pi i \left[\frac{3-2i}{-1-2i+1-2i} \right]$$

$$= 2\pi i \left(\frac{3-2i}{-4i} \right)$$

$$= -\frac{\pi}{2} (3-2i)$$

⑤ Find $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$

C is $|z| = \frac{3}{2}$

Soln:

$$(z-1)(z-2) = 0$$

$$\begin{array}{l|l} z-1=0 & z-2=0 \\ z=1 & z=2 \\ |z|=1 < \frac{3}{2} & |z|=2 < \frac{3}{2} \\ z=1, 2 < \frac{3}{2} & \text{inside } C \end{array}$$

$$\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$$

$$\frac{1}{(z-1)(z-2)} = \frac{A(z-2) + B(z-1)}{(z-1)(z-2)}$$

Put $z=2 \Rightarrow 1 = B$

$z=1 \Rightarrow 1 = -A, -A = 1$
 $A = -1$

$$\int_C \frac{dz}{(z-1)(z-2)} = \frac{-1}{z-1} + \frac{1}{z-2}$$

$$\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz = \int_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-1} dz + \int_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-2} dz$$

By C.T.T,
 $= -2\pi i f(1) + 2\pi i f(2)$
 $= -2\pi i f(1) + 2\pi i f(2)$
 $= -2\pi i (-1) + 2\pi i (1)$
 $= 2\pi i + 2\pi i$
 $= 4\pi i$

⑥ Evaluate $\int_C \frac{z dz}{(z-1)(z-2)}$

where C is $|z-2| = \frac{1}{2}$ by using Cauchy's integral formula.

Soln:
 let $f(z) = \frac{z}{(z-1)(z-2)^2}$

$z-1=0 \Rightarrow z=1 \Rightarrow$ outside

$|z|=1 > \frac{1}{2}$

$z-2=0 \Rightarrow z=2 \Rightarrow$ Inside
 $|z-2| = \frac{1}{2}$

$$\frac{z}{(z-1)(z-2)^2} = \frac{A}{z-1} + \frac{B}{z-2} + \frac{C}{(z-2)^2}$$

$$\frac{z}{(z-1)(z-2)^2} = \frac{A(z-2)^2 + B(z-1)(z-2) + C(z-1)}{(z-1)(z-2)^2}$$

$$z = A(z-2)^2 + B(z-1)(z-2) + C(z-1)$$

Put $z=1 \Rightarrow 1 = A(1-2)^2$
 $1 = A(-1)^2$
 $A = 1$

$z=2 \Rightarrow 2 = C(2-1)$
 $C = 2$

Equating coeff of z^2

$0 = A + B$

$1 + B = 0$
 $B = -1$

$$\int_C \frac{z dz}{(z-1)(z-2)^2} = \int_C \frac{1}{z-1} dz + \int_C \frac{-1}{z-2} dz + \int_C \frac{2}{(z-2)^2} dz$$

$$1 + 2x^2 = 1 + 2x^2 + 3x^2 - 4x^3 + \dots$$

Define essential singularity
with an example.

A point $z=a$ is an essential singularity if

(i) $z=a$ is a singular point

(ii) $z=a$ should not be a pole (or) removable singularity.

Eg: $f(z) = e^{\frac{1}{z-2}}$

(i) $z=2$ is singular point

(ii) $z=2 \Rightarrow f(z) = e^{\frac{1}{z-2}}$
 $= e^{\frac{1}{2-2}}$
 $= e^{\frac{1}{0}}$
 $= e^{\infty} = \infty$
 (Not defined)

$\therefore z=2$ not pole & removable singularity.

$\therefore z=2$ is an essential singularity

① Find the Singular

Point for $f(z) = \frac{1}{z}$

Soln:

$z=0$ is a singular point
order = 1

② Identify the Nature of

Singularity of $f(z) = \frac{e^{z^2}}{z^3}$

Soln

Let $f(z) = \frac{e^{z^2}}{z^3}$

$$z^3 = 0$$

$$z = 0$$

Hence $z=0$ is a pole of order 3

③ Find the Singularities

of $f(z) = \frac{z+4}{z^2+2z+2}$

$$z^2+2z+2=0$$

$$a=1; b=2; c=2$$

$$z = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{-2 \pm \sqrt{4-4(1)(2)}}{2(1)}$$

$$= \frac{-2 \pm \sqrt{4-8}}{2}$$

$$= \frac{-2 \pm \sqrt{-4}}{2}$$

$$= \frac{-2 \pm 2i}{2}$$

$$z = -1 \pm i$$

$\therefore$ singular points $z = -1 \pm i$

④ Determine the Poles

of $f(z) = \frac{z^2}{(z-1)(z-2)^2}$

investigates their orders.
Soln:

$$z-1=0$$

$$z=1$$

Pole of order 1

$$z-2=0$$

$$z=2$$

Pole of order 2

⑤ obtain the Poles of $\frac{z+4}{z^2+2z+5}$

Soln

Given $f(z) = \frac{z+4}{z^2+2z+5}$

$$z^2+2z+5=0$$

$$a=1; b=2; c=5$$

$$z = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{-2 \pm \sqrt{2^2-4(1)(5)}}{2(1)}$$

$$= \frac{-2 \pm \sqrt{4-20}}{2}$$

$$= \frac{-2 \pm \sqrt{-16}}{2}$$

$$= \frac{-2 \pm 4i}{2}$$

$$\boxed{z = -1 \pm 2i}$$

Pole : $z = -1 \pm 2i$

⑥ Classify the nature of singularity, given

$$f(z) = \begin{cases} e^z, & \text{if } z \neq 0 \\ 0, & \text{if } z = 0 \end{cases}$$

Soln :

Essential Singularity

④ Find the Singularities of $f(z) = \frac{\cot z}{(z-a)^3}$

Soln :

$$\begin{aligned} \text{Given } f(z) &= \frac{\cot z}{(z-a)^3} \\ &= \frac{\cos z}{\sin z (z-a)^3} \end{aligned}$$

$\tan z = \frac{\sin z}{\cos z}$
 $\cot z = \frac{1}{\tan z}$

Denominator $= 0$

$$\sin z (z-a)^3 = 0$$

$$z = 0$$

$$\sin 0 = 0, \pi, 2\pi, 3\pi, \dots$$

$$(z-a)^3 = 0$$

$$z = a, \dots$$

$z-a=0$
 $z=a$

$$\therefore \sin z (z-a)^3 = 0, \pi, 2\pi, \dots, a, a, a.$$

② write the Singularity of the function $\frac{1}{1-e^z}$

Soln :

$$1-e^z = 0$$

$$e^z = 1$$

$z=0$ is the Singular point

③ Find the Singular Point of

$$f(z) = \frac{1}{z \sin z}$$

Soln :

$$z \sin z = 0$$

$$z=0; \sin 0 = 0, \pi, 2\pi, \dots$$

Points : $0, \pi, 2\pi, 3\pi, \dots$

① Find the residue of $f(z) = \frac{4}{z^3(z-2)}$ at a simple pole.

Soln:

Given $f(z) = \frac{4}{z^3(z-2)}$

Poles:-

$$\begin{array}{l|l} z^3 = 0 & z-2 = 0 \\ \hline z = 0 & z = 2 \end{array} \quad \begin{array}{l} \text{order} = 3 \\ \text{order} = 1 \end{array}$$

$\therefore z = 2$ is a simple pole.

Formula

$$R(2) = \lim_{z \rightarrow 2} (z-2) f(z)$$

$$= \lim_{z \rightarrow 2} \left[(z-2) \frac{4}{z^3(z-2)} \right]$$

$$= \lim_{z \rightarrow 2} \left(\frac{4}{z^3} \right)$$

$$= \frac{4}{(2)^3} = \frac{4}{8} = \frac{1}{2}$$

② Find the Singular Point of $\frac{z+2}{(z+1)^2}$ & hence find its residue.

Soln:

Given $f(z) = \frac{z+2}{(z+1)^2}$

Pole:

$$\begin{array}{l|l} z+1=0 & \\ \hline z=-1 & \text{order} = 2 \end{array}$$

Simple Pole

$$z = -1$$

Formula

$$R(-1) = \lim_{z \rightarrow -1} (z+1)^2 f(z)$$

$$= \lim_{z \rightarrow -1} \left[(z+1)^2 \frac{(z+2)}{(z+1)^2} \right]$$

$$= \lim_{z \rightarrow -1} (z+2)$$

$$= -1+2 = 1$$

③ Find the residue of $f(z) = \frac{50z}{(z+4)(z-1)^2}$ at $z=1$.

Soln

Poles

$$z=1$$

order = 2

Formula

$$R(1) = \lim_{z \rightarrow 1} \frac{d}{dz} \left[(z-1)^2 f(z) \right]$$

$$= \lim_{z \rightarrow 1} \frac{d}{dz} \left[(z-1)^2 \frac{50z}{(z+4)(z-1)^2} \right]$$

$$= \lim_{z \rightarrow 1} \frac{d}{dz} \left(\frac{50z}{z+4} \right) \quad u = \frac{vdu - u dv}{v^2}$$

$$= \lim_{z \rightarrow 1} \left[\frac{(z+4) \cdot 50 - 50z}{(z+4)^2} \right]$$

$$= \frac{(1+4) \cdot 50 - 50(1)}{(1+4)^2}$$

$$= \frac{250 - 50}{25} = \frac{200}{25} = 8$$

④ Calculate the residue of $f(z) = \frac{e^{2z}}{(z+1)^2}$ at its Poles.

Poles:

$$\begin{array}{l|l} z+1=0 & \\ \hline z=-1 & \end{array}$$

order = 2

Formula

$$R(-1) = \lim_{z \rightarrow -1} \frac{d}{dz} \left[(z+1)^2 f(z) \right]$$

$$= \lim_{z \rightarrow -1} \frac{d}{dz} \left[(z+1)^2 \frac{e^{2z}}{(z+1)^2} \right]$$

$$= \lim_{z \rightarrow -1} \frac{d}{dz} (e^{2z})$$

$$= \lim_{z \rightarrow -1} 2e^{2z} = 2e^{2(-1)}$$

$$= 2e^{-2}$$

⑤ Find the residue of $\tan z$ at $z = \pi/2$

Soln

$$f(z) = \tan z = \frac{\sin z}{\cos z}$$

Poles

$$\cos z = 0$$

$$z = \pm (2n+1)\pi/2; n=0,1,2,\dots$$

$$n=0,$$

$$z = (2(0)+1)\pi/2 = \pi/2$$

$$n=1$$

$$z = (2(1)+1)\pi/2 = 3\pi/2$$

$$n=2$$

$$z = (2(2)+1)\pi/2 = 5\pi/2$$

$$P = \pi/2, 3\pi/2, 5\pi/2, \dots$$

Formula

$$R(\pi/2) = \frac{\phi(\pi/2)}{\psi'(\pi/2)}$$

$$\phi(z) = \sin z$$

$$\psi(z) = \cos z$$

$$\psi'(z) = -\sin z$$

$$\sin(\pi/2) = 1$$

$$\sin(\pi/2) = 1$$

$$= \frac{1}{-1} = -1$$

⑥ Find the residue of $\cot z$ at the pole $z=0$

Soln

$$f(z) = \cot z = \frac{\cos z}{\sin z}$$

Poles

$$\sin z = 0$$

$$z = n\pi, n=0,1,2,\dots$$

$$n=0$$

$$z = 0$$

$$n=1$$

$$z = \pi$$

$$n=2$$

$$z = 2\pi, \dots$$

$$\therefore z = 0, \pm\pi, \pm 2\pi, \dots$$

$$R(n\pi) = \frac{\phi(n\pi)}{\psi'(n\pi)}$$

$$\phi(z) = \cos z$$

$$\psi(z) = \sin z$$

$$\psi'(z) = \cos z$$

$$R(z=0) = \frac{\phi(0)}{\psi'(0)}$$

$$= \frac{\cos 0}{\cos 0}$$

$$= 1$$

Laurent's series

Type 1

8 marks

$$\textcircled{1} \frac{1}{(z+1)(z+3)} \text{ for } \dots$$

$$\textcircled{1} 1 < |z| < 3; \textcircled{2} |z| < 1$$

Soln

$$\text{Let } \frac{1}{(z+1)(z+3)} = \frac{A}{z+1} + \frac{B}{z+3}$$

$$\frac{1}{(z+1)(z+3)} = \frac{A(z+3) + B(z+1)}{(z+1)(z+3)}$$

$$1 = A(z+3) + B(z+1)$$

$$\text{Put } z = -1$$

$$1 = A(-1+3) + 0$$

$$1 = 2A$$

$$A = \frac{1}{2}$$

$$\text{Put } z = -3$$

$$1 = A(0) + B(-3+1)$$

$$1 = B(-2)$$

$$B = -\frac{1}{2}$$

$$\frac{1}{(z+1)(z+3)} = \frac{A}{z+1} + \frac{B}{z+3}$$

$$= \frac{\frac{1}{2}}{z+1} + \frac{(-\frac{1}{2})}{z+3}$$

→ (A)

(i) $1 < |z| < 3$

$$1 < |z| \text{ \& } |z| < 3$$

$$\frac{1}{|z|} < 1 \text{ \& } \frac{|z|}{3} < 1 \rightarrow \textcircled{2}$$

→ (1)

From (A),

$$\frac{\frac{1}{2}}{z+1} - \frac{\frac{1}{2}}{z+3} = \frac{\frac{1}{2}}{z(1+\frac{1}{z})} - \frac{\frac{1}{2}}{3(1+\frac{z}{3})}$$

$$= \frac{1}{2z} \left(1 + \frac{1}{z}\right)^{-1} - \frac{1}{2 \times 3} \left(1 + \frac{z}{3}\right)^{-1}$$

$$= \frac{1}{2z} \left[1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \dots\right]$$

$$- \frac{1}{6} \left[1 - \frac{z}{3} + \frac{z^2}{9} - \dots\right]$$

(By Binomial expansion)

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + \dots$$

(ii) $|z| < 1$

From (A),

$$\frac{\frac{1}{2}}{z+1} - \frac{\frac{1}{2}}{z+3} = \frac{\frac{1}{2}}{1+z} - \frac{\frac{1}{2}}{3+z}$$

$$= \frac{\frac{1}{2}}{(1+z)} - \frac{\frac{1}{2}}{3(1+\frac{z}{3})}$$

$$= \frac{1}{2} (1+z)^{-1} - \frac{1}{6} \left(1 + \frac{z}{3}\right)^{-1}$$

$$= \frac{1}{2} \left[1 - z + z^2 - z^3 + \dots\right] - \frac{1}{6} \left[1 - \frac{z}{3} + \frac{z^2}{9} - \dots\right]$$

Expand $\frac{z^2-1}{(z+2)(z+3)}$ in the appropriate series in the

regions (i) $2 < |z| < 3$

(ii) $|z| > 3$

Soln:

Let $f(z) = \frac{z^2-1}{(z+2)(z+3)}$

$$= 1 + \frac{-5z-7}{(z+2)(z+3)}$$

$$\frac{-5z-7}{(z+2)(z+3)} = \frac{A}{z+2} + \frac{B}{z+3}$$

$$-5z-7 = A(z+3) + B(z+2)$$

put $z = -3$ put $z = -2$

$$15-7 = -B$$

$$-B = 8$$

$$\boxed{B = -8}$$

$$10-7 = A$$

$$\boxed{A = 3}$$

$$f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3}$$

(i) $2 < |z| < 3$

$$\Rightarrow |z| > 2, |z| < 3$$

$$f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3}$$

$$= 1 + \frac{3}{z(1+\frac{2}{z})} - \frac{8}{3(1+\frac{z}{3})}$$

$$= 1 + \frac{3}{z} \left(1 + \frac{2}{z}\right)^{-1} - \frac{8}{3} \left(1 + \frac{z}{3}\right)^{-1}$$

$$= 1 + \frac{3}{z} \left[1 - \left(\frac{2}{z}\right) + \left(\frac{2}{z}\right)^2 - \dots\right]$$

$$- \frac{8}{3} \left[1 - \left(\frac{z}{3}\right) + \left(\frac{z}{3}\right)^2 - \dots\right]$$

(ii) $|z| > 3$

$$f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3}$$

$$= 1 + \frac{3}{z(1+\frac{2}{z})} - \frac{8}{z(1+\frac{3}{z})}$$

$$= 1 + \frac{3}{z} \left(1 + \frac{2}{z}\right)^{-1} - \frac{8}{z} \left(1 + \frac{3}{z}\right)^{-1}$$

$$= 1 + \frac{3}{z} \left[1 - \left(\frac{2}{z}\right) + \left(\frac{2}{z}\right)^2 - \dots \right] - \frac{8}{z} \left[1 - \left(\frac{3}{z}\right) + \left(\frac{3}{z}\right)^2 - \dots \right]$$

Typical

③ Find the Laurent's series expansion of

$$f(z) = \frac{6z+5}{(z+1)z(z-2)} \text{ in}$$

$$1 < |z+1| < 3$$

Soln

$$\text{Given } f(z) = \frac{6z+5}{(z+1)z(z-2)}$$

$$\frac{6z+5}{(z+1)z(z-2)} = \frac{A}{z+1} + \frac{B}{z} + \frac{C}{z-2}$$

$$6z+5 = A(z)(z-2) + B(z+1)(z-2) + C(z)(z+1)$$

$$\underline{z=-1},$$

$$-6+5 = A(-1)(-1-2) + 0$$

$$-1 = 3A$$

$$\boxed{A = -\frac{1}{3}}$$

$$\underline{z=0},$$

$$0+5 = 0+B(1)(-2)+0$$

$$5 = -2B$$

$$\boxed{B = -\frac{5}{2}}$$

$$\underline{z=2},$$

$$12+5 = 0+C(3)(2)$$

$$17 = 6C$$

$$\boxed{C = \frac{17}{6}}$$

$$f(z) = \frac{-\frac{1}{3}}{z+1} + \frac{-\frac{5}{2}}{z} + \frac{\frac{17}{6}}{z-2}$$

$$1 < |z+1| < 3$$

$$|z+1| > 1 ; |z+1| < 3$$

$$f(z) = -\frac{1}{3(z+1)} - \frac{5}{2z} + \frac{17}{6(z-2)}$$

$$= \frac{-1}{3(z+1)} - \frac{5}{2 \left[\frac{z+1}{3} \right]} + \frac{17}{6(z-2)}$$

$$= \frac{-1}{3(z+1)} - \frac{5}{2(z+1) \left[1 - \frac{1}{z+1} \right]}$$

$$+ \frac{17}{6(-3)} \left[1 - \left(\frac{z+1}{3} \right) \right]^{-1}$$

$$= \frac{-1}{3(z+1)} - \frac{5}{2(z+1)} \left[1 - \frac{1}{z+1} \right]^{-1}$$

$$- \frac{17}{18} \left(1 - \frac{z+1}{3} \right)^{-1}$$

$$= \frac{-1}{3(z+1)} - \frac{5}{2(z+1)} \sum_{n=0}^{\infty} \left(\frac{1}{z+1} \right)^n$$

$$- \frac{17}{18} \sum_{n=0}^{\infty} \left(\frac{z+1}{3} \right)^n$$

① Expand $\cos z$ in a Taylor's Series about the point $z = \pi/4$

Solve Function	Value at $z = \pi/4$
$f(z) = \cos z$	$f(\pi/4) = \cos 45^\circ = \frac{1}{\sqrt{2}}$
$f'(z) = -\sin z$	$f'(45^\circ) = -\sin 45^\circ = -\frac{1}{\sqrt{2}}$
$f''(z) = -\cos z$	$f''(45^\circ) = -\cos 45^\circ = -\frac{1}{\sqrt{2}}$

$$f(z) = f(45^\circ) + \frac{f'(45^\circ)}{1!} z + \frac{f''(45^\circ)}{2!} z^2 + \dots$$

$$= \frac{1}{\sqrt{2}} + \left(-\frac{1}{\sqrt{2}}\right) z + \left(-\frac{1}{\sqrt{2}}\right) \frac{z^2}{2} + \dots$$

	0°	30°	45°	60°	90°
$\sin \theta$	$\frac{\sqrt{0}}{4} = 0$	$\frac{\sqrt{1}}{4} = \frac{1}{2}$	$\frac{\sqrt{2}}{4} = \frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{2}$	$\frac{\sqrt{4}}{4} = 1$
$\cos \theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0

Cauchy - Integral Theorem

Example (or)

⑥ Evaluate $\int_C \frac{z dz}{(z-1)(z-2)^2}$

Where C is the circle $|z-2| = \frac{1}{2}$ using Cauchy's integral formula.

Soln:-

Let $f(z) = \frac{z}{(z-1)(z-2)^2}$

$z-1=0 \Rightarrow z=1 \Rightarrow$ outside

$|z|=|1|=1 > \frac{1}{2}$

$z-2=0 \Rightarrow z=2 \Rightarrow$ Inside

$|z-2| = \frac{1}{2}$

$2 > \frac{1}{2}$

By C.I.F

$$\int_C \frac{f(z)}{(z-a)^{n+1}} dz = \frac{2\pi i}{n!} f^{(n)}(a)$$

$$\int_C \frac{\left(\frac{z}{z-1}\right)}{(z-2)^2} dz = \frac{2\pi i}{1!} f'(2)$$

$a=2$ $\hookrightarrow$ ①

$f(z) = \frac{z}{z-1}$ $\left(\frac{u}{v}\right)$

$\frac{u}{v} = \frac{vdu - u dv}{v^2}$

$f'(z) = \frac{(z-1)(1) - z(1)}{(z-1)^2}$

$f'(2) = \frac{(2-1) - 2}{(2-1)^2}$

$= \frac{1-2}{1} = -1$

$f'(2) = -1$

① $\Rightarrow \int_C \frac{z}{(z-1)(z-2)^2} dz$

$= 2\pi i (-1)$

$= -2\pi i$

① State and Prove
Cauchy's Integral
formula
Statement..

$$f'(a) = \frac{1!}{2\pi i} \int_C \frac{f(z)}{(z-a)^2} dz$$

Proof..

By Cauchy Integral formula

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a} dz$$

$$f(a) = \frac{1}{2\pi i} \int_C f(z) (z-a)^{-1} dz$$

Differentiating both sides w.r.to

$$f'(a) = \frac{1}{2\pi i} \int_C f(z) (-1) (z-a)^{-2} dz$$

$$= \frac{1}{2\pi i} \int_C f(z) (z-a)^{-2} dz$$

$$f'(a) = \frac{1!}{2\pi i} \int_C \frac{f(z)}{(z-a)^2} dz$$

$$f''(a) = \frac{2!}{2\pi i} \int_C \frac{f(z)}{(z-a)^3} dz$$

$$f'''(a) = \frac{3!}{2\pi i} \int_C \frac{f(z)}{(z-a)^4} dz$$

⋮

$$f^n(a) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-a)^{n+1}} dz$$

Hence the proof

② Evaluate $\int_C \frac{z+1}{z^2+2z+4} dz$

Where C is the circle
 $|z+1+i|=2$ using Cauchy's
integral formula.

Soln..

$$\text{Let } f(z) = \frac{z+1}{z^2+2z+4}$$

$$z^2+2z+4=0$$

$$z = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$\begin{aligned} a &= 1 \\ b &= 2 \\ c &= 4 \end{aligned}$$

$$= \frac{-2 \pm \sqrt{4-16}}{2 \times 1}$$

$$= \frac{-2 \pm \sqrt{-12}}{2}$$

$$\begin{aligned} 4 \times 3 \\ 2^2 \times 3 \end{aligned}$$

$$= \frac{-2 \pm 2\sqrt{3}i}{2}$$

$$z_1 = -1 \pm \sqrt{3}i$$

$$-1 + \sqrt{3}i \Rightarrow \text{outside}$$

$$-1 - \sqrt{3}i \Rightarrow \text{Inside}$$

$$\int_C \frac{z+1}{z^2+2z+4} dz = \int_C \frac{z+1}{(z-(-1+\sqrt{3}i))(z-(-1-\sqrt{3}i))} dz$$

$$f(z) = \frac{z+1}{z-(-1+\sqrt{3}i)} \text{ is analytic}$$

inside C .

By C.I.F

$$\int_C \frac{f(z)}{z-a} dz = 2\pi i f(a)$$

$$= 2\pi i f(-1-\sqrt{3}i)$$

$$= 2\pi i \left[\frac{-1-\sqrt{3}i+1}{-1-\sqrt{3}i+1-i\sqrt{3}} \right]$$

$$= 2\pi i \left[\frac{-\sqrt{3}i}{-2-i\sqrt{3}} \right]$$

$$= \pi i$$

Unit - IV

Formulas

① Taylor's Series

$$f(z) = f(a) + \frac{f'(a)}{1!} (z-a) + \frac{f''(a)}{2!} (z-a)^2 + \dots$$

② Cauchy's Integral formula

$$\int_C \frac{f(z)}{z-a} dz = 2\pi i f(a)$$

③ Laurent's Series

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n}$$

$$\text{where } a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-a)^{n+1}} dz$$

$$b_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-a)^{-n+1}} dz$$

① find the Laurent's

expansion of

$$f(z) = \frac{7z-2}{z(z-2)(z+1)} \quad \text{valid}$$

$$\text{in } 1 < |z+1| < 3.$$

Soln:

$$\text{let } f(z) = \frac{7z-2}{z(z-2)(z+1)}$$

$$\frac{7z-2}{z(z-2)(z+1)} = \frac{A}{z} + \frac{B}{z-2} + \frac{C}{z+1}$$

$$\frac{7z-2}{z(z-2)(z+1)} = \frac{A(z-2)(z+1)}{z(z-2)(z+1)} + \frac{B(z)(z+1)}{z(z-2)(z+1)} + \frac{C(z)(z-2)}{z(z-2)(z+1)}$$

$$7z-2 = A(z-2)(z+1) + Bz(z+1) + C(z)(z-2)$$

$$\text{put } z=0$$

$$-2 = A(-2)(1) + 0 + 0$$

$$A = \frac{-2}{-2}$$

$$\boxed{A=1}$$

$$\text{put } z=2$$

$$7(2)-2 = A(0) + B(2)(3) + C(0)$$

$$14-2 = 6B$$

$$6B = 12$$

$$B = \frac{12}{6}$$

$$\boxed{B=2}$$

$$\text{put } z=-1$$

$$-7-2 = A(0) + B(0) + C(-1)(-1-2)$$

$$-9 = 3C$$

$$\boxed{C=-3}$$

$$\frac{1}{z} + \frac{2}{z-2} - \frac{3}{z+1}$$

Region: $1 < |z+1| < 3$

Lower part:

$$1 < |z+1| \Rightarrow \left| \frac{1}{z+1} \right| < 1$$

Upper part:

$$|z+1| < 3 \Rightarrow \left| \frac{z+1}{3} \right| < 1$$

From ②

$$\text{Dr} \Rightarrow z+1$$

$$f(z) = \frac{1}{(z+1)-1} + \frac{2}{(z+1)-3} = \frac{3}{z+1}$$

$$= \frac{1}{(z+1)(1-\frac{1}{z+1})} + \frac{2}{3(\frac{z+1}{3}-1)} = \frac{3}{z+1}$$

$$= \frac{1}{(z+1)(1-x)} - \frac{3}{z+1} + \frac{2}{3(y-1)}$$

$$= \frac{1}{z+1} (1-x)^{-1} - \frac{3}{z+1} + \frac{2}{3} (1-y)^{-1}$$

$$x = \frac{1}{z+1}$$

$$y = \frac{z+1}{3}$$

$$= \frac{1}{z+1} \sum_{n=0}^{\infty} x^n - \frac{3}{z+1} + \frac{2}{3} \sum_{n=0}^{\infty} y^n$$

$$= \frac{1}{z+1} \sum_{n=0}^{\infty} \left(\frac{1}{z+1} \right)^n - \frac{3}{z+1} + \frac{2}{3} \sum_{n=0}^{\infty} \left(\frac{z+1}{3} \right)^n$$

Formulas

$$(1+x)^{-1} = \sum_{n=0}^{\infty} (-x)^n \quad \left. \vphantom{\sum_{n=0}^{\infty}} \right\} \text{if } |x| < 1$$

$$(1-x)^{-1} = \sum_{n=0}^{\infty} x^n$$

② Find the Laurent series

$$f(z) = \frac{z+2}{(z+1)(z+4)}$$

$$1 < |z| < 4$$

So

$$\text{Let } \frac{z+2}{(z+1)(z+4)} = \frac{A}{z+1} + \frac{B}{z+4} \rightarrow \textcircled{1}$$

$$\frac{z+2}{(z+1)(z+4)} = \frac{A(z+4) + B(z+1)}{(z+1)(z+4)}$$

$$z+2 = A(z+4) + B(z+1)$$

Put $z = -4$

$$-4+2 = A(0) + B(-4+1)$$

$$-2 = B(-3)$$

$$B = \frac{2}{3}$$

Put $z = -1$

$$-1+2 = A(-1+4) + B(0)$$

$$1 = A(3)$$

$$A = \frac{1}{3}$$

From ①

$$f(z) = \frac{(1/3)}{z+1} + \frac{(2/3)}{z+4} \rightarrow \textcircled{2}$$

Region: $1 < |z| < 4$

Lower part:

$$1 < |z| \Rightarrow \frac{1}{|z|} < 1 \Rightarrow \left| \frac{1}{z} \right| < 1$$

Upper part

$$|z| < 4 \Rightarrow \frac{|z|}{4} < 1$$

$$\Rightarrow \left| \frac{z}{4} \right| < 1$$

From (2)

$$\begin{cases} x = \frac{1}{2} \\ y = \frac{z}{4} \end{cases}$$

$$f(z) = \frac{(1/3)}{z(1+\frac{1}{2})} + \frac{(2/3)}{4(\frac{z}{4}+1)}$$

$$= \frac{1}{32(1+x)} + \frac{2}{6(y+1)}$$

$$= \frac{1}{32}(1+x)^{-1} + \frac{1}{6}(1+y)^{-1}$$

$$= \frac{1}{32} \sum_{n=0}^{\infty} (-x)^n + \frac{1}{6} \sum_{n=0}^{\infty} (-y)^n$$

$$= \frac{1}{32} \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n + \frac{1}{6} \sum_{n=0}^{\infty} \left(-\frac{z}{4}\right)^n$$

Unit - V

Ordinary Differential Equations (ODE)

Two marks

① Solve

$$\frac{d^2x}{dt^2} + b \frac{dx}{dt} + 9x = 0$$

Soln

$$\text{Given } \frac{d^2x}{dt^2} + b \frac{dx}{dt} + 9x = 0$$

$$(D^2 + bD + 9)x = 0$$

A.E is

$$m^2 + b m + 9 = 0$$

$$m = -3, -3$$

$$\text{C.F} = (Ax + B)e^{mx}$$

$$\text{C.F} = (Ax + B)e^{-3x}$$

3 3
opposite sign
-3, -3

② Solve $(D^4 - 2D^3 + D^2)y = 0$

Soln

Given *

A.E is

$$m^4 - 2m^3 + m^2 = 0$$

$$m^2(m^2 - 2m + 1) = 0$$

$$m^2 = 0 \quad | \quad m^2 - 2m + 1 = 0$$

$$m = 0, 0$$

$$m = 1, 1$$

opposite sign
-1, -1
1, 1

$$y = (Ax + B)e^{mx}$$

$$(e^0 = 1)$$

$$= (Ax + B)e^{0x} + (Cx + D)e^x$$

$$y = Ax + B + (Cx + D)e^x$$

③ Solve $(D^2 - 6D + 13)y = 0$

Soln

Given *

A.E is

$$m^2 - 6m + 13 = 0$$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$a = 1$$

$$b = -6$$

$$c = 13$$

$$= \frac{6 \pm \sqrt{36 - 4(1)(13)}}{2(1)}$$

$$= \frac{6 \pm \sqrt{36 - 52}}{2}$$

$$= \frac{6 \pm \sqrt{-16}}{2}$$

$$= \frac{6 \pm 4i}{2}$$

$$m = 3 \pm 2i$$

$$\text{C.F} = y = e^{\alpha x} (A \cos \beta x + B \sin \beta x)$$

$$y = e^{3x} (A \cos 2x + B \sin 2x)$$

④ Solve $(D^4 - 1)y = 0$

A.E is $m^4 - 1 = 0$

$$(m^2)^2 - (1)^2 = 0$$

$$a^2 - b^2 = (a+b)(a-b)$$

$$m^2 + 1 = 0 ; m^2 - 1 = 0$$

$$m^2 = -1 \quad m^2 = 1$$

$$m = \pm i \quad m = \pm 1$$

$$C.F = A e^{m_1 x} + B e^{m_2 x}$$

$$+ e^{\alpha x} [C \cos \beta x + D \sin \beta x]$$

$$\therefore F = A e^x + B e^x$$

$$+ e^0 [\cos x + D \sin x]$$

② Find the Particular integral of $(D^3 - D)y = e^{2x}$.

Soln

$$P.I = \frac{1}{D^3 - 1} e^{2x}$$

$$= \frac{1}{(2)^3 - 1} e^{2x} \quad (D=2)$$

$$= \frac{1}{8-1} e^{2x}$$

$$P.I = \frac{1}{7} e^{2x}$$

Type: 1

① Find the Particular integral of $(D^2 - 2D + 1)y = e^x$

Soln:

Given $\otimes$

$$P.I = \frac{1}{D^2 - 2D + 1} e^x$$

$$(D=1)$$

$$= \frac{1}{1-2+1=0} \quad \text{ordinary rule fails.}$$

$$= \frac{x}{2D-2} e^x$$

$$(D=1)$$

$$= \frac{x}{2-2=0} e^x \quad \text{ordinary rule fails.}$$

$$P.I = \frac{x^2}{2} e^x$$

Type: 2

$$\frac{1}{f(D)} = \sin ax \text{ (or)}$$

$$\frac{1}{f(D)} = \cos ax$$

$$\text{Formula } D^2 = -a^2$$



① Find the P.I of $(D^2 + 4)y = \cos 2x$.

Soln

$$P.I = \frac{1}{D^2 + 4} \cos 2x$$

$$D^2 = -a^2 = -2^2 = -4$$

$$= \frac{1}{-4+4=0} \cos 2x$$

ordinary rule fails.

$$= \frac{x}{2D} \cos 2x$$

$$= \frac{x}{2} \int \cos 2x dx$$

$$= \frac{x}{2} \left(\frac{\sin 2x}{2} \right) = \frac{x}{4} \sin 2x$$

② Find the P.I of

$$\frac{d^2y}{dx^2} + 4y = \sin 2x$$

Soln:

$$P.I = \frac{1}{D^2 + 4} \sin 2x$$

$$= \frac{1}{-4 + 0} \sin 2x$$

$$= \frac{x}{2D} \sin 2x$$

$$= \frac{x}{2} \cdot \frac{1}{D} \sin 2x$$

$$= \frac{x}{2} \int \sin 2x \, dx$$

$$= \frac{x}{2} \left(-\frac{\cos 2x}{2} \right)$$

$$= -\frac{x}{4} \cos 2x$$

Variation of Parameters

① solve $\frac{d^2y}{dx^2} + y = \tan x$

by the method of variation of parameters.

Soln:

Given $(D^2 + 1)y = \tan x$

A.E is

$$m^2 + 1 = 0$$

$$m^2 = -1$$

$$m = \pm i$$

$$C.F = e^{ix} [A \cos \beta x + B \sin \beta x]$$

$$C.F = e^0 [A \cos x + B \sin x]$$

$f_1 = \cos x$	$f_2 = \sin x$	
$f_1' = -\sin x$	$f_2' = \cos x$	$x = \tan x$

$$\begin{vmatrix} f_1 & f_1' \\ f_2 & f_2' \end{vmatrix} = \begin{vmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{vmatrix}$$

$$= \cos^2 x + \sin^2 x$$

$$= 1$$

$$P = - \int \frac{f_2 x}{f_1 f_2' - f_1' f_2} dx$$

$$= - \int \sin x \tan x \, dx$$

$$= - \int \sin x \left(\frac{\sin x}{\cos x} \right) dx$$

$$e = 1$$

$$= - \int \frac{\sin^2 x}{\cos x} dx$$

$$= - \int \frac{(1 - \cos^2 x)}{\cos x} dx$$

$$= - \int \left(\frac{1}{\cos x} - \frac{\cos^2 x}{\cos x} \right) dx$$

$$= - \int (\sec x - \cos x) dx$$

$$= - [\log(\sec x + \tan x) - \sin x]$$

$$P = -\log(\sec x + \tan x) + \sin x$$

$$Q = \int \frac{f_1 x}{f_1 f_2' - f_1' f_2} dx$$

$$= \int \frac{\cos x \tan x}{1} dx$$

$$= \int \cos x \left(\frac{\sin x}{\cos x} \right) dx$$

$$= \int \sin x dx$$

$$Q = -\cos x$$

$$P.I = P f_1 + Q f_2$$

$$= -\cos x \log(\sec x + \tan x) + \cos x \sin x - \sin x \cos x$$

$$P.I = -\cos x \log(\sec x + \tan x)$$

$$y = C.F + P.I$$

$$y = A \cos x + B \sin x - \cos x \log(\sec x + \tan x)$$

$$\sin^2 x + \cos^2 x = 1$$

$$\sin^2 x = 1 - \cos^2 x$$

$$\frac{1}{\cos x} = \sec x$$

$$(2) \text{ Solve } (D^2 + a^2)y = \tan ax$$

by the method of variation of parameters.

Soln:

$$\text{Given } (D^2 + a^2)y = \tan ax$$

$$\text{A.E is } m^2 + a^2 = 0$$

$$m^2 = -a^2$$

$$m = \pm ai$$

$$C.F = e^{ax} [A \cos \beta x + B \sin \beta x]$$

$$C.F = e^0 [A \cos ax + B \sin ax]$$

$f_1 = \cos ax$	$f_2 = \sin ax$
$f_1' = -a \sin ax$	$f_2' = a \cos ax$

$x = \tan ax$

$$P.I = \int \frac{f_2 x}{f_1 f_2' - f_1' f_2} dx$$

$$= \int \frac{\sin ax \cdot x}{\cos ax \cdot a \cos ax - (-a \sin ax) \sin ax} dx$$

$$= \int \frac{\sin ax \cdot x}{a \cos^2 ax + a \sin^2 ax} dx$$

$$= \int \frac{\sin ax \cdot x}{a (\cos^2 ax + \sin^2 ax)} dx$$

$$= \int \frac{\sin ax \cdot x}{a (1)} dx$$

$$= \int \frac{\sin ax \cdot x}{a} dx$$

$$= \frac{1}{a} \int \sin ax \cdot x dx$$

$$P = - \int \frac{f_2 x}{f_1 f_2' - f_1' f_2} dx$$

$$= - \int \left[\frac{\sin ax \tan ax}{a} \right] dx$$

$$P = - \int \frac{\sin ax}{a} \left(\frac{\sin ax}{\cos ax} \right) dx$$

$$= - \frac{1}{a} \int \frac{\sin^2 ax}{\cos ax} dx$$

$$= - \frac{1}{a} \int \frac{1 - \cos^2 ax}{\cos ax} dx$$

$$= - \frac{1}{a} \left[\int \frac{1}{\cos ax} - \frac{\cos^2 ax}{\cos ax} \right] dx$$

$$= - \frac{1}{a} \int (\sec ax - \cos ax) dx$$

$$= - \frac{1}{a} \left[\frac{1}{a} \log(\sec ax + \tan ax) \right.$$

$$\left. - \frac{\sin ax}{a} \right]$$

$$= - \frac{1}{a^2} \left[\log(\sec ax + \tan ax) - \sin ax \right]$$

$$Q = \int \frac{f_1 x}{f_1 f_2' - f_1' f_2} dx$$

$$= \int \frac{\cos ax \tan ax}{a} dx$$

$$= \frac{1}{a} \int \cos ax \left(\frac{\sin ax}{\cos ax} \right) dx$$

$$= \frac{1}{a} \int \sin ax dx$$

$$= + \frac{1}{a} \left[- \frac{\cos ax}{a} \right]$$

$$Q = - \frac{1}{a^2} \cos ax$$

$$P.I = P f_1 + Q f_2$$

$$= - \frac{1}{a^2} \left[\log(\sec ax + \tan ax) - \sin ax \right]$$

$$- \frac{1}{a^2} [\cos ax] \sin ax$$

$$P.I = - \frac{\cos ax}{a^2} \left[\log(\sec ax + \tan ax) - \sin ax \right]$$

$$= - \frac{\cos ax}{a^2} \left[\log(\sec ax + \tan ax) - \sin ax \right]$$

$$y = C.F + P.I$$

$$y = A \cos ax + B \sin ax$$

$$- \frac{\cos ax}{a^2} \left[\log(\sec ax + \tan ax) - \sin ax \right]$$

problems Based on
cauchy-Euler

(2 marks)

Q.1

(1) solve

$$x^2 \frac{d^2 y}{dx^2} - x \frac{dy}{dx} + y = 0$$

Soln:

Given $(x^2 D^2 - xD + 1)y = 0$

$$\begin{array}{l|l} \text{put } x = e^z & xD = D' \\ \log x = z & x^2 D^2 = D'^2 - D' \end{array}$$

$$[D'^2 - D' + 1]y = 0$$

$$[D'^2 - 2D' + 1]y = 0$$

A.E is

$$m^2 - 2m + 1 = 0 \quad \begin{array}{c} 1 \\ \wedge \\ -1 -1 \end{array}$$

$$m = 1, 1$$

$$C.F = (Az + B)e^z$$

Q.2

(2) Solve $x \frac{d^2 y}{dx^2} + \frac{dy}{dx} = 0$

Soln:

Given $x D^2 y + D y = 0$

$$X \cdot X \Rightarrow x^2 D^2 y + x D y = 0$$

$$(x^2 D^2 + x D)y = 0$$

put $x = e^z \Rightarrow \log x = z$

$$\begin{array}{l} xD = D' \\ x^2 D^2 = D'^2 - D' \end{array}$$

$$[D'^2 - D' + D']y = 0$$

$$D^2 y = 0$$

A.E is

$$m^2 = 0$$

$$m = 0, 0$$

$$y = (Az + B)e^{mx}$$

$$e^0 = 1$$

$$= (Az + B)e^0$$

$$= Az + B$$

$$y = A(\log x) + B$$

Unit-V Ordinary Differential Equations

2 marks
Second order linear d.f

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = v.$$

Complementary Functions

Roots of Auxiliary Equations.	C.F
Roots are Real & different m_1, m_2 $m_1 \neq m_2$	$Ae^{m_1 x} + Be^{m_2 x}$
Roots are Real & equal $m_1 = m_2 = m$	$(Ax+B)e^{mx}$ (or) $(A+Bx)e^{mx}$
Roots are Imaginary $\alpha \pm i\beta$	$e^{\alpha x} [A \cos \beta x + B \sin \beta x]$

Problems

① solve $(D^2 - 4)y = 1$

Soln

The auxiliary equation is

$$m^2 - 4 = 0$$

$$m^2 = 4$$

$$m = \pm 2$$

$$C.F = Ae^{2x} + Be^{-2x}$$

$$P.I = \frac{1}{D^2 - 4} (1) = \frac{1}{D^2 - 4} e^{0x}$$

$$= \frac{1}{0-4} = -\frac{1}{4}$$

$$y = C.F + P.I$$

$$= Ae^{2x} + Be^{-2x} - \frac{1}{4}$$

② solve $(D^2 - 3D + 2)y = 0$

$$y = Ae^x + Be^{2x}$$

③ solve $(D^2 + 2D - 1)y = 0$

$$y = A e^{(-1+\sqrt{2})x} + B e^{(-1-\sqrt{2})x}$$

④ $(D^3 - 6D^2 + 11D - 6)y = 0$

$$y = Ae^x + Be^{2x} + Ce^{3x}$$

Particular Integral

1) Find the P.I of

$$(D^2 - 4D + 4)y = e^{2x}$$

Soln

$$P.I = \frac{1}{D^2 - 4D + 4} e^{2x}$$

$$= \frac{1}{2D - 4} x e^{2x}$$

$$P.I = \frac{1}{2} x^2 e^{2x}$$

② $\frac{d^2 y}{dx^2} - 6 \frac{dy}{dx} + 8y = 0$

$$y = Ae^{4x} + Be^{2x}$$

③ $(D^2 + 3D + 2)y = 5e^{-4x}$

$$C.F = Ae^{-x} + Be^{-2x}$$

$$P.I = \frac{1}{D^2 + 3D + 2} 5e^{-4x}$$

$$= \frac{5}{(-4)^2 + 3(-4) + 2} e^{-4x}$$

$$P.I = \frac{5e^{-4x}}{6}$$

$$y = C.F + P.I$$

$$= Ae^{-x} + Be^{-2x} + \frac{5e^{-4x}}{6}$$

$$(1) (D^2 - 5D + 6)y = e^{4x}$$

$$y = Ae^{2x} + Be^{3x} + \frac{e^{4x}}{2}$$

(2) Find the P.I of

$$(D^2 + 12D - 13)y = e^x$$

Soln

$$P.I = \frac{x e^x}{14}$$

(3) Find the P.I of

$$(D^2 + 16)y = \cos 4x$$

Soln

$$P.I = \frac{1}{F(D)} (f(x))$$

$$= \frac{1}{D^2 + 16} \cos 4x$$

$$= \frac{x}{2D} \cos 4x$$

$$= \frac{x}{2} \frac{1}{D} (\cos 4x)$$

$$= \frac{x}{2} \int \cos 4x \, dx$$

$$= \frac{x}{2} \frac{\sin 4x}{4}$$

$$= \frac{x \sin 4x}{8}$$

ordinary rule

fails when
 $D^2 = -a^2$
 $D^2 = -16$

(special rule)

(2) Solve $(D^2 + 16)y = \sin 9x$

$$m = \pm 4i$$

$$C.F = A \cos 4x + B \sin 4x$$

$$P.I = \frac{1}{F(D)} (f(x))$$

$$= \frac{1}{D^2 + 16} \sin 9x$$

$$= \frac{1}{-9^2 + 16} \sin 9x$$

$$= \frac{1}{-81 + 16} \sin 9x$$

$$= \frac{1}{-65} \sin 9x$$

$$= -\frac{1}{65} \sin 9x$$

$$y = C.F + P.I$$

$$y = A \cos 4x + B \sin 4x - \frac{1}{65} \sin 9x$$

(1) Solve $(3D^2 + D - 14)y = 5 + 13e^x$

$$\begin{array}{ccc|c} 3 & 1 & -14 & \\ 0 & 6 & 14 & \\ \hline 3 & 7 & 0 & \end{array}$$

$$3x + 7 = 0$$

$$3x = -7$$

$$x = -\frac{7}{3}$$

$$x = 2, -\frac{7}{3}$$

$$C.F = Ae^{2x} + Be^{-\frac{7}{3}x}$$

$$P.I = \frac{1}{3D^2 + D - 14} (5 + 13e^{2x})$$

$$= \frac{1}{3D^2 + D - 14} 5e^{0x} + \frac{13}{3D^2 + D - 14} e^{2x}$$

$$= \frac{1}{3(0)^2 + 0 - 14} 5e^{0x} + \frac{13}{3(2)^2 + 2 - 14} e^{2x}$$

$$= -\frac{5}{14} + \frac{13}{0} e^{2x}$$

$$= -\frac{5}{14} + 13x e^{2x}$$

$$= -\frac{5}{14} + \frac{13x}{6(2) + 1} e^{2x}$$

$$= -\frac{5}{14} + x e^{2x}$$

$$\textcircled{1} (D^2 - 4D + 13)y = e^{2x}$$

$$m = 2 \pm 3i$$

$$y = e^{2x} \left[A \cos 3x + B \sin 3x \right] + \frac{e^{2x}}{9}$$

Variation of Parameters

$$C.F. = A f_1 + B f_2$$

$A, B \rightarrow \text{constants}$

$f_1, f_2 \rightarrow \text{function of } x.$

$$P.I. = P f_1 + Q f_2$$

$$P = - \int \frac{f_2 x}{f_1 f_2' - f_1' f_2} dx$$

$$Q = \int \frac{f_1 x}{f_1 f_2' - f_1' f_2} dx$$

$$y = A f_1 + B f_2 + P.I.$$

Note :-

Wronskian Method

$$W = \begin{vmatrix} f_1 & f_1' \\ f_2 & f_2' \end{vmatrix} = f_1 f_2' - f_2 f_1'$$

$\textcircled{1}$ Solve by the method of Variation of Parameters.

$$2 \frac{d^2 y}{dx^2} + 8y = \tan 2x$$

Soln : $\checkmark$
given

$$\div 2 \Rightarrow \frac{d^2 y}{dx^2} + 4y = \frac{1}{2} \tan 2x$$

The A.E is

$$m^2 + 4 = 0$$

$$m^2 = -4$$

$$m = \pm 2i$$

$$C.F. = A \cos 2x + B \sin 2x$$

$f_1 = \cos 2x$	$f_2 = \sin 2x$	$f_3 = \tan 2x$
$f_1' = -2 \sin 2x$	$f_2' = 2 \cos 2x$	$X = \tan 2x$

$$\begin{vmatrix} f_1 & f_1' \\ f_2 & f_2' \end{vmatrix} = \begin{vmatrix} \cos 2x & -2 \sin 2x \\ \sin 2x & 2 \cos 2x \end{vmatrix}$$

$$f_1 f_2' - f_2 f_1' = 2 \cos^2 2x + 2 \sin^2 2x$$

$$= 2 (\cos^2 2x + \sin^2 2x)$$

$$= 2(1)$$

$$= 2$$

$$P = - \int \frac{f_2 X}{f_1 f_2' - f_1' f_2} dx$$

$$\tan x = \frac{\sin x}{\cos x}$$

$$\sin^2 x = 1 - \cos^2 x$$

$$= - \int \frac{\sin 2x \tan 2x}{2} dx$$

$$= - \frac{1}{2} \int \sin 2x \times \frac{\sin 2x}{\cos 2x} dx$$

$$= - \frac{1}{2} \int \frac{\sin^2 2x}{\cos 2x} dx$$

$$= - \frac{1}{2} \int \frac{(1 - \cos^2 2x)}{\cos 2x} dx$$

$$= - \frac{1}{2} \int \left(\frac{1}{\cos 2x} - \frac{\cos^2 2x}{\cos 2x} \right) dx$$

$$= - \frac{1}{2} \int (\sec 2x - \cos 2x) dx$$

$$= - \frac{1}{2} \left[\frac{1}{2} \log(\sec 2x + \tan 2x) - \frac{\sin 2x}{2} \right]$$

$$P = - \frac{1}{4} (\log(\sec 2x + \tan 2x) + \frac{\sin 2x}{4})$$

$$Q = \int \frac{f_1 X}{f_1 f_2' - f_1' f_2} dx$$

$$= \int \frac{\cos 2x \tan 2x}{2} dx$$

$$Q = \frac{1}{2} \int \cos 2x \frac{\sin 2x}{\cos 2x} dx$$

$$= \frac{1}{2} \int \sin 2x dx$$

$$= -\frac{\cos 2x}{2 \cdot 2}$$

$$Q = -\frac{\cos 2x}{4}$$

$$P.I = P f_1 + Q f_2$$

$$= \frac{\cos 2x}{4} \left[\log(\sec 2x + \tan 2x) \right]$$

$$+ \frac{\cos 2x \sin 2x}{4} - \frac{\sin 2x \cos 2x}{4}$$

$$= -\frac{\cos 2x}{4} \log(\sec 2x + \tan 2x)$$

$$y = A f_1 + B f_2 + P.I$$

$$= A \cos 2x + B \sin 2x$$

$$- \frac{\cos 2x}{4} \log(\sec 2x + \tan 2x)$$

3) Solve the d.E

$$\frac{d^2 y}{dx^2} + 2 \frac{dy}{dx} + y = \frac{e^{-x}}{x^2} \text{ by}$$

the method of V.P.

Soln

$$A.E.D \quad m^2 + 2m + 1 = 0$$

$$m = -1, -1$$

$$C.F = (Ax + B) e^{-x}$$

$$C.F = Ax e^{-x} + B e^{-x}$$

$f_1 = x e^{-x}$ u.v = u dv + v du	$f_2 = e^{-x}$
$f_1' = -x e^{-x} + e^{-x}$	$f_2' = -e^{-x}$

$$\begin{vmatrix} f_1 & f_1' \\ f_2 & f_2' \end{vmatrix} = \begin{vmatrix} x e^{-x} & -x e^{-x} + e^{-x} \\ e^{-x} & -e^{-x} \end{vmatrix}$$

$$= -x e^{-2x} + x e^{-2x} - e^{-2x}$$

$$= -e^{-2x}$$

$$P = - \int \frac{f_2 x}{f_1 f_2' - f_2' f_1} dx$$

$$= - \int \left(\frac{e^{-x}}{e^{-2x}} \right) \left(\frac{e^{-x}}{e^{-2x}} \right) dx$$

$$= + \int e^x dx = - \int \frac{1}{x^2} dx$$

$$P = - \frac{1}{x}$$

$$Q = \int \frac{f_1 x}{f_1 f_2' - f_2' f_1} dx$$

$$= \int \frac{x e^{-x} \left(\frac{e^{-x}}{x^2} \right)}{-e^{-2x}} dx$$

$$= - \int \frac{1}{x} dx$$

$$Q = -\log x$$

$$P.I = P f_1 + Q f_2$$

$$= -\frac{1}{x} (x e^{-x}) + \log x e^{-x}$$

$$P.I = -e^{-x} - \log x e^{-x}$$

$$y = Ax e^{-x} + B e^{-x} - e^{-x} - \log x e^{-x}$$

(3) Find the wronskian of y_1, y_2 of $y'' - 2y' + y = e^x \log x$ soln:

$$C.F = A x e^x + B e^x$$

$$\begin{vmatrix} y_1 & y_1' \\ y_2 & y_2' \end{vmatrix} = \begin{vmatrix} x e^x & x e^x + e^x \\ e^x & e^x \end{vmatrix}$$

$$= x e^{2x} - x e^{2x} - e^{2x}$$

$$= -e^{2x}$$

(4) $\frac{d^2 y}{dx^2} + y = \operatorname{cosec} x$

$$m^2 + 1 = 0$$

$$m^2 = -1$$

$$m = \pm i$$

$$C.F = A \cos x + B \sin x$$

$f_1 = \cos x$	$f_2 = \sin x$	$X = \operatorname{cosec} x$
$f_1' = -\sin x$	$f_2' = \cos x$	

$$\begin{vmatrix} f_1 & f_1' \\ f_2 & f_2' \end{vmatrix} = \begin{vmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{vmatrix}$$

$$= \cos^2 x + \sin^2 x$$

$$= 1$$

$$P = - \int \frac{f_2 X}{f_1 f_2' - f_2 f_1'} dx$$

$$= - \int \frac{\sin x \operatorname{cosec} x}{1} dx$$

$$= - \int \sin x \cdot \frac{1}{\sin x} dx$$

$$= - \int dx = -x$$

$$P = -x$$

$$Q = \int \frac{f_1 X}{f_1 f_2' - f_2 f_1'} dx$$

$$Q = \int \frac{\cos x \operatorname{cosec} x}{1} dx$$

$$= \int \cos x \left(\frac{1}{\sin x} \right) dx$$

$$= \int \cot x dx$$

$$Q = \log(\sin x)$$

$$P.I = -x \cos x + \sin x \log(\sin x)$$

$$y = A \cos x + B \sin x - x \cos x + \sin x \log(\sin x)$$

Formulas

$$\int \sec x dx = \log(\sec x + \tan x)$$

$$\int \cot x dx = \log(\sin x)$$

$$\int \tan x dx = -\log(\cos x) \text{ (or) } \log \sec x$$

$$\cos^2 \theta + \sin^2 \theta = 1$$

$$\sin^2 \theta = 1 - \cos^2 \theta$$

$$\cos^2 \theta = 1 - \sin^2 \theta$$

$$\int \operatorname{cosec} x dx = -\log(\operatorname{cosec} x + \cot x)$$

$$\int \sin x dx = -\cos x$$

$$\int \cos x dx = \sin x$$

$$\int \frac{1}{x} dx = \log x$$

$$\tan \theta = \frac{\sin \theta}{\cos \theta}$$

$$\frac{1}{\cos \theta} = \sec \theta$$

$$\frac{\cos x}{\sin x} = \cot x$$

$$\textcircled{1} (D^2 + a^2)y = \sec ax$$

Soln

$$y = A \cos ax + B \sin ax$$

$$- \frac{\cos ax}{a^2} (\log \sec ax)$$

$$+ \frac{x \sin ax}{a}$$

$$- \log \cos ax$$

$$\textcircled{2} (D^2 + 4)y = \cot 2x$$

Soln:

$$y = A \cos 2x + B \sin 2x$$

$$+ \frac{1}{4} \sin 2x \log (\sec 2x - \cot 2x)$$

$$\textcircled{3} (2D^2 - D - 3)y = 25 e^x$$

$$y = A e^{3/2 x} + B e^{-x} - 4 e^x + 10 x e^x$$

$$\textcircled{4} \frac{d^2 y}{dx^2} + a^2 y = \tan ax$$

$$y = A \cos ax + B \sin ax$$

$$- \frac{1}{a^2} \left[\cos ax \log (\sec ax + \tan ax) \right]$$

$$\textcircled{5} \frac{d^2 y}{dx^2} + 4y = \sec 2x$$

$$y = A \cos 2x + B \sin 2x$$

$$+ \frac{1}{4} \cos 2x \log \cos 2x + \frac{x}{2} \sin 2x$$

$$\textcircled{6} (D^2 + 1)y = x \sin x$$

(May/June - 2010)

$$y = A \cos x + B \sin x$$

$$- \cos x \left(\frac{x^2}{4} - \frac{x \sin 2x}{4} - \frac{\cos 2x}{8} \right)$$

$$+ \sin x \left(-\frac{x \cos 2x}{4} + \frac{\sin 2x}{8} \right)$$

Cauchy Homogeneous

Equation

The equn is

$$x^n \frac{dy}{dx^n} + a_1 x^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + \dots$$

Put $x = e^z$ $\tan(y) = f(x)$

$$\log x = z \Rightarrow \frac{dz}{dx} = \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}$$

$$= \frac{1}{x} \frac{dy}{dz}$$

$$x \frac{dy}{dx} = \frac{dy}{dz}$$

$$\boxed{x \frac{dy}{dx} = D' y}$$

$$\boxed{x^2 \frac{dy}{dx} = D'(D'-1)}$$

$$\boxed{= D'^2 - D'}$$

① Solve $x^2 \frac{d^2y}{dx^2} + 4x \frac{dy}{dx} + 2y = 0$

Soln:

$$x^2 + \frac{1}{x^2}$$

Put $x = e^z$

$$\log x = z$$

$$x \frac{dy}{dx} = D' y$$

$$x^2 \frac{d^2y}{dx^2} = (D'^2 - D') y ; \text{ where}$$

$$D' = \frac{d}{dz}$$

$$(D'^2 - D') y + 4D' y + 2y = 0$$

$$(D'^2 + 3D' + 2) y = 0$$

$$m = -1, -2$$

$$C.F. = A e^{m_1 x} + B e^{m_2 x}$$

$$\boxed{x^2 = e^{2z}} = A e^{-z} + B e^{-2z}$$

$$\log a^n = n \log a$$

$$= A e^{-\log x} + B e^{-2 \log x}$$

$$= A e^{\log \frac{1}{x}} + B e^{\log \frac{1}{x^2}}$$

$$C.F. = \frac{A}{x} + \frac{B}{x^2}$$

$$P.I. = \frac{1}{D'^2 + 3D' + 2} (e^{2z} + e^{-2z})$$

$$= \frac{1}{(2)^2 + 3(2) + 2} e^{2z} + \frac{1}{2(-2) + 3} x e^{-2z}$$

$$= \frac{e^{2z}}{12} + \frac{x e^{-2z}}{2(-2) + 3}$$

$$= \frac{e^{2z}}{12} - \frac{x e^{-2z}}{1}$$

$$= \frac{e^{2 \log x}}{12} - \log x e^{-2 \log x}$$

$$= \frac{e^{\log x^2}}{12} - \log x e^{\log x^{-2}}$$

$$P.I. = \frac{x^2}{12} - \frac{\log x}{x^2}$$

$$y = C.F. + P.I.$$

$$y = \frac{A}{x} + \frac{B}{x^2} + \frac{x^2}{12} - \frac{\log x}{x^2}$$

② Solve the equn

$$\frac{d^2y}{dx^2} + \frac{1}{x} \frac{dy}{dx} = \frac{12 \log x}{x^2}$$

Soln

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} = 12 \log x$$

Put $z = \log x$; $x = e^z$. Then

$$x \frac{dy}{dx} = D'y \quad \&$$

$$x^2 \frac{d^2y}{dx^2} = (D'^2 - D')y$$

$$[D'^2 - D' + D']y = 12x$$

$$D'^2 y = 12x$$

$$m^2 = 0$$

$$m = 0 \text{ (twice)}$$

$$C.F = (Ax + B)e^{mx}$$

$$= (Ax + B)e^0$$

$$e^0 = 1$$

$$C.F = Ax + B = A \log x + B$$

$$P.I = \frac{1}{D^2} (12x)$$

$$= 12 \cdot \frac{1}{D} \left\{ \frac{1}{D} x \right\}$$

$$= 12 \cdot \frac{1}{D} \int x dx$$

$$= 12 \cdot \frac{1}{D} \left(\frac{x^2}{2} \right)$$

$$= 12 \int \frac{x^2}{2} dx$$

$$= \frac{12}{2} \cdot \frac{x^3}{3}$$

$$= 2x^3$$

$$P.I = 2(\log x)^3$$

$$y = C.F + P.I$$

$$y = A \log x + B + 2(\log x)^3$$

(3) Transform the eqn

$$(2x+3)^2 \frac{d^2y}{dx^2} + 2(2x+3) \frac{dy}{dx} - 12y = 12x$$

into a D.F with constant coefficients.

Soln: $z = \log x$

Put $\log(2x+3) = z$

$$2x+3 = e^z$$

$$2x = e^z - 3$$

$$x = \frac{e^z - 3}{2}$$

Put

$$(2x+3) \frac{dy}{dx} = 2D'y$$

$$(2x+3)^2 \frac{d^2y}{dx^2} = 2^2(D'^2 - D')y$$

$$[4(D'^2 - D') - 2(2D')]y = 3 \left(\frac{e^z - 3}{2} \right)$$

$$4(D'^2 - 2D' - 3)y = 3(e^z - 3)$$

$$(D'^2 - 2D' - 3)y = \frac{3}{4}(e^z - 3)$$

Soln

(4) $(3x^2 D^2 + 5x D + 7)y = \frac{2}{x \log x}$

Put $z = \log x$

$$x = e^z$$

$$x D = D'y$$

$$x^2 D^2 = (D'^2 - D')y$$

$$(3(D'^2 - D') + 5D' + 7)y = \frac{2}{2e^z}$$

$$(3D'^2 + 2D' + 7)y = \frac{2e^{-z}}{2}$$

(5) solve $x^2 y'' - xy' + y = 0$

$y = (A \log x + B) x$

(6) solve

$(1+x)^2 \frac{d^2 y}{dx^2} + (1+x) \frac{dy}{dx} + y$

$= 2 \sin[\log(1+x)]$

Soln

put $1+x = e^z$

$\log(1+x) = z$

$(1+x) \frac{dy}{dx} = D y$

$(1+x)^2 \frac{dy}{dx^2} = D^2 - D$

$[D^2 - D + D + 1] y = 2 \sin z$

$(D^2 + 1) y = 2 \sin z$

$m^2 + 1 = 0$

$m = \pm i$

C.F = $A \cos z + B \sin z$

$= A \cos(\log(1+x)) + B \sin(\log(1+x))$

P.I = $\frac{1}{D^2 + 1} 2 \sin z$

$= \frac{2z}{2D'} \sin z$

$= z \int \sin z dz$

$= -z \cos z$

$= -(\log(1+x)) \cos(\log(1+x))$

$y = C.F + P.I$

$y = A \cos z + B \sin z$

$= \log(1+x) \cos \log(1+x)$

(7) solve $(x^2 D^2 - xD + 1) y = \left(\frac{\log x}{x}\right)^2$

Soln

put $x = e^z$

$\log x = z$

$x D = D'$

$x^2 D^2 = D'^2 - D'$

$[D'^2 - D' - D' + 1] y = \left(\frac{z}{e^z}\right)^2$

$[D'^2 - 2D' + 1] y = z^2 e^{-2z}$

A.E is

$m^2 - 2m + 1 = 0$
 $m = 1, 1$

C.F = $(Az + B) e^{-z}$

P.I = $\frac{1}{(D'-1)^2} z^2 e^{-2z}$

$= \frac{e^{-2z} z^2}{(D'-1)^2}$

$= \frac{e^{-2z} z^2}{(D'-2)^2}$

$= \frac{e^{-2z} z^2}{(D'-3)^2}$

$= \frac{e^{-2z} z^2}{(3-D')^2}$

$= \frac{e^{-2z} z^2}{3^2 \left(1 - \frac{D'}{3}\right)^2}$

$= \frac{e^{-2z} z^2}{9 \left(1 - \frac{D'}{3}\right)^2}$

$= \frac{e^{-2z}}{9} \left[1 + 2 \frac{D'}{3} + 3 \left(\frac{D'}{3}\right)^2\right] z^2$

$= \frac{e^{-2z}}{9} \left[z^2 + \frac{4z}{3} + \frac{2}{3}\right]$

$= \frac{(e^{-z})^2}{9 \times 3} [3z^2 + 4z + 2]$

$= \frac{x^2}{27} [3(\log x)^2 + 4 \log x + 2]$

$y = C.F + P.I$
 $y = (A \log x + B) x + \frac{x^2}{27} [3(\log x)^2 + 4 \log x + 2]$

$$x^2 \frac{d^2 y}{dx^2} - 3x \frac{dy}{dx} + 4y = x^2 + \cos(\log x)$$

Soln

Put $x = e^z$

$\log x = z$

$x \frac{dy}{dx} = D'$

$x^2 \frac{d^2 y}{dx^2} = D'^2 - D'$

$[D'^2 - D' - 3D' + 4]y = e^{2z} + \cos z$

$[D'^2 - 4D' + 4]y = \dots$

A.E is

$m^2 - 4m + 4 = 0$

$m = 2, 2$

$\begin{matrix} A \\ \wedge \\ -2-2 \end{matrix}$

C.F = $(Ax + B)e^{2x}$

P.I = $\frac{1}{D'^2 - 4D' + 4} e^{2x} + \frac{1}{D'^2 - 4D' + 4} \cos z$

= $\frac{x e^{2x}}{2D - 4} + \frac{1}{-1 - 4D' + 4} \cos z$

= $\frac{x^2 e^{2x}}{2} + \frac{1}{3 - 4D'} \cos z$

= $\frac{(\log x)^2 x^2}{2} + \frac{1}{3 - 4D'} \times \frac{3 + 4D'}{3 + 4D'} \cos z$

= $\dots + \frac{3 + 4D'}{9 - 16D'^2} \cos z$ (a+b)

= $\dots + \frac{3 + 4D'}{9 - 16(-1)} \cos z$

= $\dots + \frac{1}{25} (3 + 4D') \cos z$

= $\dots + \frac{1}{25} [3 \cos z + 4(-\sin z)]$

= $\dots + \frac{1}{25} (3 \cos z - 4 \sin z)$

$y = C.F + P.I$

= $\frac{(\log x)^2 x^2}{2} + \frac{1}{25} (3 \cos \log x - 4 \sin \log x)$

Simultaneous linear Equations

① solve $\frac{dy}{dt} = x ; \frac{dx}{dt} = y$

soln

where $D = \frac{d}{dt}$

$x - Dy = 0 \rightarrow ①$

$Dx - y = 0 \rightarrow ②$

① $\times D \Rightarrow Dx - D^2 y = 0$

② $\times 1 \Rightarrow Dx - y = 0$
(-) (+)

② $\times 1 \Rightarrow Dx - y = 0$

① $\times D \Rightarrow Dx - D^2 y = 0$
(-) (+)

$D^2 y - y = 0$

$(D^2 - 1)y = 0$

The A.E is

$m^2 - 1 = 0$

$m^2 = 1$

$m = \pm 1$

$y = Ae^t + Be^{-t}$

From ① $\Rightarrow x = Ae^t - Be^{-t}$

② Solve the simultaneous Dif. eqns

$\frac{dx}{dt} + 2x + 3y = 2e^{2t}$

$\frac{dy}{dt} + 3x + 2y = 0$

soln

$(D+2)x + 3y = 2e^{2t} \rightarrow ①$

$3x + (D+2)y = 0 \rightarrow ②$

① $\times (D+2) \Rightarrow (D+2)^2 x + 3(D+2)y = 2e^{2t}(D+2)$

② $\times 3 \Rightarrow 9x + 3(D+2)y = 0$

$(D+2)^2 x - 9x = 2(D+2)e^{2t} = 2[D(e^{2t}) + 2e^{2t}]$

$(D^2 + 4D + 4 - 9)x = 2(2 + 4)e^{2t}$
 $= 8e^{2t}$ ($D=2$)

$(D^2 + 4D - 5)x$

$m^2 + 4m - 5 = 0$

$m = 1, -5$



C.F = $Ae^{5t} + Be^{-t}$

P.I = $8 \frac{e^{2t}}{D^2 + 4D - 5}$ ($D=2$)

$= \frac{8e^{2t}}{4 + 8 - 5} = \frac{8e^{2t}}{7}$

$= \frac{8}{7} e^{2t}$

$x = C.F + P.I$

$x = Ae^{5t} + Be^{-t} + \frac{8}{7} e^{2t}$

From ①,

$(D+2)x + 3y = 2e^{2t}$

$3y = 2e^{2t} - (D+2)x$

$= 2e^{2t} - (D+2)[Ae^{5t} + Be^{-t} + \frac{8}{7} e^{2t}]$

$= 2e^{2t} - AD(e^{5t}) - BDe^{-t} - \frac{8}{7} D(e^{2t})$

$- \frac{8}{7} D(e^{2t})$

$- 2Ae^{5t} - 2Be^{-t} - \frac{16}{7} e^{2t}$

$= 2e^{2t} + 2Ae^{5t} - Be^{-t} - \frac{16}{7} e^{2t}$
 $- 2Ae^{5t} - 2Be^{-t} - \frac{16}{7} e^{2t}$

$= 3Ae^{5t} - 3Be^{-t} - \frac{18}{7} e^{2t}$

$\div 3$
 $y = Ae^{5t} - Be^{-t} - \frac{6}{7} e^{2t}$

$= e^{5t} - e^{-t} - \frac{6}{7} e^{2t}$

$= \frac{32}{7} + \frac{2}{7}$
 $= \frac{-32}{7}$
 $= \frac{-18}{7}$