NOV-23

① In a certain binary communication channel, owing to noise, the probability that a transmitted zero is received as a zero is 0.95 and the probability that transmitted one is received as one is 0.9 If the probability that a zero is transmitted is 0.4 find the probability that (i) a one is received (ii) a one was transmitted given that one was received.

Soln

A → event of transmitting one.

$\bar{A}$ → " " " zero.

B → " " receiving one.

$\bar{B}$ → " " " zero

Given $P(\bar{A}) = 0.4$; $P(B/A) = 0.9$; $P(\bar{B}/\bar{A}) = 0.95$

$P(A) = 1 - P(\bar{A})$

$= 1 - 0.4$

$\boxed{P(A) = 0.6}$

$P(B/\bar{A}) = 1 - P(\bar{B}/\bar{A})$

$= 1 - 0.95$

$\boxed{P(B/\bar{A}) = 0.05}$

By Total probability theorem

$$P(B) = P(A) \times P(B/A) + P(\bar{A}) \times P(B/\bar{A})$$

$$= 0.6 \times 0.9 + 0.4 \times 0.05$$

$$= 0.56$$

By Baye's theorem

$$P(A/B) = \frac{P(A) \times P(B/A)}{P(B)}$$

$$= \frac{0.6 \times 0.9}{0.56}$$

$$= \frac{0.54}{0.56} = \frac{27}{28}$$

2 marks

APR-11
APR-19

① Define discrete r.v

A r.v 'x' is said to be discrete if has finite number of values.

Eg:

* The number of children in a particular village
* The number of doctors working in a particular district.

② Define Continuous r.v (APR-19)

A r.v 'x' which takes all possible values in a given interval is called a c.r.v

Eg:

Age, height, weight, length, pressure, temperature, time, voltage.

③ Define r.v (NOV-11)

A function is a real number to each outcome in the sample space for random experiment

④ Test whether $f(x) = \begin{cases} |x| & ; -1 \le x \le 1 \\ 0 & ; \text{otherwise} \end{cases}$ (Nov-14, APR-15)

can be the p.d.f of a c.r.v.

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$f(x) = \begin{cases} -x & ; -1 \le x \le 0 \\ x & ; 0 \le x \le 1 \\ 0 & ; \text{otherwise} \end{cases}$$

$$L.H.S = \int_{-\infty}^{\infty} f(x)\,dx$$

$$= \int_{-1}^{1} |x|\,dx$$

$$= 2\int_{0}^{1} x\,dx$$

$$= 2\left[\frac{x^2}{2}\right]_0^1 = \frac{2}{2}[1-0] = 1$$

(APR-19)

⑤ Show that for any events A & B in S,

$$P(B) = P(B/A)\ P(A) + P(B/\bar{A})\ P(\bar{A}).$$

Soln

$$R.H.S = P(B/A)\ P(A) + P(B/\bar{A})\ P(\bar{A})$$

$$= \frac{P(A \cap B)}{\cancel{P(A)}}\ \cancel{P(A)} + \frac{P(\bar{A} \cap B)}{\cancel{P(\bar{A})}}\ \cancel{P(\bar{A})}$$

$$= P(A \cap B) + \underbrace{P(\bar{A} \cap B)}_{\downarrow}$$

$$= \cancel{P(A \cap B)} + P(B) - \cancel{P(A \cap B)}$$

$$= P(B)$$

$$= L.H.S$$

$$\therefore L.H.S = R.H.S$$

(Nov-19)

⑥ If A & B are mutually exclusive events $P(A) = 0.29$ & $P(B) = 0.43$ then find $P(\bar{A})$ & $P(A \cup B)$.

Soln:

Given $P(A) = 0.29$

$P(B) = 0.43$

$$P(\bar{A}) = 1 - P(A) = 1 - 0.29 = 0.71$$

$$P(A \cup B) = P(A) + P(B) = 0.29 + 0.43 = 0.72.$$

(APR-19)

⑦ The p.d.f of the r.v X is given by

$$f(x) = \begin{cases} K(1-x^2) & ; 0 < x < 1 \\ 0 & ; \text{otherwise} \end{cases}$$

Find the value of k.

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_0^1 K(1-x^2)\,dx = 1$$

$$K \int_0^1 (1-x^2)\,dx = 1$$

$$K \left[x - \frac{x^3}{3}\right]_0^1 = 1$$

$$K\left(1 - \frac{1}{3}\right) = 1$$

$$K\left(\frac{3-1}{3}\right) = 1 \Rightarrow K\left(\frac{2}{3}\right) = 1$$

$$K = 3/2$$

APR-19

(8) The M.G.F of r.v X is given $M(t) = e^{3(e^t - 1)}$

what is $P(X=0)$

$$M_X(t) = e^{3(e^t - 1)}$$

$$= e^{3e^t} e^{-3}$$

$$= \frac{1}{e^3}\left[1 + 3\frac{e^t}{1!} + \frac{(3e^t)^2}{2!} + \dots\right]$$

$$M_X(t) = \frac{1}{e^3} + \frac{3}{e^3}\frac{e^t}{1!} + \frac{9e^{2t}}{2} + \dots$$

$$\boxed{P(0) = P(X=0) = \frac{1}{e^3}}$$

NOV-18

(9) Find the mean of a normal distribution.

Soln:

Mean $= E(x) = \mu$.

(10) A r.v X is uniformly distributed between 3 & 15. Find the variance of X.

Soln:

Given $a = 3$; $b = 15$

$$\text{variance} = \frac{1}{12}(b-a)^2$$

$$= \frac{1}{12}(15-3)^2$$

$$= \frac{1}{\cancel{12}}(12)^{\cancel{2}}$$

$$\boxed{\text{Variance} = 12}$$

APR-19

11) What are the limitations of poisson distrib

* n is indefinitely large
  (i,e) $n \to \infty$

* p is very small such that $p \to 0$

* $np = \lambda$ (a finite quantity)

$$p = \frac{\lambda}{n}$$

$$q = 1 - p$$

$$q = 1 - \lambda$$

(12) Show that the function $f(x) = \begin{cases} \frac{x^2}{3} ; & -1 < x < 2 \\ 0 ; & \text{otherwise} \end{cases}$ is a p.d.f

Soln :

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-1}^{2} \frac{x^2}{3}\,dx$$
$$= \frac{1}{3}\left[\frac{x^3}{3}\right]_{-1}^{2}$$
$$= \frac{1}{9}\left[(2)^3 - (-1)^3\right]$$
$$= \frac{1}{9}(8+1)$$
$$= \frac{1}{9} \times 9$$
$$= 1$$

(13) A continuous r.v X has a p.d.f $f(x) = k$, $0 \le x \le 1$. Find k.

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$
$$\int_{0}^{1} k\,dx = 1$$
$$k[x]_0^1 = 1$$
$$k(1-0) = 1$$
$$\boxed{k = 1}$$

APR-19

(14) An experiment succeds twice as often as it fails. Find the chance that in the next 4 trails, there shall be atleast one success.

$n = 4$; $p = 2q = 2(1-p)$

$p = 2 - 2p$

$3p = 2$

$\boxed{p = \frac{2}{3}}$

$q = 1 - p$

$= 1 - \frac{2}{3}$

$\boxed{q = \frac{1}{3}}$

$$P(X \ge 1) = 1 - P[X < 1]$$
$$= 1 - P[X = 0]$$
$$= 1 - 4C_x \left(\frac{2}{3}\right)^x \left(\frac{1}{3}\right)^{4-x}$$
$$= 1 - 4C_0 \left(\frac{2}{3}\right)^0 \left(\frac{1}{3}\right)^{4-0}$$
$$= 1 - \frac{1}{81}$$
$$= \frac{81-1}{81} = \frac{80}{81}$$

APR-19

(15) For a binomial distribution Mean is 2 & variance is 4/3, Find the 1st term of the distribution.

Mean $= np = 2 \rightarrow (1)$

variance $npq = 4/3 \rightarrow (2)$

$$\frac{(2)}{(1)} = \frac{npq}{np} = \frac{4/3}{2}$$
$$q = \frac{4}{3} \times \frac{1}{2}$$
$$\boxed{q = \frac{2}{3}}$$

$p + q = 1$

$p = 1 - p$

$= 1 - 2/3 = 1/3$

$\boxed{p = 1/3}$

From (1)

$np = 2$

$n\left(\frac{1}{3}\right) = 2$

$\boxed{n = 6}$

$$P(X=x) = nC_x\, p^x q^{n-x}$$
$$P(X=0) = 6C_0 \left(\frac{1}{3}\right)^0 \left(\frac{2}{3}\right)^{6-0}$$
$$= \frac{2^6}{3^6} = \frac{64}{729}$$

(16) Let $M_x(t) = \frac{1}{1-t}$, $|t| < 1$, M.G.F of r.v. X

Find $E(x)$ & $E(x^2)$. NOV-19

$$M_x(t) = \frac{1}{1-t} = (1-t)^{-1} = 1+t+t^2+\ldots$$

$$= 1 + \frac{t}{1!}(1) + \frac{t^2}{2!}(2) + \ldots$$

$E(x) = 1$

$E(x^2) = 2$.

(17) Let A & B be two events such that $P(A) = \frac{1}{3}$, $P(B) = 3/4$ & $P(A \cap B) = \frac{1}{4}$. Compute $P(A/B)$ & $P(\bar{A} \cap \bar{B})$ APR-19

$$P(A/B) = \frac{P(A\cap B)}{P(B)} = \frac{(1/4)}{(3/4)} = \frac{1}{3}$$

$$P(\bar{A} \cap \bar{B}) = P(\overline{A \cup B})$$

$$= 1 - P(A \cup B)$$

$$= 1 - [P(A) + P(B) - P(A\cap B)]$$

$$= 1 - \frac{1}{3} - 3/4 + 1/4$$

$$= \frac{3-1}{3} - \frac{3+1}{4}$$

$$= \frac{2}{3} - 2/4 = \frac{8-6}{12} = \frac{2}{12} = 1/6$$

NOV-19

(18) A bag contains 8 white & 4 black balls. If 5 balls are drawn at random, what is the probability that 3 are white & 2 are black?

White	Black	Total
8	4	12

$S = \{$ 5 balls are taken out of 12 $\}$

$$n(S) = 12C_5 = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8}{1\times 2\times 3\times 4\times 5} = 792.$$

(3 are white & 2 are black)

$n(A) = 8C_3 \times 4C_2$

$=$

$\boxed{n(A) = 336}$

$P(A) = \frac{n(A)}{n(S)} = \frac{336}{792} = 0.4242.$

NOV-15
APR-19

(19) ~~Let~~ X be the random variable which denote the number of heads in three tosses of a fair coin. Determine the p.m.f of X.

$P[X=x] = 3C_x \left(\frac{1}{2}\right)^3 \quad ; x = 0, 1, 2, 3.$

x	0	1	2
$P(X=x)$	$3C_0 (1/2)^3 = (1/2)^3$	$3C_1 (1/2)^3 = 3 (1/2)^3$	$3C_2 (1/2)^3 = \frac{3\times 2}{1\times 2} (1/2)^3 = 3(1/2)^3$

3
$3C_3 (1/2)^3 = (1/2)^3$

Unit-II (2 marks)

① Find the acute angle between the two lines of regression.

Soln:

$$\tan\theta = \left|\left(\frac{1-r^2}{r}\right)\left(\frac{\sigma_x \sigma_y}{\sigma_x^2+\sigma_y^2}\right)\right|$$

② If X & Y are independent r.v's with variance 2 & 3, then find the variance of $3x+4y$.

Soln

Given $Var(X) = 2$

$var(Y) = 3$.

$$Var(3x+4y) = a^2\, var(X) + b^2\, var(y)$$

$$= 3^2 \times 2 + 4^2 \times 3$$

$$= 9\times 2 + 16 \times 3$$

$$= 18 + 48$$

$$\boxed{Var(3x+4y) = 66}$$

③ Define covariance & coefficient of correlation between two r.v's X & Y.

Soln:

Covariance:

$$Cov(X,Y) = E(XY) - E(X)\,E(Y)$$

coefficient of correlation:

$$r(x,y) = \frac{Cov(x,y)}{\sigma_x \sigma_y}$$

where $\sigma_x = \sqrt{\frac{1}{n}\Sigma x^2 - \bar{x}^2}$

$\sigma_y = \sqrt{\frac{1}{n}\Sigma y^2 - \bar{y}^2}$

(4) Let X & Y be two independent R.V's with $var(X)=9$ & $var(y)=3$. Find $var(4x-2y+6)$

Soln:

Given $var(x)=9$ &

$var(y)=3$.

$$var(4x-2y+6) = 4^2 var(x) + (-2)^2 var(y)$$
$$= 16\times 9 + 4\times 3$$
$$= 144+12$$
$$\boxed{var(4x-2y+6) = 156}$$

(5) State the central limit theorem.
(or)
Lindberg - Levy's form

Statement:

Let $X_1, X_2 \dots X_n$ be a sequence of independent identically distributed r.v's with $E(X_i)=\mu$ & $var(X_i)=\sigma^2$, $i=1,2\dots n$ & if $S_n = X_1+X_2+\dots+X_n$, then under certain general conditions S_n follows a normal distribution with mean $= n\mu$ & variance $= n\sigma^2$ as $n\to\infty$.

(6) Find the value of k if $f(x,y) = kxy\, e^{-(x^2+y^2)}$, $x\geq 0, y\geq 0$ is the joint pdf.

Soln:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, dy\, dx = 1$$

$$\int_{0}^{\infty}\int_{0}^{\infty} k\, xy\, e^{-(x^2+y^2)} dy\, dx = 1$$

$$k \int_{0}^{\infty} x e^{-x^2} dx \int_{0}^{\infty} y e^{-y^2} dy = 1$$

Put $x^2=t$	$x\to 0$, $t\to 0$
$2x\,dx = dt$	$x\to\infty$
$x\,dx = \frac{1}{2}dt$	$t\to\infty$

$$k \int_{0}^{\infty} e^{-t}\frac{1}{2}dt \int_{0}^{\infty} e^{-t}\frac{1}{2}dt$$

$$K \cdot \frac{1}{2}\left(\frac{e^{-t}}{-1}\right)_0^{\infty} \frac{1}{2}\left(\frac{e^{-t}}{-1}\right)_0^{\infty} = 1$$

$e^{\infty} = 0$

$e^{0} = 1$

$$\frac{K}{4} (e^{\infty} - e^{0})(e^{\infty} - e^{0}) = 1$$

$$\frac{K}{4}(0-1)(0-1) = 1$$

$$\frac{K}{4} = 1$$

$$\boxed{K = 4}$$

7) The joint Pmf of two dimensional r.v X & Y is given by $f(x,y) = \begin{cases} kxy \, ; & x = 1,2,3 \, ; \, y = 1,2,3 \\ 0 & ; \text{otherwise} \end{cases}$

Find the value of k.

$f(x,y) = kxy$

X \ Y	1	2	3	Total
1	K	2K	3K	6K
2	2K	4K	6K	12K
3	3K	6K	9K	18K
Total	6K	12K	18K	36K / 36K

$f(1,1) = K$

$f(1,2) = K(1)(2) =$

$f(1,3) = K(1)(3) =$

$f(2,1) = K(2)(1) =$

$f(2,2) = K(2)(2) =$

$f(2,3) = K(2)(3)$

$f(3,1) = K(3)(1)$

$f(3,2) = K(3)(2)$

$f(3,3) = K(3)(3)$

Total probability

$36K = 1$

$\boxed{K = }$

8) If the joint Pdf of a bivariate r.v (X,Y) is given by $f(x,y) = \begin{cases} K(1-x)(1-y) \text{ if } & 0 < x, y < 1 \\ 0 & ; \text{otherwise} \end{cases}$

Find the value of K.

Soln:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, dx\, dy = 1$$

$$\int_0^1 \int_0^1 K(1-x)(1-y)\, dx\, dy = 1$$

$$K\int_0^1 (1-x)\, dx \int_0^1 (1-y)\, dy = 1$$

$$K \left(x - \frac{x^2}{2}\right)_0^1 \left(y - y^2\right)_0^1 = 1$$

$k\left(1-\frac{1}{2}-0\right)\left(1-\frac{1}{2}-0\right)=1$

$k\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)=1$

$\frac{K}{4}=1$

$\boxed{K=4}$

(9) State the equations of two regression lines & angle between them?

Regression line :-

$$y-\bar{y}=r\frac{\sigma_y}{\sigma_x}(x-\bar{x})$$

$$x-\bar{x}=r\frac{\sigma_x}{\sigma_y}(y-\bar{y}).$$

Angle

$$\tan\theta=\left|\frac{1-r^2}{r}\left(\frac{\sigma_x\sigma_y}{\sigma_x^2+\sigma_y^2}\right)\right|$$

(10) If the joint pdf of (x,y) is given by

$$f(x,y)=\begin{cases} e^{-(x+y)} & ; x \& y \geq 0 \\ 0 & ; \text{otherwise}\end{cases}$$

Check whether X & Y are independent.

soln:-

Marginal density of X & Y :-

$f(x)=\int_{-\infty}^{\infty}f(x,y)\,dy$

$=\int_0^{\infty}e^{-(x+y)}\,dy$

$=e^{-x}\int_0^{\infty}e^{-y}\,dy$

$=e^{-x}\left[-e^{-y}\right]_0^{\infty}$

$=-e^{-x}\left(e^{-\infty}-e^{0}\right)$

$=-e^{-x}(0-1)$

$\boxed{f(x)=e^{-x}}$

$e^{-\infty}=0$
$e^{0}=1$

$f(y)=\int_{-\infty}^{\infty}f(x,y)\,dx$

$=\int_0^{\infty}e^{-(x+y)}\,dx$

$=e^{-y}\int_0^{\infty}e^{-x}\,dx$

$=e^{-y}\left[e^{-x}\right]_0^{\infty}$

$=-e^{-y}\left[e^{-\infty}-e^{0}\right]$

$=-e^{-y}(0-1)$

$\boxed{f(y)=e^{-y}}$

Independent :-

$f(x)\,f(y)=f(x,y)$

$e^{-x}\,e^{-y}=e^{-(x+y)}.$

x,y are independent,

(11) Let X & Y be r.v's with joint density functi[on]

$$f_{xy}(x,y) = \begin{cases} 4xy \; ; & 0 \le x \le 1 ; \; 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Find $E(xy)$.

Soln

$$E(xy) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy \, f(x,y) \, dx \, dy$$

$$= \int_0^1 \int_0^1 xy \, (4xy) \, dx \, dy$$

$$= 4 \int_0^1 x^2 \, dx \int_0^1 y^2 \, dy$$

$$= 4 \left[\frac{x^3}{3}\right]_0^1 \left[\frac{y^3}{3}\right]_0^1$$

$$= \frac{4}{9} (1-0)(1-0)$$

$$\boxed{E(xy) = \frac{4}{9}}$$

(12) Let (x,y) be a two dimensional r.v. Define covariance of (x,y). If X & Y are independent, what will be the covariance of (x,y).

Soln:

$$\text{cov}(x,y) = E(xy) - E(x)\,E(y)$$

If X & Y are independent, then $\text{cov}(x,y) = 0$

(13) What do you mean by correlation between two random variables?

Soln

Degree of relationship &

nature "

(14) The joint P.m.f of a two dimensional r.v(is given by $P(x,y) = K(2x+y)$; $x = 1, 2$ & $y = 1, 2$ where K is a constant. Find the value of

x \ y	1	2	Total
1	3K	4K	7K
2	5K	6K	11K
Total	8K	10K	$\frac{18K}{18K} = 1$

$P(x,y) = K(2x+y)$

$P(1,1) = K(2(1)+1)$

$P(1,2) = K(2(1)+2) =$

$P(2,1) = K(2(2)+1) =$

$P(2,2) = K(2(2)+2) =$

Total probability $18K = 1$

$K = \frac{1}{18}$

Unit-III

Two marks

① State the four types of a stochastic Process

(i) Discrete time, discrete state random Process

(ii) " ", continuous " " "

(iii) continuous time, discrete " " "

(iv) " ", continuous " " "

② Define ⓐ wide sense stationary random Process
ⓑ Ergodic random process.

(or) When a random process said to be Ergodic process?

ⓐ A random process $X(t)$ is said to be wide sense stationary if its mean is constant & its autocorrelation depends only on time difference.

(i) $E(x(t)) = \text{Constant}$

(ii) $R_{xx}(t, t+\tau) = R_{xx}(\tau)$.

ⓑ A random process $X(t)$ is called ergodi if its ensemble averages are equal to appropriate time averages.

③ State any two Properties of a Poisson Proc

(i) The poisson Process is a markov process,

(ii) The sum of two independent Poisson Proce is again a poisson process.

(iii) The Difference of two independent Poisson processes is not a Poisson Process.

④ State the postulates of a Poisson Process.

Let $X(t)$ = number of times an event A.

t = time

The sequence $\{X(t), t \in (0,\infty)\}$ forms a poisso process with parameter λ.

* Eve… in non-overlapping intervals are independent of each other

* $P[X(t)=1$, for t in $(x, x+h)] = \lambda h + o(h)$
* $P[X(t)=0$, " " $] = 1-\lambda h + o(h)$
* $P[X(t)=$ 2 or more " " $] = o(h)$.

④ Is poisson process stationary? Justify.

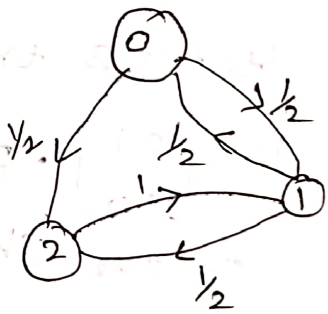
Poisson process is not a stationary process, as its statistical properties (mean, autocorrelation, ...) are time dependent.

⑤ consider a markov chain with state $\{0,1,2\}$ and transition probability matrix

$$P = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 1 & 0 & 0 \end{bmatrix}$$

. Draw the state transition Diagram.

Given, $P = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & \frac{1}{2} & 0 & \frac{1}{2} \\ 2 & 1 & 0 & 0 \end{array}$



⑥ Give an example of evolutionary random process. Poisson process.

⑦ what is markov process?
(or) Define Markov process.

A random process $X(t)$ is said to be a Markov process if for any $t_1 < t_2 < t_3 < \dots < t_n$

$$\frac{P[X(t_n)] \le x_n}{X(t_{n-1})} = x_{n-1}$$

$$X(t_{n-2}) = x_{n-2}, \dots, X(t_1) = x_1 = P\left[\frac{X(t_n) \le x_n}{X(t_{n-1}) = x_{n-1}}\right].$$

(i,e) the conditional distribution of $X(t_n)$ for given values of $X(t_1), X(t_2) \dots X(t_{n-1})$ depends only on $X(t_{n-1})$.

(8) Give an example of an ergodic Process.

* A Markov Chain finite state Space.
* A stochastic process $X(t)$ is ergodic if its time average tends to ensemble average as $T \to \infty$.

(9) State Chapman Kolmogorow theorem

If 'p' is the tpm of a homogenous Markov Chain, then the n-step tpm $p(n)$ is equal to p^n.

$$\therefore [p^n_{ij}] = [P_{ij}]^n$$

(10) Consider a markov Chain with two states & transition Probability matrix $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Find the stationary distribution of the Chain.

Soln:

$$P = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{array}$$

Stationary State distribution :

$$\pi = \pi_1 \ \pi_2$$

$$a=1 \ ; \ b=1 \ ; \ a+b = 1+1 = 2$$

$$\pi = \begin{bmatrix} \frac{b}{a+b} & \frac{a}{a+b} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

1) Define a K^{th} order Stationary process. When will it become a strict sense stationary process?
(or)
Define a stationary process

A random process is said to be K^{th} order stationary process if all its ~~fite~~ finite dimensional distributions are invariant under translation of time for all $t_1, t_2 \ldots t_K$ & for $n = 1, 2 \ldots K$ only and not for $n > K$, then the process is called the K^{th} order stationary. It becomes a SSS process when $K \to \infty$.

(1) Using limiting behaviour of homogeneous markov chain, find steady state probability of the chain given by the transanetion probability matrix.

$$P = \begin{bmatrix} 0.1 & 0.6 & 0.3 \\ 0.5 & 0.1 & 0.4 \\ 0.1 & 0.2 & 0.7 \end{bmatrix}$$

Soln:

Let $P = [\ * \ ]$

Let $\pi = (\pi_1\ \pi_2\ \pi_3)$

$\pi P = \pi \rightarrow (1)$

$\pi_1 + \pi_2 + \pi_3 = 1 \rightarrow (2)$

$\pi_3 = 1 - \pi_1 - \pi_2 \rightarrow (3)$

From (1),

$\pi P = \pi$

$$(\pi_1\ \pi_2\ \pi_3) \begin{bmatrix} 0.1 & 0.6 & 0.3 \\ 0.5 & 0.1 & 0.4 \\ 0.1 & 0.2 & 0.7 \end{bmatrix} = (\pi_1\ \pi_2\ \pi_3)$$

$0.1\pi_1 + 0.5\pi_2 + 0.1\pi_3 = \pi_1 \rightarrow (4)$

$0.6\pi_1 + 0.1\pi_2 + 0.2\pi_3 = \pi_2 \rightarrow (5)$

$0.3\pi_1 + 0.4\pi_2 + 0.7\pi_3 = \pi_3 \rightarrow (6)$

~~From (4)~~, Sub (3) in (4).

$0.1\pi_1 + 0.5\pi_2 + 0.1(1 - \pi_1 - \pi_2) = \pi_1$

$\cancel{0.1\pi_1} + 0.5\pi_2 + 0.1 - \cancel{0.1\pi_1} - 0.1\pi_2 = \pi_1$

$-\pi_1 + 0.4\pi_2 + 0.1 = 0 \rightarrow (7)$

sub ③ in ⑤

$0.6\pi_1 + 0.1\pi_2 + 0.2(1-\pi_1-\pi_2) = \pi_2$

$0.6\pi_1 + 0.1\pi_2 + 0.2 - 0.2\pi_1 - 0.2\pi_2 - \pi_2 = 0$

$0.4\pi_1 - 1.1\pi_2 + 0.2 = 0 \rightarrow ⑧$ $\boxed{-1-0.1}$

$⑦ \times 0.4 \Rightarrow -0.4\pi_1 + 0.16\pi_2 = -0.04$

$⑧ \times 1 \Rightarrow \quad 0.4\pi_1 - 1.1\pi_2 = -0.2$

$(-) \quad (+) \quad (+)$

$-0.94\pi_2 = -0.24$

$\pi_2 = \frac{0.24}{0.94} \times \frac{100}{100} = \frac{24}{94}$

$\boxed{\pi_2 = 0.2553}$

From ⑧

$0.4\pi_1 - 1.1\pi_2 + 0.2 = 0$

$0.4\pi_1 - 1.1(0.2553) + 0.2 = 0$

$0.4\pi_1 - 0.28083 + 0.2 = 0$

$0.4\pi_1 - 0.08083 = 0$

$0.4\pi_1 = 0.08083$

$\pi_1 = \frac{0.08083}{0.4}$

$\boxed{\pi_1 = 0.202075 \simeq 0.2021}$

From ③

$\pi_3 = 1 - \pi_1 - \pi_2$

$= 1 - 0.2021 - 0.2553$

$\boxed{\pi_3 = 0.5426}$

$\therefore \pi = (\pi_1 \quad \pi_2 \quad \pi_3)$

$\pi = (0.2021, 0.2553, 0.5426)$

Type: 4 V_3R

(1) Show that the vectors form a basis for $V_3(R)$. $u_1 = (2,1,0)$, $u_2 = (-3,-3,1)$; $u_3 = (-2,1,-1)$

Soln:

To find the set is linearly independent or not.

coefficient matrix:

$$|A| = \begin{vmatrix} 2 & 1 & 0 \\ -3 & -3 & 1 \\ -2 & 1 & -1 \end{vmatrix} = 2(3-1)-1(3+2)+0$$
$$= 2(2)-1(5)$$
$$= 4-5$$
$$= -1 \neq 0$$

It is a linearly independent

$\therefore$ It is a basis

---

(1) Define Bases

A set $S = \{u_1, u_2 \ldots u_n\}$ of vectors is a basis of V if it has the following two properties

(1) S is linearly independent.

(2) $L(S) = V$; (i,e) every element of V is a linearly combination of finite elements of S.

---

(2) Define Dimension

Let V be a finite dimensional vector space over a field F, the number of elements in any basis of $V(F)$ is called dimension of V. It is denoted by $\dim V$.

(1) $\dim\{0\} = 0$
(2) $\dim F^n = n$
(3) $\dim M_{m\times n}(F) = mn$
(4) $\dim P_n(F) = n+1$.

Linear Independence and Linear Dependence (2, 5, & 8 marks)

1) Examine whether or not the vectors $u_1 = (1,1,2)$ $u_2 = (2,3,1)$ & $u_3 = (4,5,5)$ in R^3 are linearly dependent.

Soln:

$$|A| = \begin{vmatrix} 1 & 1 & 2 \\ 2 & 3 & 1 \\ 4 & 5 & 5 \end{vmatrix} = 1(15-5) - 1(10-4) + 2(10-12)$$

$= 1(10) - 1(6) + 2(-2)$

$= 10 - 6 - 4$

$= 10 - 10$

$= 0.$

$\therefore$ It is a linearly dependent.

---

2) Show that the vectors $u_1 = (1,1,1)$, $u_2 = (1,2,3)$ & $u_3 = (1,5,8)$ span R^3.

Soln:

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 5 & 8 \end{vmatrix} = 1(16-15) - 1(8-3) + 1(5-2)$$

$= 1(1) - 1(5) + 1(3)$

$= 1 - 5 + 3$

$= -5 + 4$

$= -1 \neq 0.$

It is a linearly independent

---

3) Determine whether the vectors $u = (1,1,2)$, $v = (1,0,1)$ and $w = (2,1,3)$ span the vector space R^3

$$|A| = \begin{vmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{vmatrix} = 1(0-1) - 1(3-2) + 2(1-0)$$

$= -1 - 1 + 2$

$= 0$

It is a linearly dependent

---

4) Show that the vectors $(1,1,0,0)$, $(0,1,-1,0)$, $(0,0,0,3)$ in $V_4(R)$ are linearly independent.

Soln:

Let $u_1 = (1,1,0,0)$, $u_2 = (0,1,-1,0)$

$u_3 = (0,0,0,3)$

Let $a_1 u_1 + a_2 u_2 + a_3 u_3 = 0$

$a_1(1,1,0,0) + a_2(0,1,-1,0) + a_3(0,0,0,3) = (0,0,0,0)$

$a_1 = 0$

$a_1 + a_2 = 0$

$-a_2 = 0$

$3a_3 = 0$.

Solution :

$\therefore a_1 = 0$

$a_2 = 0$

$a_3 = 0$.

$\therefore$ It is a linearly dependent.

Linear combinations & Linear system of equations

(1) Express $v = (3,7,-4)$ in R^3 as a linear combination of the vectors $u_1 = (1,2,3)$; $u_2 = (2,3,7)$; $u_3 = (3,5,6)$.

Soln. :

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 7 \\ 3 & 5 & 6 \end{vmatrix} = 1(18-35) - 2(12-21) + 3(10-9)$$

$= 1(-17) - 2(-9) + 3(1)$

$= -17 + 18 + 3$

$= -17 + 21$

$= 4 \neq 0$.

$\therefore$ It is a linearly independent.

H.W

(2) Consider the vectors $u = (1,2,-1)$ & $v = (6,4,2)$ in R^3. Show that $w = (9,2,7)$ is a linearly combination of u & v.

(4) Determine $(0,1),(0,-3)$ forms a basis of R^2.

Soln

$$|A| = \begin{vmatrix} 0 & 1 \\ 0 & -3 \end{vmatrix} = 0.$$

It is a linearly dependent.

$\therefore$ It is a not basis

(5) Let $u_1=(1,2)$, $u_2=(2,1)$. verify whether $x=(2,2)$ is a linear combination of u_1 & u_2.

Soln:

$$|A| = \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = 1-4 = -3 \neq 0.$$

$\because$ It is a linear independent.

$\therefore$ It is a basis.

(or)

Let $x = a_1u_1 + a_2u_2$

$(2,2) = a_1(1,2) + a_2(2,1)$

$a_1 + 2a_2 = 2 \rightarrow (1)$

$2a_1 + a_2 = 2 \rightarrow (2)$

$(1)\times 2 \Rightarrow 2a_1 + 4a_2 = 4$

$(2)\times 1 \Rightarrow 2a_1 + a_2 = 2$

$(-) \quad (-) \quad (-)$

$3a_2 = 2$

$\boxed{a_2 = 2/3}$

$a_2 = 2/3$ sub (1)

$a_1 + 2a_2 = 2$

$a_1 + 2(2/3) = 2$

$a_1 = 2 - 4/3$

$a_1 = \frac{6-4}{3} = 2/3$

$\boxed{a_1 = 2/3}$

Solution:

$a_1 = a_2 = 2/3$

?) Let $u_1 = (2,1)$, $u_2 = (-4,-2)$ verify whether $x = (3,5)$ is a linear combination of u_1 & u_2.

Soln:

$$|A| = \begin{vmatrix} 2 & 1 \\ -4 & -2 \end{vmatrix} = -4+4 = 0.$$

It is a linear Dependent.

It is a not basis

Unit-V

Two marks

Inner Products and Norms

(1) Show that the vectors $u = (-2,3,1,4)$ & $v = (1,2,0,-1)$ are orthogonal in R^4 (X)

Soln:

Let $u = (-2,3,1,4)$, $v = (1,2,0,-1)$.

$\bar{u}\cdot\bar{v} = (-2,3,1,4)\cdot(1,2,0,-1)$

$= -2+6+0-4$

$\boxed{\bar{u}\cdot\bar{v} = 0.}$

$\therefore$ $\bar{u}$ & $\bar{v}$ are orthogonal.

(2) Prove that the vectors $u=(1,1)$ & $v=(1,-1)$ are orthogonal w.r.to the Euclidean inner product on R^2 (X)

Soln: Let $u=(1,1)$; $v=(1,-1)$

$\bar{u}\cdot\bar{v} = (1,1)\cdot(1,-1)$

$= 1-1$

$\boxed{\bar{u}\cdot\bar{v} = 0}$

(3) Let $u=(1,3,5)$, $v=(4,5,5)$ in R^3 (X) find the inner product (u,v).

Soln:

Let $u=(1,3,5)$; $v=(4,5,5)$

$\bar{u}\cdot\bar{v} = (1,3,5)\cdot(4,5.5)$

$= 4+15+25$

$\boxed{\bar{u}\cdot\bar{v} = 44}$

(4) Evaluate $(3u-2v, w)$, when $u = (1,3,-4,2)$, $v = (4,-2,2,1)$ & $w = (5,-1,-2,6)$.

Soln.:

$$3u-2v = 3(1,3,-4,2) - 2(4,-2,2,1)$$

$$= (3,9,-12,6) - (8,-4,4,2)$$

$$\boxed{3u-2v = (-5,13,-16,4)}$$

Angle between two vectors

$$\cos\theta = \frac{(u,v)}{\|u\|\,\|v\|}$$

(1) Given $u = (2,3,5)$, $v = (1,-4,3)$ in R^3 then find the angle θ between u & v.

Soln.:

$$\cos\theta = \frac{(u,v)}{\|u\|\,\|v\|}$$

$$u.v = (2,3,5).(1,-4,3)$$

$$= 2-12+15$$

$$\boxed{u.v = 5}$$

$$\|u\| = \sqrt{u,u}$$

$$= \sqrt{(2,3,5)(2,3,5)}$$

$$= \sqrt{4+9+25}$$

$$\boxed{\|u\| = \sqrt{38}}$$

$$\|v\| = \sqrt{v,v}$$

$$= \sqrt{(1,-4,3)(1,-4,3)}$$

$$= \sqrt{1+16+9}$$

$$\boxed{\|v\| = \sqrt{26}}$$

$$\cos\theta = \frac{u.v}{\|u\|\,\|v\|} = \frac{5}{\sqrt{38}\sqrt{26}}.$$

H.W

(2) Find the angle between two vectors $u=(4,3,1,-2)$ & $v=(-2,1,2,3)$.

Soln:

$$\cos\theta = \frac{(u,v)}{\|u\|\,\|v\|}$$

$$u\cdot v = (4,3,1,-2)\cdot(-2,1,2,3)$$

$$= -8+3+2-6$$

$$\boxed{u\cdot v = -9}$$

$$\|u\| = \sqrt{u\cdot u}$$
$$= \sqrt{(4,3,1,-2)(4,3,1,-2)}$$
$$= \sqrt{16+9+1+4}$$
$$\boxed{\|u\| = \sqrt{30}}$$

$$\|v\| = \sqrt{v\cdot v}$$
$$= \sqrt{(-2,1,2,3)(-2,1,2,3)}$$
$$= \sqrt{4+1+4+9}$$
$$\boxed{\|v\| = \sqrt{18}}$$

$$\cos\theta = \frac{u\cdot v}{\|u\|\,\|v\|}$$

$$\boxed{\cos\theta = \frac{-9}{\sqrt{30}\,\sqrt{18}}}$$

Distance between two vectors

$$\|x-y\| = \sqrt{(x-y)(x-y)}$$

(1) Given $x=(1,2,3)$ & $y=(1,0,1)$. find distance between x & y through inner product.

Soln:

Let $x=(1,2,3)$ & $y=(1,0,1)$

$$x-y = (1,2,3)-(1,0,1)$$
$$x-y = (0,2,2)$$
$$\|x-y\| = \sqrt{(x-y)(x-y)}$$
$$= \sqrt{(0,2,2)(0,2,2)}$$
$$= \sqrt{0+4+4}$$
$$\boxed{\|x-y\| = \sqrt{8}}$$

1. State Rank-nullity theorem (Dimension theorem)

Let V & W be vector spaces, and let $T: V \to W$ be linear. If V is finite-dimensional, then

$$\text{nullity}(T) + \text{rank}(T) = \text{Dim}(V).$$

---

2. Define kernal of T.

Let V & W be vector spaces over a field F and $T: V \to W$ be a linear transformation. is called kernal of T.

$$\text{Ker } T = \{ v \mid v \in V \text{ \& } T(v) = 0 \}.$$

---

3. Find the Kernel & range of the identity operator.

Let $I: V \to V$ be the identity operator.

Since $I(v) = v$, $R(I) = V$ &

$$\text{Ker}(I) = \{0\}.$$

---

4. Find a basis for the null space of the matrix $A = \begin{bmatrix} 5 & 2 \\ 1 & 0 \end{bmatrix}$.

Soln:

Let $A = \begin{bmatrix} 5 & 2 \\ 1 & 0 \end{bmatrix}$

$$\sim \begin{bmatrix} 5 & 2 \\ 0 & -2 \end{bmatrix} \quad R_2 \to 5R_2 - R_1$$

Which is echelon form of matrix

$\therefore$ The set of non-zero vectors

$$\{(5,2); (0,-2)\}.$$

It is a basis.

(5) state Cauchy - Schwarz inequality

$|(x,y)| \leq \|x\| \, \|y\|$; where $x, y \in V$ & $c \in F$.

(6) state triangle inequality

$\|x+y\| \leq \|x\| + \|y\|$

orthogonal set

(1) Check whether $\{u_1, u_2, u_3\}$ is an orthogonal set where $u_1 = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$; $u_2 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$; $u_3 = \begin{pmatrix} 1/2 \\ 2 \\ -7/2 \end{pmatrix}$

soln :-

$(u_1, u_2) = 3(-1) + (1 \times 2) + 1 \times 1$

$= -3 + 2 + 1$

$= -3 + 3$

$(u_1, u_2) = 0$

$(u_1, u_3) = 3(1/2) + 1 \times 2 + 1 \times (-7/2)$

$= 3/2 + 2 - 7/2$

$= -\frac{\cancel{4}^{2}}{\cancel{2}} + 2$

$= -2 + 2$

$(u_1, u_3) = 0$

$(u_2, u_3) = -1 \times \frac{1}{2} + 2 \times 2 + 1 \times (-7/2)$

$= -1/2 + 4 - 7/2$

$= -\frac{\cancel{8}^{4}}{\cancel{2}} + 4$

$= -4 + 4$

$(u_2, u_3) = 0$

$\therefore \{u_1, u_2, u_3\}$ is an orthogonal set.

② Define Adjoint matrix

Let $A = a_{ij}$ be an $m \times n$ matrix with Complex value. The adjoint matrix of A is the $n \times m$ matrix.

$$A^* = b_{ij} \text{ such that } b_{ij} = \overline{a_{ji}}$$

(i.e) $A^* + \overline{A^t}$

Basis & Dimension

① verify Rank – Nullity theorem for the matrix $\begin{bmatrix} 1 & 2 & 5 \\ 7 & -1 & 2 \\ 3 & 7 & 4 \end{bmatrix}$ (or) Find basis & Dimension theorem.

Soln

Coefficient matrix:

$$\text{Let } A = \left[\begin{array}{ccc|c} 1 & 2 & 5 & 0 \\ 7 & -1 & 2 & 0 \\ 3 & 7 & 4 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 5 & 0 \\ 0 & -15 & -33 & 0 \\ 0 & 1 & -11 & 0 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 7R_1 \\ R_3 \to R_3 - 3R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 5 & 0 \\ 0 & -15 & -33 & 0 \\ 0 & 0 & -198 & 0 \end{array}\right] \quad R_3 \to 15R_3 + R_2$$

$x_1 + 2x_2 + 5x_3 = 0 \rightarrow (1)$

$-15x_2 - 33x_3 = 0 \rightarrow (2)$

$-198x_3 = 0 \rightarrow (3)$

$x_3 = 0 \rightarrow (3)$

sub (3) in (2)

$-15x_2 = 0$

$\boxed{x_2 = 0} \rightarrow (4)$

sub $x_2 = 0$ & $x_3 = 0$ in (1)

$x_1 + 0 + 0 = 0$

$\boxed{x_1 = 0}$

Solution

$\cancel{x_1, x_2}$ $x_1 = 0$

$x_2 = 0$

$x_3 = 0$. (x_1, x_2, x_3)

$N(T)$ is spanned by $(0, 0, 0)$

Dim of $N(T) = 0$.

$\text{Dim}(V) = 3$.

$R(T) + N(T) = 3 + 0 = 3$.

$\therefore R(T) + N(T) = \text{Dim}(V)$.

$\therefore$ Rank - Nullity theorem verified.

---

(2) Consider the matrix mapping $C : R^4 \rightarrow R^3$ where $C = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & -1 & 2 & -1 \\ 1 & -3 & 2 & -2 \end{bmatrix}$

Find a basis and dimension of the range of the linear map.

$\to (1)$

$-15x_2 - 33x_3 = 0 \to (2)$

$-198 x_3 = 0 \to (3)$

$x_3 = 0 \to (3)$

sub (3) in (2)

$-15x_2 = 0$

$\boxed{x_2 = 0} \to (4)$

sub $x_2 = 0$ & $x_3 = 0$ in (1)

$x_1 + 0 + 0 = 0$

$\boxed{x_1 = 0}$

Solution

~~x_1, x_2~~ $x_1 = 0$

$x_2 = 0$

$x_3 = 0$.

(x_1, x_2, x_3)

N(T) is spanned by $(0, 0, 0)$

Dim of N(T) = 0.

Dim (V) = 3.

R(T) + N(T) = 3 + 0 = 3.

$\therefore$ R(T) + N(T) = Dim(V).

$\therefore$ Rank - Nullity theorem verified.

---

(2) Consider the matrix mapping

$c: R^4 \to R^3$ where $c = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & -1 & 2 & -1 \\ 1 & -3 & 2 & -2 \end{bmatrix}$

Find a basis and dimension of the range of the linear map.

Soln :-

Let $C = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & -1 & 2 & -1 \\ 1 & -3 & 2 & -2 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & -5 & 2 & -3 \\ 0 & -5 & 2 & -3 \end{bmatrix}$ $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - R_1$

$\sim \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & -5 & 2 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $R_3 \to R_3 - R_2$.

One non-zero equation

2 unknowns.

$\text{Dim}(V) = 2$.

Basis $= \{(1,2,0,1), (0,-5,2,-3)\}$

It is a basis

---

① Find an orthogonal basis containg the vecto $(1,3,4)$ for $V_3(R)$ with the standard inner product.

Soln

Let (x_1, x_2, x_3) be any vector orthogonal to $(1,3,4)$.

$\therefore x_1 + 3x_2 + 4x_3 = 0$

Solution is $(1,1,-1)$ ↳①

$\therefore y_1 + y_2 - y_3 = 0 \to ②$

$(1,3,4)\ (1,1,-1)$
$= 1\times1 + 3\times1 + 4\times(-1)$
$= 1+3-4$
$= 4-4 = 0$.

$$\frac{y_1}{-3-4} = \frac{y_2}{4+1} = \frac{y_3}{1-3}$$

$$\frac{y_1}{-7} = \frac{y_2}{5} = \frac{y_3}{-2}.$$

$\therefore (-7, 5, -2).$

Basis = $\{(1,3,4), (1,1,-1), (-7,5,-2)\}$

(2) Find an orthogonal basis containing $(1,1,-1), (1,0,1)$ for $V_3(R)$ with Standard $\langle \cdot, \cdot \rangle$. → inner product

Sol:

$y_1 + y_2 - y_3 = 0$

$y_1 + 0y_2 + y_3 = 0.$

$$\begin{matrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{matrix}$$

$$\frac{y_1}{1-0} = \frac{y_2}{-1-1} = \frac{y_3}{0-1}$$

$$\frac{y_1}{1} = \frac{y_2}{-2} = \frac{y_3}{-1}$$

$\therefore (1, -2, -1).$

Basis = $\{(1,1,-1), (1,0,1), (1,-2,-1)\}$