Unit-I

Random variable

Types of random variable:

There are two types of random variables.

1) Discrete random variable
2) continuous " "

Formula :- Discrete random variable

1) $\sum_i P(x_i) = 1$

2) $\text{var}(ax+by) = a^2 \text{var}(x) + b^2 \text{var}(y)$

3) Mean $= E(x) = \sum x_i P(x_i)$ (μ)

4) $E(x^2) = \sum x_i^2 P(x_i)$

5) Variance $= \text{Var}(x) = E(x^2) - (E(x))^2$

6) $E(ax+by) = a E(x) + b E(y)$.

7) Probability mass function $P(x) = P(X=x)$.

8) Standard deviation $= \sqrt{\text{var}(x)}$. (S.D, σ^2)

Problems :-

(APR-14)

(1) X & y are independent random variables with variance 2 & 3. Find the variance of $3x+4y$.

Soln :-

Given, $\text{var}(x) = 2$

$\text{var}(y) = 3$.

$\text{Var}(ax+by) = a^2 \text{var}(x) + b^2 \text{var}(y)$

$\text{Var}(3x+4y) = 3^2 \text{var}(x) + 4^2 \text{var}(y)$

$= 9(2) + 16(3)$

$= 18 + 48$

$= 66$.

(2) If the range of x is the Set $(0,1,2,3,4)$ and $P(X=x) = 0.2$ determine the mean & Variance of the random variable. (Nov 09)

Soln:

x	0	1	2	3	4
$P(X=x)$	0.2	0.2	0.2	0.2	0.2

$$\text{Mean} = E(x) = \sum x_i P(x_i)$$

$$= 0(0.2) + 1(0.2) + 2(0.2) + 3(0.2) + 4(0.2)$$

$$= 0 + 0.2 + 0.4 + 0.6 + 0.8$$

$$\boxed{E(x) = 2}$$

$$E(x^2) = \sum x_i^2 P(x_i)$$

$$= 0^2(0.2) + 1^2(0.2) + 2^2(0.2) + 3^2(0.2) + 4^2(0.2)$$

$$= 0 + 0.2$$

$$\boxed{E(x^2) = 6}$$

$$\text{Var}(x) = E(x^2) - (E(x))^2$$

$$= 6 - (2)^2$$

$$= 6 - 4$$

$$\boxed{\text{Var}(x) = 2}$$

(3) A r.v X has the following probability function. Find the value of k.

X	0	1	2	3	4
$P(x)$	k	$3k$	$5k$	$7k$	$9k$

Soln:

$$\sum P(x_i) = 1$$

$$P(0) + P(1) + P(2) + P(3) + P(4) = 1$$

$$k + 3k + 5k + 7k + 9k = 1$$

$$25k = 1$$

$$\boxed{k = \frac{1}{25}}$$

(4) A discrete ... probability d...

x	0	
$P(x=x)$	a	

(i) Find th...

(ii) Find P(...

(iii) Find t...

Soln:

(i) Given $\sum_{i=}^{8}$

$a+$

(ii) $P(X<3)$

(or)

$= 1 - P(X \geq 3)$

$= 1 - \frac{8}{9}$

$= \frac{9-8}{9}$

$= \frac{1}{9}$

2) $P(X \geq 3)$

2) $P(0<x<3$

NOV-11

(7) A discrete random variable X has the following probability distribution.

x	0	1	2	3	4	5	6	7	8
P(X=x)	a	3a	5a	-7a	9a	11a	13a	15a	17a

(i) Find the value of a,

(ii) Find $P(X<3)$, $P(0<X<3)$, $P(X\geq 3)$

(iii) Find the distribution function of X.

Soln:

(i) Given $\sum_{i=0}^{8} P(x_i) = 1$

$$a+3a+5a+7a+9a+11a+13a+15a+17a=1$$

$$81a = 1$$

$$a = \frac{1}{81}$$

X =	0	1	2	3
P(x) =	$\frac{1}{81}$	$\frac{3}{81}$	$\frac{5}{81}$	$\frac{7}{81}$

(ii) $P(X<3) = P(X=0) + P(X=1) + P(X=2)$

$= a+3a+5a$

$= 9a$

$= 9\left(\frac{1}{81}\right)$

$= \frac{1}{9}$

4	5	6	7	8
$\frac{9}{81}$	$\frac{11}{81}$	$\frac{13}{81}$	$\frac{15}{81}$	

(or) $= 1 - P(X \geq 3)$

$= 1 - 8/9$

$= \frac{9-8}{9}$

$= 1/9$

(b) $P(0<X<3) = P(X=1) + P(X=2)$

$= 3a+5a$

$= 8a$

$= \frac{8}{81}$

(c) $P(X\geq 3) = 1 - P(X<3)$

$= 1 - \frac{1}{9}$

$= \frac{9-1}{9} = \frac{8}{9}$

(or) $= P(3) + P(4) + P(5) + P(6) + P(7) + P(8)$

$= \frac{7}{81} + \frac{9}{81} + \frac{11}{81} + \frac{13}{81} + \frac{15}{81} + \frac{17}{81}$

$= \frac{72}{81} = \frac{8}{9}$.

(iii) Distribution Function :-

x_i	$P(x_i)$	$F(x_i)$
0	$P(0) = \frac{1}{81}$	$F(0) = P(0) = \frac{1}{81}$
1	$P(1) = \frac{3}{81}$	$F(1) = F(0) + P(1) = \frac{1}{81} + \frac{3}{81} = \frac{4}{81}$
2	$P(2) = \frac{5}{81}$	$F(2) = F(1) + P(2) = \frac{4}{81} + \frac{5}{81} = \frac{9}{81}$
3	$P(3) = \frac{7}{81}$	$F(3) = F(2) + P(3) = \frac{9}{81} + \frac{7}{81} = \frac{16}{81}$
4	$P(4) = \frac{9}{81}$	$F(4) = F(3) + P(4) = \frac{16}{81} + \frac{9}{81} = \frac{25}{81}$
5	$P(5) = \frac{11}{81}$	$F(5) = F(4) + P(5) = \frac{25}{81} + \frac{11}{81} = \frac{36}{81}$
6	$P(6) = \frac{13}{81}$	$F(6) = F(5) + P(6) = \frac{36}{81} + \frac{13}{81} = \frac{49}{81}$
7	$P(7) = \frac{15}{81}$	$F(7) = F(6) + P(7) = \frac{49}{81} + \frac{15}{81} = \frac{64}{81}$
8	$P(8) = \frac{17}{81}$	$F(8) = F(7) + P(8) = \frac{64}{81} + \frac{17}{81} = \frac{81}{81} = 1$

NOV-10
APR-12
APR-15

(5) A r.v x has the following Probability function

$X = x_i$	0	1	2	3	4	5	6	7
$P(X = x_i)$	0	k	2k	2k	3k	k^2	$2k^2$	$7k^2 + k$

a) Find k

b) Evaluate $P[x < 6]$, $P[x \geq 6]$

c) If $P[x \leq c] > \frac{1}{2}$, then find the minimum value of c.

d) Find $P[x < 2]$, $P[x > 3]$, $P[1 < x < 5]$

(a) $\sum_i P(x_i) = 1$

$0 + k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 1$

$10k^2 + 9k - 1 = 0$

$k = -1, \quad k = \frac{1}{10}$

-10

$10 \quad -1 \Rightarrow (9)$

$\frac{10}{10}, \frac{-1}{10}$

$1, -\frac{1}{10}$

opposite sign

$-1, \frac{1}{10}$

x	0	1	2	3	4	5	6
$P(x)$	0	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\left(\frac{1}{10}\right)^2 = \frac{1}{100}$	$\frac{2}{100}$

(b) (i) $P[X \geq 6] = P[X=6] + P[X=7]$

$= \frac{2}{100} + \frac{17}{100}$

$= \frac{19}{100}$

7

$\frac{7}{100} + \frac{1}{10}$

$= \frac{7+10}{100}$

$= \frac{17}{100}$

(ii) $P[X<6] = 1 - P[X \geq 6]$

$= 1 - \frac{19}{100}$

$= \frac{100-19}{100}$

$= \frac{81}{100}$

$-1, -2, -3 \ldots -50, 1$

(or)

$P[X<6] = P(0) + P(1) + P(2) + P(3) + P(4) + P(5)$

$= 0 + \frac{1}{10} + \frac{2}{10} + \frac{2}{10} + \frac{3}{10} + \frac{1}{100}$

$= \frac{8}{10} + \frac{1}{100}$

$= \frac{80+1}{100}$

$= \frac{81}{100}$

(c)

x_i	$P(x_i)$	$F(x_i)$
0	0	$F(0) = P(0) = 0$
1	$\frac{1}{10}$	$F(1) = F(0) + P(1) = 0 + \frac{1}{10} = \frac{1}{10} < \frac{1}{2}$ (0.10 < 0.50)
2	$\frac{2}{10}$	$F(2) = F(1) + P(2) = \frac{1}{10} + \frac{2}{10} = \frac{3}{10} < \frac{1}{2}$ (0.30 < 0.50)
3	$\frac{2}{10}$	$F(3) = F(2) + P(3) = \frac{3}{10} + \frac{2}{10} = \frac{5}{10} = \frac{1}{2}$ (0.5 = 0.50)
4	$\frac{3}{10}$	$F(4) = F(3) + P(4) = \frac{5}{10} + \frac{3}{10} = \frac{8}{10} > \frac{1}{2}$ (0.80 > 0.50)
5	$\frac{1}{100}$	$F(5) = F(4) + P(5) = \frac{8}{10} + \frac{1}{100} = \frac{81}{100} > \frac{1}{2}$ (0.81 > 0.50)
6	$\frac{2}{100}$	$F(6) = F(5) + P(6) = \frac{81}{100} + \frac{2}{100} = \frac{83}{100} > \frac{1}{2}$ (0.83 > 0.50)
7	$\frac{17}{100}$	$F(7) = F(6) + P(7) = \frac{83}{100} + \frac{17}{100} = \frac{100}{100} = 1 > \frac{1}{2}$.

∴ The Minimum value of $c = 4$ $\because P[X \le c] \ge \frac{1}{2}$

(d) (i) $P(X<2) = P(0) + P(1)$

$= 0 + \frac{1}{10}$

$= \frac{1}{10}$

(ii) $P(X>3) = 1 - P(X \le 3)$

$= 1 - (P(0) + P(1) + P(2) + P(3))$

$= 1 - (0 + \frac{1}{10} + \frac{2}{10} + \frac{2}{10})$

$= 1 - \frac{5}{10}$

$= \frac{10-5}{10} = \frac{5}{10} = \frac{1}{2}$.

(or)

$= P(4) + P(5) + P(6) + P(7)$

$= \frac{3}{10} + \frac{1}{100} + \frac{2}{100} + \frac{17}{100}$

$= \frac{3}{10} + \frac{20}{100}$

$= \frac{30+20}{100}$

$= \frac{50}{100}$

$= \frac{1}{2}$

(iii) $P(1<x<5) = P(2) + P(3) + P(4)$

$= \frac{2}{10} + \frac{2}{10} + \frac{3}{10}$

$= \frac{7}{10}$

APR-17
APR-15
APR-19

(6) If the r.v X takes the values 1, 2, 3 & 4 such that $2P(X=1) = 3P(X=2) = P(X=3) = 5P(X=4)$. Find the Probability distribution and Cumulative distribution function of X.

Soln

X is a d.r.v

Given $2P(X=1) = 3P(X=2) = P(X=3) = 5P(X=4) = K$

X	1	2	3	4
P(x)	$\frac{K}{2}$	$\frac{K}{3}$	K	$\frac{K}{5}$

$2P(1) = K$

$P(1) = K/2$

$3P(2) = K$

$P(2) = K/3$

$1P(3) = K$

$P(3) = K/1$

$5P(4) = K$

$P(4) = K/5$

$\sum_i P(x_i) = 1$

$\frac{K}{2} + \frac{K}{3} + K + \frac{K}{5} = 1$

$\frac{15K + 10K + 30K + 6K}{30} = 1$

$\frac{61K}{30} = 1$

$61K = 30$

$K = \frac{30}{61}$

x_i	p_i
1	
2	
3	
4	

Distribution function

x_i	$P(x_i)$	$F(x_i)$
1	$\frac{15}{61}$	$F(1) = P(1) = \frac{15}{61}$
2	$\frac{10}{61}$	$F(2) = F(1) + P(2) = \frac{15}{61} + \frac{10}{61} = \frac{25}{61}$
3	$\frac{30}{61}$	$F(3) = F(2) + P(3) = \frac{25}{61} + \frac{30}{61} = \frac{55}{61}$
4	$\frac{6}{61}$	$F(4) = F(3) + P(4) = \frac{55}{61} + \frac{6}{61} = \frac{61}{61} = 1$.

1.5) Continuous random variables

Defn:

A r.v x which takes all possible values in a given interval is called a c.r.v.

Eg:

Age, height, weight

Formula:-

① $\int_{-\infty}^{\infty} f(x)\,dx = 1$

② Mean $= \mu = E(x) = \int_{-\infty}^{\infty} x\,f(x)\,dx$

③ $E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$

④ Variance $= \text{Var}(x) = E(x^2) - (E(x))^2$

⑤ $P[a \le x \le b] = F(b) - F(a)$

b) $0 \le F(x) \le 1$.

① A c.r.v x that can assume any value between $x=2$ & $x=5$ has a density function given by $f(x) = k(1+x)$. Find $P[x<4]$.

Soln:

We know that Given $f(x) = k(1+x)$

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_2^5 k(1+x)\,dx = 1$$

$$k\int_2^5 (1+x)\,dx = 1$$

$$k\left[x + \frac{x^2}{2}\right]_2^5 = 1$$

$$k\left[\left(5 + \frac{25}{2}\right) - \left(2 + \frac{4}{2}\right)\right] = 1$$

$$k\left[\left(\frac{10+25}{2}\right) - \left(\frac{4+4}{2}\right)\right] = 1$$

$$P[x<4] = \int_2^4 f(x)\,dx$$

$$= \int_2^4 k(1+x)\,dx$$

$$= \int_2^4 \left(\frac{2}{27}\right)(1+x)\,dx$$

$$= \frac{2}{27}\int_2^4 (1+x)\,dx$$

$$= \frac{2}{27}\left[x + \frac{x^2}{2}\right]_2^4$$

$$k\left[\frac{35}{2} - \frac{8}{2}\right] = 1$$

$$k\left[\frac{27}{2}\right] = 1$$

$$\boxed{k = \frac{2}{27}}$$

$$= \frac{2}{27}\left[\left(4 + \frac{16}{2}\right) - \left(2 + \frac{2^2}{2}\right)\right]$$

$$= \frac{2}{27}\left[(4+8) - (2+2)\right]$$

$$= \frac{2}{27}\left[12 - 4\right]$$

$$= \frac{2}{27}(8)$$

$$= \frac{16}{27}$$

(2) Find the value of c given the pdf of a r.v X as $f(x) = \begin{cases} c/x^3, & \text{if } 1 < x < \infty \\ 0, & \text{otherwise} \end{cases}$

Soln:

WKT $\int_{-\infty}^{\infty} f(x)\,dx = 1$

diff $\Rightarrow x^n = n x^{n-1}$

$\int x^n dx = \frac{x^{n+1}}{n+1}$

$$\int_1^{\infty} \frac{c}{x^3}\,dx = 1$$

$$c\int_1^{\infty} \left(\frac{1}{x^3}\right) dx = 1$$

$$c\int_1^{\infty} x^{-3}\,dx = 1$$

$$c\left[\frac{x^{-3+1}}{-3+1}\right]_1^{\infty} = 1$$

$$c\left[\frac{x^{-2}}{-2}\right]_1^{\infty} = 1$$

$$\frac{c}{-2}\left[(\infty)^{-2} - (1)^{-2}\right]$$

$$\frac{c}{-2}\left[\frac{1}{x^2}\right]_1^{\infty} = 1$$

$$\frac{c}{-2}\left[\frac{1}{\infty} - \frac{1}{1^2}\right] = 1$$

$$\frac{c}{-2}(0-1) = 1$$

$$\frac{c}{-2} \times -1 = 1 \Rightarrow \boxed{c = 2}$$

probability density function

NOV-17

(3) Let X be a C.r.V with the P.d.f $f(x) = \frac{1}{4}$; $2 \le x \le 6$. Find the expected value and variance of X.

Soln:-

Given $f(x) = \frac{1}{4}$, $2 < x < 6$

$$E(x) = \int_2^6 x f(x)\,dx$$

$$= \int_2^6 x \left(\frac{1}{4}\right) dx$$

$$= \frac{1}{4}\int_2^6 x\,dx$$

$$= \frac{1}{4}\left[\frac{x^2}{2}\right]_2^6$$

$$= \frac{1}{8}(36-4)$$

$$= \frac{1}{8}(32^4)$$

$$E(x) = 4$$

$$E(x^2) = \int_2^6 x^2 f(x)\,dx$$

$$= \int_2^6 x^2 \left(\frac{1}{4}\right) dx$$

$$= \frac{1}{4}\int_2^6 x^2\,dx$$

$$= \frac{1}{4}\left[\frac{x^3}{3}\right]_2^6$$

$$= \frac{1}{12}[216-8]$$

$6^3 = 6\times6\times6 = 36\times6 = 216$

$$= \frac{1}{12}[208]^{52}$$

$$E(x^2) = \frac{52}{3}$$

$$Var(X) = E(x^2) - [E(x)]^2$$

$$= \frac{52}{3} - (4)^2$$

$$= \frac{52}{3} - \frac{16}{1}$$

$$= \frac{52-48}{3}$$

$$Var(X) = \frac{4}{3}$$

) Show that t is a p.d.

Soln:-

WKT $\int_{-\infty}^{\infty} f(x)$

May-20

(5) If X is then find (

Soln:-

(1) $\int_{-1}^{1}$

$\int_{-1}^{1} k$

$k\left[\frac{x}{}\right.$

$\frac{k}{3}$

$\frac{k}{3}$

$E(x^2) =$

$= \int_{-1}$

$= k$

④ Show that the function $f(x) = \begin{cases} \frac{x^2}{3}, & -1 < x < 2 \\ 0, & \text{otherwise} \end{cases}$ is a p.d.f

Soln:-

WKT $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-1}^{2} \frac{x^2}{3}\,dx$

$= \frac{1}{3}\left[\frac{x^3}{3}\right]_{-1}^{2} = \frac{1}{9}\left[8-(-1)\right]$

$= \frac{1}{9}(8+1) = \frac{9}{9} = 1$

May-2011

⑤ If X is a c.r.v with p.d.f $f(x) = \begin{cases} kx^2, & -1 < x < 1 \\ 0, & \text{elsewhere} \end{cases}$

then find ① the value of k

② The mean & variance of X

③ $P\left(\frac{1}{3} \le x < 4\right)$.

Soln:-

① $\int_{-1}^{1} f(x)\,dx = 1$

$\int_{-1}^{1} kx^2\,dx = 1$

$k\left[\frac{x^3}{3}\right]_{-1}^{1} = 1$

$\frac{k}{3}\left[1-(-1)\right] = 1$

$\frac{k}{3}(2) = 1$

$\boxed{k = 3/2}$

② Mean $= E(x) = \int_{-1}^{1} x f(x)\,dx$

$= \int_{-1}^{1} x\,kx^2\,dx$

$= k\int_{-1}^{1} x^3\,dx$

$= k\left[\frac{x^4}{4}\right]_{-1}^{1}$

$= k/4\,(1-1)$

$\boxed{E(x) = 0}$

$E(x^2) = \int_{-1}^{1} x^2 f(x)\,dx$

$= \int_{-1}^{1} x^2\,kx^2\,dx$

$= k\int_{-1}^{1} x^4\,dx$

$= k\left[\frac{x^5}{5}\right]_{-1}^{1}$

$= \frac{k}{5}\left[1-(-1)\right]$

$E(x^2) = k/5\,(2)$

$= \frac{2k}{5}$

$= 2/5 \times 3/2$

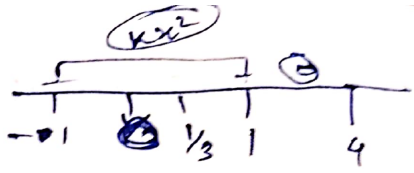
$\boxed{E(x^2) = 3/5}$

$\text{var}(x) = E(x^2) - [E(x)]^2$

$= 3/5 - 0$

$\boxed{\text{var}(x) = 3/5}$

(3) $P\left(\frac{1}{3} \le x < 4\right)$



$$= \int_{1/3}^{1} f(x)\,dx + \int_{1}^{4} f(x)\,dx$$

$$= \int_{1/3}^{1} kx^2\,dx + 0$$

$$= k\left[\frac{x^3}{3}\right]_{1/3}^{1}$$

$$= \frac{k}{3}\left[1 - (1/3)^3\right]$$

$$= \frac{k}{3}\left(1 - \frac{1}{27}\right) = \frac{k}{3}\left(\frac{27-1}{27}\right)$$

$$= \left(\frac{3}{2} \times \frac{1}{3}\right)\left(\frac{13}{\frac{26}{27}}\right)$$

$$= \frac{13}{27}$$

1.6) Moments - Moment Generating Functions

Binomial Expansions :-

$$(1-x)^{-1} = 1+x+x^2+x^3+\dots$$
$$(1+x)^{-1} = 1-x+x^2-x^3+\dots$$
$$(1-x)^{-2} = 1+2x+3x^2+4x^3+\dots$$
$$(1+x)^{-2} = 1-2x+3x^2-4x^3+\dots$$

M.G.F :-

$$M_X(t) = E(e^{tx})$$

$$= \begin{cases} \int_{-\infty}^{\infty} e^{tx} f(x)\,dx, & \text{for a C.P.d} \\ \sum_{x=-\infty}^{\infty} e^{tx} P(x), & \text{for a D.P.d} \end{cases}$$

Problems

① Find the M.G.F for the distribution where

$$f(x) = \begin{cases} \frac{2}{3} & \text{at } x=1 \\ \frac{1}{3} & \text{at } x=2 \\ 0 & \text{otherwise} \end{cases}$$

Soln :-

$$M_X(t) = E(e^{tx}) = \sum_{x=0}^{\infty} e^{tx} P(x)$$

$$= 0 + e^t\left(\frac{2}{3}\right) + e^{2t}\left(\frac{1}{3}\right) + 0 + \dots$$

$$= \frac{2}{3}e^t + \frac{1}{3}e^{2t}$$

$$= \frac{e^t}{3}\left[2+e^t\right]$$

(2) If a R.V X has the MGF, $M_X(t) = \frac{3}{3-t}$ compute $E(x^2)$ & S.D of X.

soln:

$$M_X(t) = \frac{3}{3-t} = \frac{3(1)}{3(1-\frac{t}{3})} = \frac{1}{(1-t/3)} = \left(1-\frac{t}{3}\right)^{-1}$$

$(1-x)^{-1} = 1+x+x^2+x^3+\cdots$

$$= 1+\left(\frac{t}{3}\right)+\left(\frac{t}{3}\right)^2+\cdots$$

$$= 1+\frac{t}{1!}\left(\frac{1}{3}\right)+\frac{t^2}{2!}\left(\frac{2}{9}\right)+\cdots$$

$E(x) = \frac{1}{3}$

$E(x^2)$ = coefficient of $\frac{t^2}{2!}$

$= \frac{2}{9}$.

$Var(X) = E(x^2) - (E(x))^2$

$= \frac{2}{9} - \frac{1}{9} = \frac{1}{9}$

$S.D = \sqrt{Var(x)} = \sqrt{\frac{1}{9}} = \frac{1}{3}$

(3) If the R.V X has the MGF, $M_X(t) = \frac{2}{2-t}$. find the variance of X.

Soln:

$$M_X(t) = \frac{2}{2-t} = \frac{2(1)}{2(1-\frac{t}{2})} = \frac{1}{(1-\frac{t}{2})} = (1-t/2)^{-1}$$

$(1-x)^{-1} = 1+x+x^2+\cdots$

$$= 1+\left(\frac{t}{2}\right)+\left(\frac{t}{2}\right)^2+\cdots$$

$$= 1+\frac{t}{1!}\left(\frac{1}{2}\right)+\frac{t^2}{2!}\left(\frac{2}{4}\right)+\cdots$$

$E(x) = \frac{1}{2}$

$E(x^2) = 2/4 = \frac{1}{2}$.

$Var(x) = E(x^2) - (E(x))^2$

$= \frac{1}{2} - \left(\frac{1}{2}\right)^2$

$= \frac{1}{2} - \frac{1}{4}$

$= \frac{4-2}{8} = \frac{2}{8} = \frac{1}{4}$.

APR-17
NOV-10
APR-12

(3) Find the M.G.F of a r.v x having the Probability p.d.f

$$f(x) = \begin{cases} \frac{x}{4}e^{-x/2} & ; x>0 \\ 0 & , \text{elsewhere} \end{cases}$$

Also deduce the first four moments about the origin.

Given $f(x) = \frac{x}{4}e^{-x/2}$; $x>0$.

$$M_X(t) = E(e^{tx})$$

$$= \int_0^\infty e^{tx} f(x)\,dx$$

$$= \int_0^\infty e^{tx}\, \frac{x}{4} e^{-x/2}\,dx$$

$$= \frac{1}{4}\int_0^\infty x\, e^{tx - x/2}\,dx$$

$$= \frac{1}{4}\int_0^\infty x\, e^{-\left(\frac{1}{2}-t\right)x}\,dx$$

$$= \frac{1}{4}\,\frac{1!}{\left(\frac{1}{2}-t\right)^2}$$

$\because \int_0^\infty x^n e^{-ax}\,dx = \frac{n!}{a^{n+1}}$

$$= \frac{1}{4}\,\frac{1!}{\left(\frac{1-2t}{2}\right)^2}$$

$$= \frac{1}{4}\,\frac{4}{(1-2t)^2}$$

$$= \frac{1}{(1-2t)^2}$$

$$= (1-2t)^{-2}$$

$\because (1-x)^{-2} = 1+2x+3x^2+\cdots$

$$= 1+2(2t)+3(2t)^2+4(2t)^3+5(2t)^4+\cdots$$

$$= 1+4t+12t^2+32t^3+80t^4+\cdots$$

$$= 1+\frac{t}{1!}(4)+\frac{t^2}{2!}(24)+\frac{t^3}{3!}(192)+\frac{t^4}{4!}(1920)+\cdots$$

$\therefore \mu_1' = 4\ ;\ \mu_2' = 24,\ \mu_3' = 192,\ \mu_4' = 1920$

---

(4) A r.v X has pdf

$$f_X(x) = \begin{cases} kx^2 e^{-x} & ; x>0 \\ 0 & ; \text{otherwise} \end{cases}$$

Find the rth moment of X about origin. Hence, find the mean & variance.

Soln:

WKT $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_0^\infty kx^2 e^{-x}\,dx = 1$$

$\because \int_0^\infty x^n e^{-ax}\,dx = \frac{n!}{a^{n+1}}$

$$k\frac{(2!)}{(1)^3} = 1$$

$$2k = 1 \Rightarrow \boxed{k = \frac{1}{2}}$$

$$\mu_r' = E(x^r) = \int_0^\infty x^r f(x)\,dx$$

$$= \int_0^\infty x^r \frac{1}{2} x^2 e^{-x}\,dx$$

$$= \frac{1}{2}\int_0^\infty x^{r+2} e^{-x}\,dx$$

$$\mu_r' = \frac{1}{2}(r+2)!$$

$\because \int_0^\infty x^n e^{-x}\,dx = \frac{n!}{a^{n+1}}$

Put $r=1 \Rightarrow \mu_1' = \frac{1}{2}(1+2)! = \frac{1}{2}(3!) = \frac{1}{2}(1\times 2\times 3)$

Mean $\therefore \mu_1' = 3$

Put $r=2 \Rightarrow \mu_2' = \frac{1}{2}(2+2)! = \frac{1}{2}(4!) = \frac{1}{2}(1\times 2\times 3\times 4)$

$= 12$

Variance $= \mu_2' - (\mu_1')^2$

$= 12 - (3)^2$

$= 12 - 9$

Variance $= 3$

Discrete Distributions :-

* Binomial distribution
* Poisson "
* Geometric "

(7) Binomial Distributions :-

$$P(X=x) = {}^nC_x\, p^x\, q^{n-x} \quad ; \quad x = 0, 1, 2 \ldots n, \qquad p+q=1.$$

Binomial			
MGF	Mean	Variance	S.D
$q+pe^t$	np	npq	$\sqrt{npq}$

Problems

① Check whether the following data follow a binomial distribution or not. Mean = 3, Variance

Soln :-

Given Mean $= np = 3 \rightarrow (1)$

Variance $= npq = 4 \rightarrow (2)$

$$\frac{(2)}{(1)} \Rightarrow \frac{npq}{np} = \frac{4}{3} \quad 1 \cdot \frac{1}{3}$$

$$q = 1\tfrac{1}{3} > 1$$

$\therefore q > 1$ which is not possible $0 < q < 1$

② Under what conditions binomial distribution tends to Poisson distribution?

Soln :-

$n \to \infty$, $p \to 0$, $\lambda = np$ is finite

③ If a fo

$P(X \le 1)$,

of heads

Soln :-

Given

$P(X$

$P[X \le 1]$

④ The M

distrik

distrik

Soln :-

Mea

S.

(3)/(1)

③ If a fair coin is tossed twice, find $P(X \leq 1)$, where X denotes the number of heads in each experiment

Soln:

$P = \frac{2}{4} = \frac{1}{2}$ → single time
→ Head, Tail

Given $p = \frac{1}{2}$; $q = \frac{1}{2}$, $n = 2$

$p + q = 1$
$q = 1 - p$
$q = 1 - 1/2$
$q = \frac{1}{2}$

$P(X = x) = {}^nC_x \; p^x \; q^{n-x}$

$P[X \leq 1] = P[X = 0] + P[X = 1]$

$= {}^2C_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^2 + {}^2C_1 \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^1$

$^2C_0 = 1$
$^2C_1 = \frac{2}{1} = 2$

$= \frac{1}{4} + 2 \times \frac{1}{2} \times \frac{1}{2}$

$= \frac{1}{4} + \frac{1}{2} = \frac{2+4}{8} = \frac{6}{8} = \frac{3}{4}$.

④ The Mean & Standard deviation of binomial distribution are 5 & 2. Determine the distribution.

Soln:

$\text{Mean} = np = 5 \rightarrow ①$

$S.D = \sqrt{npq} = 2 \rightarrow ②$

$(npq)^{1/2 \times 2} = 2^2$

$\sqrt{\quad} = \frac{1}{2}$

$npq = 4 \rightarrow ③$

$\frac{③}{①} \Rightarrow \frac{npq}{np} = \frac{4}{5}$

$q = \frac{4}{5}$

$p = 1 - q$

$= 1 - 4/5$

$= \frac{5-4}{5}$

$p = 1/5$

Sub ①,

$np = 5$

$n(1/5) = 5$

$n = 25$

$P(X = x) = {}^nC_x \; p^r q^{n-x}$

$= 25C_x \left(\frac{1}{5}\right)^x \left(\frac{4}{5}\right)^{25-x}$

(5) In 256 sets of twelve tosses of a coin, how many cases may one expect eight heads and four tails?

Soln:

$N = 256$ (sets)

$n = 12$ (tosses)

Head	Tails	Total
8	4	12

$X = 8, x = 4$

$P = \frac{1}{2}$ → single time
→ Head tail

$q = 1 - P$

$= 1 - \frac{1}{2}$

$\boxed{q = \frac{1}{2}}$

$P(X = x) = {}^{n}C_x \, p^x \, q^{n-x}$

$P(x = 8) = N \left[{}^{n}C_x \, p^x \, q^{n-x} \right]$

$= 256 \left[{}^{12}C_8 \left(\frac{1}{2}\right)^8 \left(\frac{1}{2}\right)^{12-8} \right]$

$256 \left[495 \left(\frac{1}{256} \right) \frac{1}{16} \right]$

$= \frac{495}{16}$

$= 30.937$ times

$P(X = 8) \simeq 31$ times

${}^{12}C_8 = \frac{12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5}{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8}$

$= 11 \times 5 \times 9$

$= 495$

$2^8 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$

$2^8 = 256$

(6) 6 dice are thrown 729 times. How many times do you expect atleast three dice to show 5 or 6?

$p = \frac{2}{6}$ $\quad q = 1-p = 1-\frac{1}{3}$ $\quad n = 6$.

$p = \frac{1}{3}$ $\quad q = \frac{2}{3}$

$P(X=x) = {}^nC_x \, p^x \, q^{n-x}$; $x = 0, 1, 2 \ldots 6$.

$P(X=x) = {}^6C_x \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{6-x}$

$P(X=x) = {}^nC_x \, p^x \, q^{n-x}$

$P(X \geq 3) = N\left[P(X=3) + P(X=4) + P(X=5) + P(X=6)\right]$

$= 729\left[{}^6C_3 \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^3 + {}^6C_4 \left(\frac{1}{3}\right)^4 \left(\frac{2}{3}\right)^2 + {}^6C_5 \left(\frac{1}{3}\right)^5 \left(\frac{2}{3}\right) + {}^6C_6 \left(\frac{1}{3}\right)^6 \left(\frac{2}{3}\right)^0\right]$

$= 729\left[20 \times \frac{1}{27}\left(\frac{8}{27}\right) + 15 \times \frac{1}{81} \times \left(\frac{4}{9}\right) + 6 \times \frac{1}{243} \times \frac{2}{3} + 1 \times \frac{1}{729} \times 1\right]$

$= 729\left[\frac{160}{729} + \frac{60}{729} + \frac{12}{729} + \frac{1}{729}\right]$

$P(X \geq 3) = \frac{233}{729} \times 729 = 233$ times

In 729 times, when 6 dice are thrown

$= \frac{729 \times 233}{729}$

$= 233$ times.

$^6C_3 = \frac{6 \times 5 \times 4}{1 \times 2 \times 3} = 5 \times 4 = 20$

$^6C_4 = \frac{6 \times 5 \times 4 \times 3}{1 \times 2 \times 3 \times 4} = 5 \times 3 = 15$

$^6C_5 = \frac{6 \times 5 \times 4 \times 3 \times 2}{1 \times 2 \times 3 \times 4 \times 5} = 6$

$^6C_6 = \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{1 \times 2 \times 3 \times 4 \times 5 \times 6} = 1$

(7) 10 coins are thrown simultaneously. Find the Probability of getting atleast 7 heads.

Soln

Given $p = \frac{1}{2}$; $q = \frac{1}{2}$; $n = 10$

$P(X = x) = {}^{n}C_x \, p^x \, q^{n-x}$

$P(X \geq 7) = P(X=7) + P(X=8) + P(X=9) + P(X=10)$

$= {}^{10}C_7 \left(\frac{1}{2}\right)^7 \left(\frac{1}{2}\right)^3 + {}^{10}C_8 \left(\frac{1}{2}\right)^8 \left(\frac{1}{2}\right)^2$

$+ {}^{10}C_9 \left(\frac{1}{2}\right)^9 \left(\frac{1}{2}\right) + {}^{10}C_{10} \left(\frac{1}{2}\right)^{10}$

$= \left(\frac{1}{2}\right)^{10} \left[{}^{10}C_7 + {}^{10}C_8 + {}^{10}C_9 + {}^{10}C_{10}\right]$

$= \frac{1}{2^{10}} \left[120 + 45 + 10 + 1\right]$

$= \frac{176}{1024}$

$= 0.1718$

${}^{10}C_7 = \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4}{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7}$

$= 10 \times 3 \times 4 = 120$

${}^{10}C_8 = \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3}{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8}$

$= 5 \times 9 = 45$

${}^{10}C_9 = \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9}$

$= 10$

${}^{10}C_{10} = \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10}$

$\boxed{{}^{10}C_{10} = 1}$

Poisson Distribution

$$P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!} \; ; \; x = 0, 1, 2 \cdots n \; ; \; \lambda = np$$

Poisson

MGF	Mean	Variance	S.D
$e^{\lambda(e^t-1)}$	λ	λ	$\sqrt{\lambda}$

① If 3% of the electric bulbs manufactured by a company are defective, find the probability that in a sample of 100 bulbs exactly 5 bulbs are defective ($e^{-3} = 0.0498$).

Soln :-

Given $p = 3\% = \frac{3}{100}$; $n = 100$; $\lambda = np$

$= 100 \times \frac{3}{100}$

$\lambda = 3$

$$P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!}$$

$$P(X=5) = \frac{e^{-3}3^5}{5!} = \frac{0.498 \times 243}{120}$$

$= 0.1008$

$3^5 = 3\times3\times3\times3\times3$

$9\times9\times3$

81×3

243

$5! = 1\times2\times3\times4\times5$

$= 120$

---

② If X is Poisson variate $P(X=2) = 9P(X=4) + 90P(X=6)$ find Mean & Variance of X.

Soln :-

$$P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!} \; ; \; x = 0, 1, 2 \cdots n$$

$$P(X=2) = 9P(X=4) + 90P(X=6)$$

$4! = 1\times2\times3\times4$

$$\frac{e^{-\lambda}\lambda^2}{2!} = 9\frac{e^{-\lambda}\lambda^4}{4!} + 90\frac{e^{-\lambda}\lambda^6}{6!}$$

$$\frac{e^{-\lambda}\lambda^2}{2!} = e^{-\lambda}\lambda^2\left(\frac{9\lambda^2}{4!} + \frac{90\lambda^4}{6!}\right)$$

$$\frac{1}{2} = \frac{9\lambda^2}{1\times2\times3\times4} + \frac{90\lambda^4}{1\times2\times3\times4\times5\times6}$$

$$\frac{1}{2} = \frac{3\lambda^2}{8} + \frac{\lambda^4}{8}$$

$$\frac{1}{2} = \frac{3\lambda^2 + \lambda^4}{8}$$

$$\lambda^4 + 3\lambda^2 = 4$$

$$\lambda^4 + 3\lambda^2 - 4 = 0$$

$$\lambda = 1, -1, \pm 2i$$

1	1	0	3	0	−4
	0	1	1	4	4
−1	1	1	4	4	0
	0	−1	0	−4	
−4	1	0	4	0	
	0	−4			

$\lambda = 1, -1 \quad \lambda^2 = -4$

$\lambda = \pm 2i$

Mean = variance = $\lambda = 1$

$S.D = \sqrt{\lambda} = \sqrt{1} = 1$

---

(3) The moment generating function of a r.v X is given by $M_X(t) = e^{3(e^t - 1)}$. Find $P(X=1)$

Soln:

Given $M_X(t) = e^{3(e^t - 1)}$

M.G.F

$$M_X(t) = e^{\lambda(e^t - 1)}$$

$$\boxed{\lambda = 3}$$

$$P(X = x) = \frac{e^{-\lambda} \lambda^x}{x!}$$

$$P(X=1) = \frac{e^{-3} 3^1}{1!} = 0.0498 \times 3 = 0.1494.$$

---

(4) In a book of 520 pages, 390 typographical errors occur. Assuming Poisson's law for the number of errors per page, find the probability that a random sample of 5 pages will contain no error.

Soln:

$$\lambda = \frac{\text{Errors}}{\text{Pages}} = \frac{390}{520} = 0.75$$

$$\boxed{\lambda = 0.75}$$

$$P(X=x) = \frac{e^{-\lambda} \lambda^x}{x!}$$

$\therefore P[\text{5 pages contain no error}] = P[X=0]^5$ (no error)

$$= \left[\frac{e^{-0.75}(0.75)^0}{0!}\right]^5 = (e^{-0.75})^5$$

$$= e^{3.75}$$

$$= 0.0235.$$

(5) If 2% electric bulbs produced by a Company are defective, find the probability that in a sample of 200 bulbs exactly 4 are defective.

Soln

$p = 2\% = \frac{2}{100}$; $x = 4$; $n = 200$, $\lambda = np$

$= 200 \times \frac{2}{100}$

$\lambda = 4$

$P(X=x) = \frac{e^{-\lambda} \lambda^x}{x!}$

$P(X=4) = \frac{e^{-4} 4^4}{4!} = \frac{e^{-4} \times 4 \times 4 \times 4 \times 4}{1 \times 2 \times 3 \times 4}$

$e^{-4} = 0.0183$

$= \frac{32}{3} e^{-4} = \frac{32 \times 0.0183}{3}$

$= 0.1952$

APR-15
NOV-16
APR-18
NOV-17

(6) messages arrive at a switch board in a Poisson manner at an average rate of six per hour. Find the probability for each of the following events.

(1) exactly two messages arrive with in one hour

(2) No message arrives with in one hour.

(3) atleast three messages arrives with one hour.

Soln:

Mean $\lambda = 6$

$P(X=x) = \frac{e^{-\lambda} \lambda^x}{x!}$

$P(X=1) = e^{-6}(6)^1$

$= 6 \cdot e^{-6}$

$= 0.0149.$

(1) $P(X=2) = \frac{e^{-6} 6^2}{2!} = \frac{e^{-6} \times 6 \times 6^3}{1 \times 2} = 18 e^{-6} (0.0450)$

(2) $P(X=0) = \frac{e^{-6} 6^0}{0!} = e^{-6} = 0.0025$

$0! = 1$
$6^0 = 1$

(3) $P(X \geq 3) = 1 - P(X < 3)$

~~$= 1 - (0.0025 + 0.0149 + 0.0450)$~~

$= 1 - 25(0.0025) = 1 - [P(X=0) + P(X=1) + P(X=2)]$

$= 1 - 0.0625 = 1 - \left[\frac{e^{-6} 6^0}{0!} + \frac{e^{-6} 6^1}{1!} + \frac{e^{-6} 6^2}{2!}\right]$

$= 0.9376.$

$= 1 - e^{-6}(1 + 6 + 18)$

$= 1 - 25 e^{-6}.$

Geometric Distribution

Table - I		Table - II	
$P(X=x) = q^{x-1} p$; $x = 1, 2, \ldots$		$P(X=x) = q^x p$; $x = 0, 1, 2$	
$M_X(t)$	$\frac{p e^t}{(1-qe^t)}$	$M_X(t)$	$\frac{p}{(1-qe^t)}$
Mean	$\frac{1}{p}$	Mean	$\frac{q}{p}$
Variance	$\frac{q}{p^2}$	Variance	$\frac{q}{p^2}$

APR-17
APR-10
NOV-12
APR-15

(1) If the probability that an applicant for a driver's licence will pass the road test on any given trial is 0.8. What is the probability tha he will finally pass the test

a) on the fourth trial

b) in fewer than 4 trials ?

Soln

Given $P = 0.8$ & $q = 0.2$ ($p+q=1$, $q = 1-p$, $= 1-0.8$)

$P(X=r) = q^{x-1} p$; $r = 1, 2, 3 \ldots \infty$ ($q = 0.2$)

(a) $P(X=4) = (0.2)^{4-1} (0.8) = (0.2)^3 (0.8) = 0.0064$

(b) $P(X<4) = P(X=1) + P(X=2) + P(X=3)$

$= (0.2)^{1-1} (0.8)^{\phi} + (0.2)^{2-1} (0.8) + (0.2)^{3-1} (0.8)$

$= (1 + 0.2 + (0.2)^2)(0.8)$

$= 0.992$.

APR-17

① X is a geometric variate taking values 1, 2, ... ∞, then find P(X is odd).

$$P(X=r) = q^{r-1} p$$

$$P(X \text{ is odd}) = P(X=1) + P(X=3) + P(X=5)$$

$$= p + q^2 p + q^4 p + \ldots$$

$$= p(1 + q^2 + q^4 + \ldots)$$

$(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots$

$(1-q^2)^{-1} = 1 + q^2 + (q^2)^2 + (q^2)^3 + \ldots$

$= 1 + q^2 + q^4 + q^6 \ldots$

$$= p(1-q^2)^{-1}$$

$$= p\left(\frac{1}{1-q^2}\right) = p\,\frac{1}{(1+q)(1-q)} = \cancel{p}\,\frac{1}{\cancel{p}(1+q)}$$

$p + q = 1$

$p = 1 - q$

$a^2 - b^2 = (a+b)(a-b)$

$$= \frac{1}{1+q}.$$

APR-13
NOV-17

③ If the probability that a target is destroyed on any one shot is 0.5, what is the probability that it would be destroyed on 6th attempt?

Soln

Given $p = 0.5$ (or) $\frac{1}{2}$; $p + q = 1$

$q = 1 - 0.5$

$q = 0.5$ (or) $\frac{1}{2}$

$$P(X=r) = q^{r-1} p$$

$$P(X=6) = q^5 p$$

$$= \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)$$

NOV-16
APR-17

$$= \frac{1}{2^6} = \frac{1}{64} = 0.156$$

④ Balls are tossed at random into 50 boxes. Find the expected number of tosses required to get the first ball in the fourth box.

$$p = \frac{1}{50}$$

Mean:

APR-16 $E(y) = \frac{1}{p} = \frac{1}{50}$

⑤ Let $P(X=x) = \left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^{x-1}$; $x = 1, 2, 3 \ldots$ be the

$P(q)^{x-1}$

① $P(X>4)$

② $P(X>4 / x>2)$

③ $E(x)$ ④ $\text{Var}(x)$.

(1) $P[X>k] = q^k = \left(\frac{1}{4}\right)^k$

(2) $P(X>4) = \left(\frac{1}{4}\right)^4 = \frac{1}{256}$

(2) $P[X>m+n / X>M] = \frac{q^{m+n}}{q^m} = q^n$

$P[X>4/X>2) = q^2$

$= \left(\frac{1}{4}\right)^2$

$= 1/16$

(3) Mean

$E(x) = \frac{1}{p} = \frac{1}{3/4} = 4/3$

(4) $\text{var}(x) = \frac{q}{p^2} = \frac{1}{4} \times \left(\frac{4}{3}\right)^2$

$= \frac{1}{9} \times \frac{4^2}{9}$

$= \frac{4}{9}$.

AMU APR-10, APR-11, NOV-15, NOV-18, APR-6.

(6) Establish the memoryless Property of geometric distributions.

Statement:

If X has a G.D, then for any two positive integers m & n.

$P[X>m+n/X>m] = P[X>n]$.

Proof:

$P[X>m+n/X>m] = \frac{P[X>m+n \cap X>m]}{P[X>m]} = \frac{P[X>m+n]}{P[X>m]}$

$P(X=r) = q^{r-1}p$; $r = 1, 2, 3 \ldots$

$P(X>k) = \sum_{k+1}^{\infty} q^{r-1} p$

$= q^{k+1-1} p + q^{k+2-1} p + q^{k+3-1} p + \ldots$

$$= q^k p + q^{k+1} p + q^{k+2} p + \cdots$$

$$= q^k p [1 + q + q^2 + \cdots]$$

$$= q^k p (1-q)^{-1}$$

$$= \frac{q^k p}{(1-q)}$$

$p + q = 1$

$p = 1 - q$

$$= q^k \left(\frac{p}{p}\right)$$

$$\boxed{P(X > k) = q^k}$$

$\therefore$ $P[X > m+n] = q^{m+n}$.

$P[X > m] = q^m$

$$\therefore P[X > m+n / X > m] = \frac{q^{m+n}}{q^m} = \frac{q^m \cdot q^n}{q^m} = q^n$$

$\therefore P[X > m+n / X > m] = P[X > n]$.

Probability & Random variables

Defn: Probability

Probability is a mathematical measurement of uncertainty.

Eg:

* Tossing a coin $S = \{H, T\}$; $n(S) = 2$
* Throwing a dice $S = \{1, 2, 3, 4, 5, 6\}$; $n(S) = 6$
* Tossing 2 coins $S = \{HH, HT, TH, TT\}$; $n(S) = 4$
* Tossing 2 dice $S = \{(1,1)\ (1,2)\ (1,3)\ (1,4)\ (1,5)\ (1,6)$
  $(2,1)\ (2,2)\ (2,3)\ (2,4)\ (2,5)\ (2,6)$
  ...
  $(6,1), (6,2), (6,3), (6,4)\ (6,5)\ (6,6)$
  $n(S) = 36$.

Axioms of Probability

① what are the axioms of probability.

S → Sample space

E → Event in a random experiment

Axiom : 1 $\quad 0 \leq P(E) \leq 1$

Axiom : 2 $\quad P(S) = 1$

Axiom : 3 For any sequence of mutually exclusive events $E_1, E_2, E_3 \ldots$

(i.e) events for which $E_i E_j = \phi$ when $i \neq j$

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$

$P(E)$ → The probability of the event E.

② Using the axiom

$P(A^c) = 1 - P(A)$

Soln:

Let $A \cup A^c =$

$P(A \cup A^c)$

$P(A) + P(A^c)$

$P(A^c)$

③ Prove that

Zero.

WKT

P

P(

Formula

1) P(A)

④ A bo

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from the

atleast

Soln:

P(atlea

is

10+4+2=16

(2) Using the axioms of probability, prove

$P(A^c) = 1 - P(A)$

Soln,

Let $A \cup A^c = S$

$P(A \cup A^c) = P(S)$

$P(A) + P(A^c) = 1$

$P(A^c) = 1 - P(A)$

(3) Prove that the probability of a null set is zero.

WKT $S \cup \phi = S$

$P(S \cup \phi) = P(S)$

$P(S) + P(\phi) = P(S)$

$P(\phi) = P(S) - P(S)$

$P(\phi) = 0$

Formula:

1) $P(A) = \frac{n(A)}{n(S)}$

(4) A lot of integrated circuit chips consists of 10 good, 4 with minor defects and 2 with major defects. Two chips are randomly chosen from the lot. What is the probability that atleast one chip is good? (APR-17)

Soln:

$P(\text{atleast one is good}) = \frac{n(A)}{n(S)} = \frac{{}^{10}C_1 \times {}^{6}C_1 + {}^{10}C_2}{{}^{16}C_2}$

(10 good, 6 defects (4+2))

$= \frac{10 \times 6 + 45}{120}$

$= \frac{60+45}{120} = \frac{105}{120} = \frac{7}{8}$

10+4+2=16

(5) Four persons are chosen at random from a group containing 3 men, 2 women & 4 children. Show that the chance that exactly two of them will be children is $\frac{10}{21}$. NOV-06

Men	Women	Children	Total
3	2	4	9

4 persons out of 9 persons $= {}^9C_4 = 126 \rightarrow (1)$

Total = (2 children out of 4 children) $\times$ (2 persons out of 5 persons)

(2 persons → 4−2=2; 5 persons → 3+2=5)

$= {}^4C_2 \times {}^5C_2$

$= 6 \times 10$

$= 60 \rightarrow (2)$

$\therefore$ Probability $= \frac{(2)}{(1)} = \frac{60}{126} = \frac{10}{21}$

(6) If A & B are events with $P(A) = \frac{3}{8}$; $P(B) = \frac{1}{2}$ & $P(A \cap B) = 1/4$. find $P(A^c \cap B^c)$ Nov 06

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= \frac{3}{8} + \frac{1}{2} - \frac{1}{4}$

$= \frac{3+4-2}{8}$

$P(A \cup B) = 5/8$

$P(A^c \cap B^c) = 1 - P(A \cup B)$

$= 1 - 5/8$

$= \frac{8-5}{8}$

$= \frac{3}{8}$

(7) If A & B are independent events with $P(A) = 0.4$ & $P(B) = 0.5$. Find $P(A \cup B)$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= P(A) + P(B) - P(A)\,P(B)$$

$$= 0.4 + 0.5 - 0.4 \times 0.5$$

$$= 0.9 - 0.2$$

$$P(A \cup B) = 0.7$$

(8) If you twice a flip a balanced coin, what is the probability of getting atleast one head?

$S = \{HH, HT, TH, TT\}$ = 3/4

$$P(\text{atleast one head}) = 1 - P(\text{no head})$$

$$= 1 - \frac{1}{4}$$

$$= \frac{4-1}{4}$$

$$= \frac{3}{4}$$

(9) Let $P(A \cup B) = 5/6$; $P(A \cap B) = \frac{1}{3}$ & $P(\bar{B}) = \frac{1}{2}$. Are the events A & B independent. Explain.

$$P(B) = 1 - P(\bar{B})$$

$$= 1 - \frac{1}{2}$$

$$P(B) = \frac{1}{2}$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$5/6 = P(A) + \frac{1}{2} - \frac{1}{3}$$

$$P(A) = 5/6 - \frac{1}{2} + \frac{1}{3}$$

$$= \frac{5-3+2}{6}$$

$$= 5/6 + 1/3 - \frac{1}{2}$$

$$= \frac{15+6}{18} - \frac{1}{2}$$

$$= \frac{21}{18} - \frac{1}{2}$$

$$= \frac{42-18}{36}$$

$$= \frac{24}{36}$$

$$= \frac{2}{3}.$$

$$P(A \cap B) = P(A)\,P(B)$$

$$\frac{1}{3} = \frac{1}{x} \cdot \frac{2}{3}$$

$$1/3 = 1/3$$

$\therefore$ A & B are independent.

(1) State and prove Bayes theorem

Statement:-

Let $B_1, B_2 \ldots B_n$ be an exhaustive & mutually exclusive random experiments and A be an event related to that B_i then

$$P(B_i/A) = \frac{P(B_i)\,P(A/B_i)}{\sum_{i=1}^{n} P(B_i)\,P(A/B_i)}$$

$i = 1, 2 \ldots n$.

Proof:

By conditional probability,

$$P(B_i/A) = \frac{P(B_i \cap A)}{P(A)} \longrightarrow (1)$$

Using multiplication rule of Probability,

$$P(B_i \cap A) = P(B_i)\,P(A/B_i) \longrightarrow (2)$$

Using Total probability theorem,

$$P(A) = \sum_{i=1}^{n} P(B_i)\,P(A/B_i) \longrightarrow (3)$$

Sub (2) & (3) in (1),

$$P(B_i/A) = \frac{P(B_i)\,P(A/B_i)}{\sum_{i=1}^{n} P(B_i)\,P(A/B_i)}$$

Hence the proof.

	Binomial	Poisson	Geometric		Uniform	Exponential	Normal
pdf	$P(x=x) = {}^nC_x p^x q^{n-x}$	$\frac{e^{-\lambda}\lambda^x}{x!}$, $x=0,1,2,\ldots,\infty$	$q^{x-1}p$, $x=1,2\ldots$	$q^x p$, $x=0,1,2\ldots$	$f(x)=\begin{cases}\frac{1}{b-a}, & a<x<b \\ 0, & \text{otherwise}\end{cases}$	$f(x)=\begin{cases}\lambda e^{-\lambda x}, & x>0 \\ 0, & \text{otherwise}\end{cases}$	$f(x)=\frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$
mgf	$q+pe^t$	$e^{\lambda(e^t-1)}$	$\frac{pe^t}{(1-qe^t)}$	$\frac{p}{(1-qe^t)}$	$\frac{e^{bt}-e^{at}}{(b-a)t}$	$\frac{\lambda}{\lambda-t}$; $\lambda>t$	
mean	np	λ	$\frac{1}{p}$	$\frac{q}{p}$	$\frac{a+b}{2}$	$\frac{1}{\lambda}$	μ
variance	npq	λ	$\frac{q}{p^2}$	$\frac{q}{p^2}$	$\frac{(b-a)^2}{12}$	$\frac{1}{\lambda^2}$	σ^2
S.D	$\sqrt{npq}$	$\sqrt{\lambda}$	Memory less property $P(X>s+t \mid X>s) = P(X>t)$		$\sqrt{\frac{(b-a)^2}{12}}$	$\sqrt{\frac{1}{\lambda^2}}$ Memory less property $P(X>t)=e^{-\lambda t}$	$\sqrt{\sigma^2}$

Discrete r.v

$E(x) = \sum x\, p(x)$

$E(x^2) = \sum x^2\, p(x)$

variance $E(x^2) - (E(x))^2$

S.D $\sqrt{\text{var}(x)}$

	Discrete r.v's	Continuous r.v's
$E(x)$	$\sum x\, p(x)$	$\int_{-\infty}^{\infty} x\, f(x)\, dx$
$E(x^2)$	$\sum x^2\, p(x)$	$\int_{-\infty}^{\infty} x^2\, f(x)\, dx$
Variance	$E(x^2) - (E(x))^2$	$E(x^2) - (E(x))^2$
S.D	$\sqrt{var(x)}$	$\sqrt{var(x)}$

Uniform Distribution
(or)
Rectangular "

Formula:

$$f(x) = \begin{cases} \frac{1}{b-a}, & a < x < b \\ 0, & \text{otherwise} \end{cases}$$

	Uniform
$M_X(t)$	$\frac{e^{bt} - e^{at}}{(b-a)t}$
Mean	$\frac{a+b}{2}$
Variance	$\frac{(b-a)^2}{12}$
S.D	$\sqrt{\frac{(b-a)^2}{12}}$

APR-06

① If X is a r.v with cdf as F(x). Show that the r.v $Y = F(x)$ is uniformly distributed in (

Soln

Given $Y = F(x)$ at $(0,1)$

$$g(y) = f(x)\left|\frac{dx}{dy}\right|$$

$$= f(x)\left|\frac{1}{\frac{dy}{dx}}\right|$$

$$= f(x)\left|\frac{1}{f(x)}\right|$$

$$= 1$$

$$g(y) = \begin{cases} 1, & 0 < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

NOV-06

② If X is uniformly distributed over (0,10) Calculate the probability that (a) $x > 6$ (b) $3 < x < 8$.

Soln: Given $f(x) = \frac{1}{b-a}, \quad a < x < b$

$= \frac{1}{10-0}, \quad 0 < x < 0$

$f(x) = \frac{1}{10}$

$= \int_a^b f(x)\,dx$

(a) $P[X>6] = \frac{1}{10}\int_6^{10} dx = \frac{1}{10}[x]_6^{10}$

$= \frac{1}{10}[10-6]$

$= \frac{4}{10} = \frac{2}{5}$

(b) $P[3<x<8] = \frac{1}{10}\int_3^8 dx = \frac{1}{10}[x]_3^8$

$= \frac{1}{10}[8-3]$

$= \frac{5}{10} = \frac{1}{2}$.

APR-10
APR-11

(3) X is uniformly distributed with mean 1 and variance $\frac{4}{3}$, find $P(X<0)$.

Soln:

Given Mean $= 1 \Rightarrow \frac{b+a}{2} = 1$

$b+a = 2 \rightarrow (1)$

variance $= 4/3 \Rightarrow \frac{(b-a)^2}{12} = \frac{4}{3}$

$(b-a)^2 = \frac{4\times 12}{3} 4$

$\sqrt{(b-a)^2} = \sqrt{4\times 4}$

$b-a = 4 \rightarrow (2)$

From (1) & (2)

$b+a = 2$

$b-a = 4$

$2b = 6$

$b = 6/2 = 3$

$\boxed{b=3}$

Sub $b=3$ in (1)

$b+a = 2$

$a = 2-b$

$a = 2-3$

$\boxed{a=-1}$

$f(x) = \frac{1}{b-a} = \frac{1}{3-(-1)} = \frac{1}{3+1} = \frac{1}{4}$.

$P(X<0) = \int_{-1}^{0} f(x)\,dx$

$= \int_{-1}^{0} \frac{1}{4}\,dx$

$= \frac{1}{4}\int_{-1}^{0} dx$

$= \frac{1}{4}[x]_{-1}^{0}$

$= \frac{1}{4}(0-(-1))$

$= 1/4$

(4) Show that (NOV-06) for the uniform distribution $f(x) = \frac{1}{2a}$, $-a < x < a$ the M.G.F about origin is $\frac{\sinh at}{at}$

Soln:-

Given $f(x) = \frac{1}{2a}$; $-a < x < a$

$\therefore M_X(t) = \int_a^b e^{tx} f(x)\, dx$

$= \int_{-a}^{a} e^{tx} \frac{1}{2a}\, dx$

$= \frac{1}{2a} \int_{-a}^{a} e^{tx}\, dx$

$= \frac{1}{2a} \left[\frac{e^{tx}}{t}\right]_{-a}^{a}$

$= \frac{1}{2at} \left[e^{at} - e^{-at}\right]$

$= \frac{1}{at} \left[\frac{e^{at} - e^{-at}}{2}\right]$

Sinhat: $\frac{e^{at} - e^{-at}}{2}$

$$\boxed{M_X(t) = \frac{1}{at} \sinh at}$$

Exponential Distribution

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & , \quad x \geq 0, \lambda > 0 \\ 0 & , \quad \text{otherwise} \end{cases}$$

Exponential

M.g.f	Mean	Variance	S.D
$\frac{\lambda}{\lambda - t}$; $\lambda > t$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\sqrt{\frac{1}{\lambda^2}}$

① Establish the memoryless property of Exponential Distribution.

Statement:

If X is E.D, then

$$P(X > s+t / X > s) = P(X > t), \forall\ s, t > 0.$$

Proof:

Unit-II

Correlation Analysis

Karl Pearson's Co-efficient of Correlation

$$r(x,y) \text{ (or) } r_{xy} = \frac{cov(x,y)}{\sigma_x \sigma_y}$$

$$\sigma_x = \sqrt{var(x)}$$

$$\sigma_y = \sqrt{var(y)}$$

$$\bar{x} = \frac{\Sigma x}{n} \; ; \; \bar{y} = \frac{\Sigma y}{n}$$

$$\sigma_x = \sqrt{\frac{1}{n}\Sigma x^2 - \bar{x}^2}$$

$$\sigma_y = \sqrt{\frac{1}{n}\Sigma y^2 - \bar{y}^2}$$

$$cov(x,y) = \frac{1}{n}\Sigma xy - \bar{x}\bar{y}$$

(or) Covariance

Problems

(1) Calculate the correlation co-efficient for the following heights (in inches) of fathers x & their sons Y.

x	65	66	67	67	68	69	70	72
y	67	68	65	68	72	72	69	71

Soln:

x	y	x^2	y^2	xy
65	67	4225	4489	4355
66	68	4356	4624	4488
67	65	4489	4225	4355
67	68	4489	4624	4556
68	72	4624	5184	4896
69	72	4761	5184	4968
70	69	4900	4761	4830
72	71	5184	5041	5112
$\Sigma x = 544$	$\Sigma y = 552$	$\Sigma x^2 = 37028$	$\Sigma y^2 = 38132$	$\Sigma xy = 37560$

$\bar{x} = \frac{\Sigma x}{n} = \frac{544}{8} = 68$

$\boxed{\bar{x} = 68}$

$\bar{y} = \frac{\Sigma y}{n} = \frac{552}{8} = 69$

$\boxed{\bar{y} = 69}$

~~$\bar{x}\bar{y} = 68 \times 69 = 4692$~~

~~$\boxed{\bar{x}\bar{y} = 4692}$~~

$\sigma_x = \sqrt{\frac{1}{n}\Sigma x^2 - \bar{x}^2}$

$= \sqrt{\frac{1}{8}(37028) - 68^2}$

$= \sqrt{4628.5 - 4624}$

$\boxed{\sigma_x = 2.121}$

$\sigma_y = \sqrt{\frac{1}{n}\Sigma y^2 - \bar{y}^2}$

$= \sqrt{\frac{1}{8}(38132) - 69^2}$

$= \sqrt{4766.5 - 4761}$

$\boxed{\sigma_y = 2.345}$

$Cov(x,y) = \frac{1}{n}\Sigma xy - \bar{x}\bar{y}$

$= \frac{1}{8}(37560) - 68 \times 69$

$= 4695 - 4692$

$\boxed{Cov(x,y) = 3}$

$r(x,y) = \frac{cov(x,y)}{\sigma_x \sigma_y}$

$= \frac{3}{(2.121)(2.345)}$

$= \frac{3}{4.973}$

$\boxed{r(x,y) = 0.6032}$

(2) Find the correlation coefficient for the following data

x	10	14	18	22	26	30
y	18	12	24	6	30	36

Soln :-

x	y	x^2	y^2	xy
10	18	100	324	180
14	12	196	144	168
18	24	324	576	432
22	b	484	36	132
26	30	676	900	780
30	36	900	1296	1080
120	126	2680	3276	2772

$$\bar{x} = \frac{\Sigma x}{n} = \frac{120}{6} = 20$$

$$\boxed{\bar{x} = 20}$$

$$\bar{y} = \frac{\Sigma y}{n} = \frac{126}{6} = 21$$

$$\boxed{\bar{y} = 21}$$

$$Cov(x,y) = \frac{\Sigma xy}{n} - \bar{x}\bar{y}$$

$$= \frac{2772}{6} - 20 \times 21$$

$$= 462 - 420$$

$$\boxed{Cov(x,y) = 42}$$

$$r(x,y) = \frac{Cov(x,y)}{\sigma_x \sigma_y}$$

$$= \frac{42}{46.667 \times 10.25}$$

$$= 0.5998$$

$$\boxed{r(x,y) = 0.60}$$

$$\sigma_x = \sqrt{\frac{\Sigma x^2}{n} - \bar{x}^2}$$

$$= \sqrt{\frac{2680}{6} - (20)^2}$$

$$= \sqrt{\frac{1340}{3} - 400}$$

$$\boxed{\sigma_x = 46.667}$$

$$\sigma_y = \sqrt{\frac{\Sigma y^2}{n} - \bar{y}^2}$$

$$= \sqrt{\frac{3276}{6} - (21)^2}$$

$$= \sqrt{546 - 441}$$

$$= \sqrt{105}$$

$$\boxed{\sigma_y = 10.25}$$

H.W

(1) Compute the coefficient of correlation between x & y using the following data

x	1	
y	8	

(2) Find the coefficient of correlation between the industrial production and the following data

x	55
y	35

H.W

(1) Compute the coefficient of correlation between X & Y using the following data.

X	1	3	5	7	8	10
Y	8	12	15	17	18	20

Ans 0.9878

(2) Find the coefficient of correlation between industrial production and export using the following data.

X	55	56	58	59	60	60	62
Y	35	38	37	39	44	43	44

Ans 0.8972

Rank correlation:-

$$r = 1 - \frac{6 \Sigma d_i^2}{n(n^2-1)} \quad ; \quad d_i = x_i - y_i$$

n = no. of items.

① Find the rank correlation coefficient from the following data.

X	1	2	3	4	5	6	7
Y	4	3	1	2	6	5	7

Soln:-

$n = 7$

X	Y	$d_i = x_i - y_i$	d_i^2
1	4	-3	9
2	3	-1	1
3	1	2	4
4	2	2	4
5	6	-1	1
6	5	1	1
7	7	0	0
		0	20

$$r = 1 - \frac{6 \Sigma d_i^2}{n(n^2-1)}$$

$$= 1 - \frac{6 \times 20}{7(49-1)}$$

$$r = 0.6429$$

$r = 0$ to 1

Regression

Coefficient of correlation:

$$r = \sqrt{b_{yx} \times b_{xy}}$$

$$\bar{x} = \frac{\Sigma x}{n} \quad ; \quad \bar{y} = \frac{\Sigma y}{n}$$

$$b_{yx} = \frac{\Sigma (x-\bar{x})(y-\bar{y})}{\Sigma (x-\bar{x})^2}$$

$$b_{xy} = \frac{\Sigma (x-\bar{x})(y-\bar{y})}{\Sigma (y-\bar{y})^2}$$

Line x on y :

$$x - \bar{x} = b_{xy}(y - \bar{y})$$

Line y on x :

$$y - \bar{y} = b_{yx}(x - \bar{x})$$

Problems

① obtain the equation of the lines of reg from the following data.

X	1	2	3	4	5	6	7
Y	9	8	10	12	11	13	14

Soln:

X	Y	$x-\bar{x}$ $x-4$	$y-\bar{y}$ $y-11$	$(x-\bar{x})^2$	$(y-\bar{y})^2$	$(x-\bar{x})(y-\bar{y})$
1	9	-3	-2	9	4	6
2	8	-2	-3	4	9	6
3	10	-1	-1	1	1	1
4	12	0	1	0	1	0
5	11	1	0	1	0	0
6	13	2	2	4	4	4
7	14	3	3	9	9	9
28	77	0	0	28	28	26

$\bar{x} = \frac{\Sigma x}{n} = \frac{28}{7} = 4$ | $\bar{y} = \frac{\Sigma y}{n} = \frac{77}{7} = 11$

$\bar{x} = 4$ | $\bar{y} = 11$

$$b_{yx} = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{\Sigma(x-\bar{x})^2} = \frac{26}{28} = 0.928$$

$b_{yx} = 0.928$

$$b_{xy} = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{\Sigma(y-\bar{y})^2} = \frac{26}{28} = 0.928$$

$b_{xy} = 0.928$

$$r = \sqrt{b_{yx} \times b_{xy}}$$

$$= \sqrt{0.928 \times 0.928}$$

$r = 0.8939$

Line x on y :-

$$x - \bar{x} = b_{xy}(y - \bar{y})$$

$$x - 4 = 0.928(y - 11)$$

$$x - 4 = 0.928y - 10.208$$

$$x - 4 - 0.928y + 10.208 = 0$$

$$x - 0.928y + 6.208 = 0$$

Line y on x :-

$$y - \bar{y} = b_{yx}(x - \bar{x})$$

$$y - 11 = 0.928(x - 4)$$

$$y - 11 = 0.928x - 3.712$$

$$y - 11 - 0.928x + 3.712 = 0$$

$$y - 0.928x - 7.288 = 0.$$

(2) Find the equation of the regression line y on x from the following data

X	3	5	6	8	9	11
Y	2	3	4	6	5	8

Soln:-

x	y	$x-\bar{x}$	$y-\bar{y}$	$(x-\bar{x})^2$	$(y-\bar{y})^2$	$(x-\bar{x})(y-\bar{y})$
3	2	-4	-3	16	9	12
5	3	-2	-2	4	4	4
6	4	-1	-1	1	1	1
8	6	1	1	1	1	1
9	5	2	0	4	0	0
11	8	4	3	16	9	12
42	28	0	0	42	24	30

$\bar{x} = \frac{\Sigma x}{n} = \frac{42}{6} = 7$

$\boxed{\bar{x} = 7}$

$\bar{y} = \frac{\Sigma y}{n} = \frac{28}{6} = 4.6 \approx 5$

$\boxed{\bar{y} = 5}$

$$b_{yx} = \frac{\Sigma (x-\bar{x})(y-\bar{y})}{\Sigma (x-\bar{x})^2}$$

$$= \frac{30}{42}$$

$$= \frac{5}{7}$$

$$\boxed{b_{yx} = 0.7}$$

Line y on x :-

$$y - \bar{y} = b_{yx}(x - \bar{x})$$

$$y - 5 = 0.7(x - 7)$$

$$y - 5 = 0.7x - 4.9$$

$$y - 5 - 0.7x + 4.9 = 0$$

$$y - 0.7x - 0.1 = 0$$

3) The two lin

$8x-$

$40x$

The vari

(i) The

(ii) Cor

Soln:- (i) Mea

Given $8\bar{x}$

$40\bar{x}$

(1) × 5 ⇒

(2) × 1 ⇒

Mean

(ii)

(A) ⇒ $8x = 10$

$x = \frac{10}{8}$

b_{xy}

(A) ⇒ $10y =$

$y =$

b_{yx}

(3) The two lines of regression are

$8x - 10y + 66 = 0 \longrightarrow (A)$

$40x - 18y - 214 = 0 \longrightarrow (B)$

The variance of x is 9. Find

(i) The mean values of x & y

(ii) Correlation coefficient between x & y.

Soln: (i) Mean...

Given $8\bar{x} - 10\bar{y} = -66 \longrightarrow (1)$

$40\bar{x} - 18\bar{y} = 214 \longrightarrow (2)$

$(1) \times 5 \Rightarrow 40\bar{x} - 50\bar{y} = -330$

$(2) \times 1 \Rightarrow 40\bar{x} - 18\bar{y} = 214$

$-32\bar{y} = -544$

$\bar{y} = \frac{-544}{-32}$

$\boxed{\bar{y} = 17}$.

$\bar{y} = 17$ in (1)

$8\bar{x} - 10\bar{y} = -66$

$8\bar{x} - 10(17) = -66$

$8\bar{x} = -66 + 170$

$8\bar{x} = 104$

$\bar{x} = \frac{104}{8} = 13$

$\boxed{\bar{x} = 13}$

Mean

$\bar{x} = 13, \bar{y} = 17.$

(ii) 1st x & y

2nd y & x

(A) $\Rightarrow 8x = 10y - 66$ $x = \frac{10}{8}y - \frac{66}{8}$ $\boxed{b_{xy} = \frac{10}{8}}$	(B) $\Rightarrow 18y = 40x - 214$ $y = \frac{40}{18}x - \frac{214}{18}$ $\boxed{b_{yx} = \frac{40}{18}}$	$r = \sqrt{b_{xy}\, b_{yx}}$ $= \sqrt{\left(\frac{10}{8}\right)\left(\frac{40}{18}\right)}$ $r = 1.66 \neq 1$
(A) $\Rightarrow 10y = 8x + 66$ $y = \frac{8}{10}x + \frac{66}{10}$ $\boxed{b_{yx} = \frac{8}{10}}$	(B) $\Rightarrow 40x = 18y + 214$ $x = \frac{18}{40}y + \frac{214}{40}$ $\boxed{b_{xy} = \frac{18}{40}}$	$r = \sqrt{b_{xy}\, b_{yx}}$ $= \sqrt{\frac{18}{10} \times \frac{8}{10}}$ $r = \pm 0.6$

$\therefore r = 0.6$

2.1) Joint Probability – marginal & Conditional distributions

(16 Marks)

Discrete Random Variables

① The joint probability mass function of (x,y) is given by $p(x,y) = k(2x+3y)$, $x = 0,1,2$; $y = 1,2,3$. Find all the marginal and conditional probability distribution. Also, find the probability distribution of $(x+y)$ & $P(x+y>3)$.

Soln:

$P(x,y) = k(2x+3y)$

$P(0,1) = k(0+3) = 3k = \frac{3}{72}$

$P(0,2) = k(0+6) = 6k = \frac{6}{72}$

$P(0,3) = k(0+9) = 9k = \frac{9}{72}$

$P(1,1) = k(2+3) = 5k = \frac{5}{72}$

$P(1,2) = k(2+6) = 8k = \frac{8}{72}$

$P(1,3) = k(2+9) = 11k = \frac{11}{72}$

$P(2,1) = k(4+3) = 7k = \frac{7}{72}$

$P(2,2) = k(4+6) = 10k = \frac{10}{72}$

$P(2,3) = k(4+9) = 13k = \frac{13}{72}$

$72k$

Total Probability $= 1$

$72k = 1$

$k = \frac{1}{72}$

Marginal Distribution

X \ Y	1	2	3	Total	$k = \frac{1}{72}$
0	3k	6k	9k	18k	$\frac{18}{72}$
1	5k	8k	11k	24k	$\frac{24}{72}$
2	7k	10k	13k	30k	$\frac{30}{72}$
Total	15k	24k	33k	72k	$\frac{72}{72} = 1$
$k = \frac{1}{72}$	$\frac{15}{72}$	$\frac{24}{72}$	$\frac{33}{72}$	$\frac{72}{72} = 1$	

(Y)

Marginal Distribution X & Y

$P(X=0) = \frac{18}{72}$, $P(X=1) = \frac{24}{72}$; $P(X=2) = \frac{30}{72}$

$P(Y=1) = \frac{15}{72}$; $P(Y=2) = \frac{24}{72}$; $P(Y=3) = \frac{33}{72}$.

Conditional Distribution :-

X	Y
$X=0$,	$Y=1$
$\frac{P(0,1)}{P(X=0)} = \frac{3}{72} \times \frac{72}{18} = \frac{1}{6}$	$\frac{P(0,1)}{P(Y=1)} = \frac{3}{72} \times \frac{72}{15} = \frac{1}{5}$
$\frac{P(0,2)}{P(X=0)} = \frac{6}{72} \times \frac{72}{18} = \frac{1}{3}$	$\frac{P(1,1)}{P(Y=1)} = \frac{5}{72} \times \frac{72}{15} = \frac{5}{15} = \frac{1}{3}$

$\frac{P(0,3)}{P(X=0)} = \frac{9}{72} \times \frac{72}{180} = \frac{1}{2}$

when $X=1$

$\frac{P(1,1)}{P(X=1)} = \frac{5}{72} \times \frac{72}{24} = \frac{5}{24}$

$\frac{P(1,2)}{P(X=1)} = \frac{8}{72} \times \frac{72}{24} = \frac{1}{3}$

$\frac{P(1,3)}{P(X=2)} = \frac{11}{72} \times \frac{72}{24} = \frac{11}{24}$

$X=2$

$\frac{P(2,1)}{P(X=2)} = \frac{7}{72} \times \frac{72}{30} = \frac{7}{30}$

$\frac{P(2,2)}{P(X=2)} = \frac{10}{72} \times \frac{72}{30} = \frac{1}{3}$

$\frac{P(2,3)}{P(X=2)} = \frac{13}{72} \times \frac{72}{30} = \frac{13}{30}$

$\frac{P(2,1)}{P(Y=1)} = \frac{7}{72} \times \frac{72}{15} = \frac{7}{15}$

$Y=2$

$\frac{P(0,2)}{P(Y=2)} = \frac{6}{72} \times \frac{72}{24} = \frac{1}{4}$

$\frac{P(1,2)}{P(Y=2)} = \frac{8}{72} \times \frac{72}{24} = \frac{1}{3}$

$\frac{P(2,2)}{P(Y=2)} = \frac{10}{72} \times \frac{72}{24} = \frac{5}{12}$

$Y=3$

$\frac{P(0,3)}{P(Y=3)} = \frac{9}{72} \times \frac{72}{33} = \frac{3}{11}$

$\frac{P(1,3)}{P(Y=3)} = \frac{11}{72} \times \frac{72}{33} = \frac{1}{3}$

$\frac{P(2,3)}{P(Y=3)} = \frac{13}{72} \times \frac{72}{33} = \frac{13}{33}$

Probability Distribution :-

X+Y	P
1 P(0,1)	$\frac{3}{72}$
2 P(0,2) + P(1,1)	$\frac{6}{72} + \frac{5}{72} = \frac{11}{72}$
3 P(0,3) + P(1,2) + P(2,1)	$\frac{9}{72} + \frac{8}{72} + \frac{7}{72} = \frac{24}{72}$
4 P(1,3) + P(2,2)	$\frac{11}{72} + \frac{10}{72} = \frac{21}{72}$
5 P(2,3)	$\frac{13}{72}$
Total	1

$P(X+Y \leq 3) = P(4) + P(5) = \frac{21}{72} + \frac{13}{72} = \frac{34}{72}$.

(8) The joint distribution of x & y is given by $f(x,y) = \frac{x+y}{21}$; $x = 1,2,3$, $y = 1,2$. Find the marginal distribution & conditional distributions

Soln:-

Marginal distribution :-

x \ y	1	2	3	Total
1	$\frac{2}{21}$	$\frac{3}{21}$	$\frac{4}{21}$	$\frac{9}{21}$
2	$\frac{3}{21}$	$\frac{4}{21}$	$\frac{5}{21}$	$\frac{12}{21}$
Total	$\frac{5}{21}$	$\frac{7}{21}$	$\frac{9}{21}$	1

$P(x,y)$ (or) $f(x,y) = \frac{x+y}{21}$

$f(1,1) = \frac{1+1}{21} = \frac{2}{21}$

$f(1,2) = \frac{1+2}{21} = \frac{3}{21}$

$f(2,1) = \frac{2+1}{21} = \frac{3}{21}$

$f(2,2) = \frac{2+2}{21} = \frac{4}{21}$

$f(3,1) = \frac{3+1}{21} = \frac{4}{21}$

$f(3,2) = \frac{3+2}{21} = \frac{5}{21}$

Marginal distribution x & y

$P(x=1) = \frac{5}{21}$; $P(x=2) = \frac{7}{21}$, $P(x=3) = \frac{9}{21}$

$P(y=1) = \frac{9}{21}$; $P(y=2) = \frac{12}{21}$.

Conditional Distribution :-

x	y

$\frac{P(1,1)}{P(x=1)} = \frac{2}{21} \times \frac{21}{5} = \frac{2}{5}$

$\frac{P(2,1)}{P(x=1)} = \frac{3}{21} \times \frac{21}{7} = \frac{3}{7}$

$\frac{P(3,1)}{P(x=2)} =$

$\frac{P(1,2)}{P(x=2)} =$

$\frac{P(3,1)}{P(x=3)} =$

$\frac{P(3,2)}{}$

X \ Y	1	2	Total
1	$\frac{2}{21}$	$\frac{3}{21}$	$\frac{5}{21}$
2	$\frac{3}{21}$	$\frac{4}{21}$	$\frac{7}{21}$
3	$\frac{4}{21}$	$\frac{5}{21}$	$\frac{9}{21}$
Total	$\frac{9}{21}$	$\frac{12}{21}$	1

Conditional distribution

X

X=1

$$\frac{P(1,1)}{P(X=1)} = \frac{2}{21} \times \frac{21}{5} = \frac{2}{5}$$

$$\frac{P(1,2)}{P(X=1)} = \frac{3}{21} \times \frac{21}{5} = \frac{3}{5}$$

X=2

$$\frac{P(2,1)}{P(X=2)} = \frac{3}{21} \times \frac{21}{7} = \frac{3}{7}$$

$$\frac{P(2,2)}{P(X=2)} = \frac{4}{21} \times \frac{21}{7} = \frac{4}{7}$$

X=3

$$\frac{P(3,1)}{P(X=3)} = \frac{4}{21} \times \frac{21}{9} = \frac{4}{9}$$

$$\frac{P(3,2)}{P(X=3)} = \frac{5}{21} \times \frac{21}{9} = \frac{5}{9}$$

Y

Y=1

$$\frac{P(1,1)}{P(Y=1)} = \frac{2}{21} \times \frac{21}{9} = \frac{2}{9}$$

$$\frac{P(2,1)}{P(Y=1)} = \frac{3}{21} \times \frac{21}{9} = \frac{1}{3}$$

$$\frac{P(3,1)}{P(Y=1)} = \frac{4}{21} \times \frac{21}{9} = \frac{4}{9}$$

Y=2

$$\frac{P(1,2)}{P(Y=2)} = \frac{3}{21} \times \frac{21}{12} = \frac{1}{4}$$

$$\frac{P(2,2)}{P(Y=2)} = \frac{4}{21} \times \frac{21}{12} = \frac{1}{3}$$

$$\frac{P(3,2)}{P(Y=2)} = \frac{5}{21} \times \frac{21}{12} = \frac{5}{12}$$

Continuous Random Variables

(1) The joint probability density function of a two-dimensional random variable (x,y) is given by $f(x,y) = xy^2 + \frac{x^2}{8}$; $0 \le x \le 2$, $0 \le y \le 1$. Compute (i) $P(X>1)$ (ii) $P(Y<\frac{1}{2})$ (iii) $P[X>1/Y<\frac{1}{2}]$ (iv) $P(X<Y)$.

Soln:

Given $f(x,y) = xy^2 + \frac{x^2}{8}$; $0 \le x \le 2$, $0 \le y$

Marginal Density of X:

$$f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^1 \left(xy^2 + \frac{x^2}{8}\right) dy$$

$$= \left[\frac{xy^3}{3} + \frac{x^2}{8}y\right]_0^1 = \left(\frac{x}{3} + \frac{x^2}{8} - 0\right)$$

$$\boxed{f(x) = \frac{x}{3} + \frac{x^2}{8}.}$$

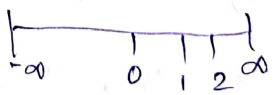
Marginal Density of Y:

$$f(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^2 \left(xy^2 + \frac{x^2}{8}\right) dx$$

$$= \left[\frac{x^2}{2}y^2 + \frac{x^3}{8\times 3}\right]_0^2 = \left[\frac{(2)^2}{2}y^2 + \frac{(2)^3}{8\times 3} - 0\right]$$

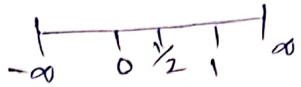
$$\boxed{f(y) = 2y^2 + \frac{1}{3}}$$

(i) $P(X>1) = \int_1^2 f(x)\,dx$



$$= \int_1^2 \left(\frac{x}{3} + \frac{x^2}{8}\right) dx = \left[\frac{x^2}{6} + \frac{x^3}{24}\right]_1^2 = \frac{4}{6} + \frac{8}{24} - \frac{1}{6} - \frac{1}{24}$$

$$= \frac{3}{6} + \frac{7}{24} = \frac{12+7}{24} = \frac{19}{24}.$$



② $P[y < \frac{1}{2}] = \int_0^{1/2} f(y)\,dy$

$= \int_0^{1/2} \left(2y^2 + \frac{1}{3}\right) dy = \left[\frac{2y^3}{3} + \frac{1}{3}y\right]_0^{1/2}$

$= \left[\frac{2}{3}\left(\frac{1}{8}\right) + \frac{1}{3}\left(\frac{1}{2}\right) - 0\right]$

$= \frac{1}{12} + \frac{1}{6} = \frac{1+2}{12} = \frac{3}{12} = \frac{1}{4}$.

③ $P[X>1 / Y<1/2] = \dfrac{\int_{x=1}^{x=2}\int_{y=0}^{y=1/2}\left(xy^2 + \frac{x^2}{8}\right) dy\,dx}{\left(\frac{1}{4}\right)}$

$= 4\int_1^2 \left[\frac{xy^3}{3} + \frac{x^2}{8}y\right]_0^{1/2} dx$

$= 4\int_1^2 \left[\frac{x\left(\frac{1}{2}\right)^3}{3} + \frac{x^2\left(\frac{1}{2}\right)}{8}\right] dx$

$= 4\int_1^2 \left(\frac{x}{24} + \frac{x^2}{16}\right) dx$

$= 4\left[\frac{x^2}{48} + \frac{x^3}{48}\right]_1^2$

$= \frac{4^1}{48_{12}}\left(x^2 + x^3\right)_1^2$

$= \frac{1}{12}\left((2^2+2^3) - (1^2+1^3)\right)$

$= \frac{1}{12}(4+8-2)$

$= \frac{1}{12_6}(6)^5$

$= \frac{5}{6}$.

④ $P[X<Y] = \int_{y=0}^{y=1}\int_{x=0}^{x=y}\left(xy^2 + \frac{x^2}{8}\right) dx\,dy$.

$= \int_0^1 \left[\frac{x^2y^2}{2} + \frac{x^3}{24}\right]_{x=0}^{x=y} dy$

$= \int_0^1 \left(\frac{y^2y^2}{2} + \frac{y^3}{24}\right) dy$

$= \left[\frac{y^5}{10} + \frac{y^4}{96}\right]_0^1 = \frac{1}{10} + \frac{1}{96} = \frac{96+10}{960} = \frac{106}{960}^{53}_{480}$

$= \frac{53}{480}$

(2) Determine the value of c that makes the function $F(x,y) = c(x+y)$ a joint probability density function over the range $0 < x < 3$ & $x < y < x+2$. Also Determine the following

(i) $P(X<1, Y<2)$

(ii) $P(Y \geq 2)$

(iii) $E(x)$.

Soln:-

Given $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(x,y)\,dx\,dy = 1$

$$c\int_0^3\int_x^{x+2}(x+y)\,dy\,dx = 1$$

$$c\int_0^3 \left[xy + \frac{y^2}{2}\right]_x^{x+2} dx = 1$$

$$c\int_0^3 \left(x(x+2) + \frac{(x+2)^2}{2} - \left(x^2 + \frac{x^2}{2}\right)\right) dx = 1$$

$$= x^2 + 2x + \frac{1}{2}(x^2+4+4x) - x^2 - \frac{x^2}{2}$$

$$= 2x + \frac{1}{2}(4) + \frac{4x}{2}$$

$$= 4x + 2$$

$$c\int_0^3 (4x+2)\,dx = 1$$

$$c\left(\frac{4x^2}{2} + 2x\right)_0^3 = 1$$

$$c\left(\frac{2\times 9}{2} + 2\times 3\right) = 1$$

$$c(18+6) = 1$$

$$c(24) = 1$$

$$\boxed{c = \frac{1}{24}}$$

(i) $P(X<1;\ Y<2) = \int_0^1\int_{y=x}^{y=2} \frac{1}{24}(x+y)\,dy\,dx$

$$= \frac{1}{24}\int_0^1 \left(2x + \frac{2^2}{2} - \left(x^2 + \frac{x^2}{2}\right)\right) dx$$

$$= \frac{1}{24}\int_0^1 \left(2x + 2 - \frac{3x^2}{2}\right) dx$$

$$= \frac{1}{24}\left[\frac{2x^2}{2} + 2x - \frac{3x^3}{6}\right]_0^1$$

$$= \frac{1}{24}\left(1^2 + 2(1) - \frac{3(1)^3}{6}\right)$$

$$= \frac{1}{24}\left(3 - \frac{3}{6}\right)$$

$$= \frac{1}{24}\left(\frac{18-3}{6}\right)$$

$$= \frac{1}{24}\left(\frac{15}{6}\right)$$

$$= \frac{5}{48}.$$

$$= \frac{1}{24}\int_0^1 \left[xy + \frac{y^2}{2}\right]_{y=x}^{y=2} dx$$

$$= \frac{1}{24}\int_0^1 \left(2x + 2 - x^2 - \frac{x^2}{2}\right) dx$$

$$= \frac{1}{24}\left[\frac{2x^2}{2} + 2x - \frac{x^3}{6}\right]_0^1$$

$$= \frac{1}{24}\left(1 + 2 - \frac{1}{6}\right)$$

$$= \frac{1}{24}\left(\frac{7}{6}\right)$$

(3) The joint PDF of the random variables x & y is defined as

$$f(x,y) = \begin{cases} 25e^{-5y} & ; 0<x<0.2 ; y>0 \\ 0 & ; \text{otherwise} \end{cases}$$

(i) Find the marginal pdf's of x & y.

(ii) What is the covariance of x & y?

(iii) What is the correlation coefficient of x & y?

(iv) Are x & y are independent.

Soln:

(i) Marginal Density of x & y.

$$f(x) = \int_0^\infty 25e^{-5y}\,dy$$

$$= 25\int_0^\infty e^{-5y}\,dy$$

$$= 25\left[\frac{e^{-5y}}{-5}\right]_0^\infty$$

$$= -5\left[e^{-5y}\right]_0^\infty$$

$e^\infty = 0$, $e^0 = 1$

$$= -5\left(e^\infty - e^0\right)$$

$$= -5(0-1)$$

$$f(x) = 5$$

$$f(y) = \int_0^{0.2} 25e^{-5y}\,dx$$

$$= 25e^{-5y}\int_0^{0.2} dx$$

$$= 25e^{-5y}\left[x\right]_0^{0.2}$$

$$= 25e^{-5y}(0.2-0)$$

$$= 25e^{-5y}(0.2)$$

$$f(y) = 5e^{-5y}$$

(ii) Covariance of x & y.

Formula:

$$Cov(x,y) = E(xy) - E(x)E(y)$$

$$E(x) = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$= \int_0^{0.2} x(5)\,dx$$

$$= 5\int_0^{0.2} x\,dx$$

$$= 5\left[\frac{x^2}{2}\right]_0^{0.2}$$

$$= \frac{5}{2}(0.2)^2$$

$$= \frac{5}{2}(0.04)$$

$$E(x) = 0.1$$

$$E(y) = \int_{-\infty}^{\infty} y f(y)\,dy$$

$$= \int_0^\infty y\left(5e^{-5y}\right)dy$$

$$= 5\int_0^\infty \underset{u}{y}\,\underset{v}{e^{-5y}}\,dy$$

$$= 5\left[y\left(\frac{e^{-5y}}{-5}\right) - (1)\frac{e^{-5y}}{25}\right]_0^\infty$$

$$= 5\left[(0-0) - \left(0 - \frac{e^0}{25}\right)\right]$$

$$= 5\left(\frac{1}{25}\right) = \frac{1}{5}$$

$$E(y) = \frac{1}{5}$$

$$E(xy) = \int_0^{\infty}\int_0^{\infty} xy\, f(x,y)\, dx\, dy$$

$$= \int_0^{\infty}\int_0^{0.2} xy\,(25e^{-5y})\, dx\, dy$$

$$= 25\int_0^{\infty}\int_0^{0.2} xy\, e^{-5y}\, dx\, dy$$

$$= 25\int_0^{\infty} y e^{-5y} dy \int_0^{0.2} x\, dx$$

$$\int_0^{\infty} y e^{-5y} dy = \frac{1}{25}$$

$$= 25 \left(\frac{1}{25}\right)\left(\frac{x^2}{2}\right)_0^{0.2}$$

$$= \frac{0.2^2}{2} = \frac{0.04}{2} = 0.02$$

$$\boxed{E(xy) = 0.02}$$

$$Cov(x,y) = E(xy) - E(x)E(y)$$

$$= 0.02 - (0.1)\left(\frac{1}{5}\right)$$

$$= 0.02 - 0.02$$

$$\boxed{Cov(x,y) = 0}$$

(iii) Correlation coefficient of x & y

Formula

$$r(x,y) = \frac{Cov(x,y)}{\sigma_x \sigma_y} = 0$$

(iv) Independent:

$$f(x)\cdot f(y) = f(x,y) \Rightarrow 5e^{-5y} \times 5e^{-5y} = 25e^{-5y}$$

$$25e^{-5y} = 25e^{-5y}.$$

x & y are independent.

(4) Given the joint p.d.f of (x,y) as

$$f(x,y) = \begin{cases} 8xy, & 0<x<y<1 \\ 0, & \text{otherwise} \end{cases}$$

Find the marginal & conditional p.d.f of x & y. Are x & y are independent?

Soln:

Given $f(x,y) = \begin{cases} 8xy, & 0<x<y<1 \\ 0, & \text{otherwise} \end{cases}$

Marginal Density function of x & y:

$$f(x) = \int_{-\infty}^{\infty} f(x,y)\, dy = \int_x^1 8xy\, dy = 4x\left[\frac{y^2}{x^2}\right]_x^1$$

$$= 4x(1-x^2)$$

$$f(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

$$= \int_{0}^{y} 8xy\,dx$$

$$= 4y\left[\frac{x^2}{2}\right]_0^y$$

$$= 4y(y^2)$$

$$\boxed{f(y) = 4y^3}$$

Conditional distribution

$$f(x/y) = \frac{f(x,y)}{f(y)} = \frac{8xy}{4y^3} = \frac{2x}{y^2}.$$

$$f(y/x) = \frac{f(x,y)}{f(x)} = \frac{8xy}{4x(1-x^2)} = \frac{2y}{1-x^2}.$$

Independent

$f(x)\cdot f(y) = f(x,y) \rightarrow$ indepen

$f(x)\cdot f(y) \neq f(x,y) \rightarrow$ NOT indepen

$$f(x)\,f(y) = 4x(1-x^2)\,4y^3$$

$$= 16xy^3(1-x^2)$$

$$\boxed{f(x)\,f(y) \neq f(x,y)}$$

$\therefore$ x & y are not independent.

---

(5) Find k if the joint p.d.f of a bivariat r.v. (x,y) is given by

$$f(x,y) = \begin{cases} k(1-x)(1-y) & ; \text{if } 0<x<1,\ 0<y \\ 0 & ; \text{otherwise} \end{cases}$$

Soln:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$$

$$\int_0^1 \int_0^1 k\,(1-x)(1-y)\,dx\,dy = 1$$

$$k \int_0^1 (1-x)\,dx \int_0^1 (1-y)\,dy = 1$$

$$k \left[x - \frac{x^2}{2}\right]_0^1 \left[y - \frac{y^2}{2}\right]_0^1 = 1$$

$$k \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{2}\right) = 1$$

$$k \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = 1$$

$$\boxed{k = 4}$$

Unit - III

Random processes

Markov process (or) Markov chain :-

① If the transition probability matrix (TPM) of a markov chain is $\begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$. find the Steady state distribution of the chain.

Soln:-

Let $P = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

Stationary State distribution :-

Formula :-

$$\pi = [\pi_1, \pi_2]$$

$$a = 1; \quad b = \frac{1}{2}$$

$$a+b = 1 + \frac{1}{2} = \frac{2+1}{2} = \frac{3}{2}$$

$$\boxed{a+b = \frac{3}{2}}$$

$$\pi = \begin{bmatrix} \frac{b}{a+b} & \frac{a}{a+b} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\frac{1}{2}}{\frac{3}{2}} & \frac{1}{\frac{3}{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} \times \frac{2}{3} & \frac{2}{3} \end{bmatrix}$$

$$\boxed{\pi = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \end{bmatrix}}$$

$$P = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}$$

$$\pi = \begin{bmatrix} \frac{b}{a+b} & \frac{a}{a+b} \end{bmatrix}$$

② Consider a markov chain with two states and tpm $\begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$. Find the stationary distribution of the chain. Find $\lim_{n \to \infty} p^n$

Soln:-

Let $P = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

Stationary distribution

$$\pi = [\pi_1 \quad \pi_2]$$

$$a = \frac{1}{4} \quad ; \quad b = \frac{1}{2}$$

$$a+b = \frac{1}{4} + \frac{1}{2} = \frac{2+4}{8} = \frac{6}{8} = \frac{3}{4}$$

$$\boxed{a+b = \frac{3}{4}}$$

$$\pi = \begin{bmatrix} \frac{b}{a+b} & \frac{a}{a+b} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1/2}{3/4} & \frac{1/4}{3/4} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} \times \frac{4}{3} & \frac{1}{4} \times \frac{4}{3} \end{bmatrix}$$

$$\boxed{\pi = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \end{bmatrix}}$$

formula :

$$\lim_{n \to \infty} P^n = \begin{bmatrix} \frac{b}{a+b} & \frac{a}{a+b} \\ \frac{b}{a+b} & \frac{a}{a+b} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1/2}{3/4} & \frac{1/4}{3/4} \\ \frac{1/2}{3/4} & \frac{1/4}{3/4} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} \times \frac{4}{3} & \frac{1}{4} \times \frac{4}{3} \\ \frac{1}{2} \times \frac{4}{3} & \frac{1}{4} \times \frac{4}{3} \end{bmatrix}$$

$$\boxed{\lim_{n \to \infty} P^n = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}}$$

③ Consider a markov chain with 3 states & tpm $P = \begin{bmatrix} 0.6 & 0.2 & 0.2 \\ 0.1 & 0.8 & 0.1 \\ 0.6 & 0 & 0.4 \end{bmatrix}$. Find the Stationary Probabilities of the chain.

Soln: Let $P = (*)$

Let $\pi = (\pi_1\ \pi_2\ \pi_3)$

$$\pi P = \pi \rightarrow (1)$$

$$\pi_1 + \pi_2 + \pi_3 = 1 \rightarrow (2)$$

$$\pi_3 = 1 - \pi_1 - \pi_2 \rightarrow (3)$$

(1) $\Rightarrow \pi P = \pi$

$$(\pi_1\ \pi_2\ \pi_3)\begin{bmatrix} 0.6 & 0.2 & 0.2 \\ 0.1 & 0.8 & 0.1 \\ 0.6 & 0 & 0.4 \end{bmatrix} = (\pi_1\ \pi_2\ \pi_3)$$

$1 - 0.6 = 0.4$

$$0.6\pi_1 + 0.1\pi_2 + 0.6\pi_3 = \pi_1 \quad (3)$$

$$\pi_1 - 0.6\pi_1 - 0.1\pi_2 - 0.6\pi_3 = 0$$

$$0.4\pi_1 - 0.1\pi_2 - 0.6\pi_3 = 0 \rightarrow (3)$$

$1 - 0.8 = 0.2$

$$0.2\pi_1 + 0.8\pi_2 = \pi_2 \quad (4)$$

$$\pi_2 - 0.2\pi_1 - 0.8\pi_2 = 0$$

$$0.2\pi_1 + 0.2\pi_2 = 0 \rightarrow (4)$$

$1 - 0.4 = 0.6$

$$0.2\pi_1 + 0.1\pi_2 + 0.4\pi_3 = \pi_3 \quad (5)$$

$$\pi_3 - 0.2\pi_1 - 0.1\pi_2 - 0.4\pi_3 = 0$$

$$0.2\pi_1 - 0.1\pi_2 + 0.6\pi_3 = 0 \rightarrow (5)$$

Next page

From (3) & (5)

(3) $\times 1 \Rightarrow 0.4\pi_1 - 0.1\pi_2 + 0.6\pi_3 = 0$

(5) $\times 2 \Rightarrow 0.4\pi_1 - 0.2\pi_2 + 0.2\pi_3 = 0$

$$0.1\pi_2 - 0.18\pi_3 = 0 \rightarrow (6)$$

From (6) & (5)

$$0.2\pi_1 + 0.2\pi_2 = 0$$

$$0.6\pi_1 - 0.1\pi_2 = 0$$

$$0.1\pi_2 = 0.18\pi_3$$

$$\pi_3 = \frac{0.1}{0.18}\pi_2$$

$$\pi_3 = 0.55\pi_2$$

From (4)

$$0.2\pi_1 + 0.2\pi_2 = 0$$

$$0.2\pi_1 = -0.2\pi_2$$

$$\pi_1 = -\frac{0.2}{0.2}\pi_2$$

$$\pi_1 = -\pi_2$$

$\pi_2 + \pi_2 + 0.05\,\pi_2 = 1$

$0.05\,\pi_2 = 1$

$\pi_2 = \frac{1}{0.05} = 20$

$\boxed{\pi_2 = 20}$

Calculator use ② & ③ & ⑤

$\pi_1 = \frac{2}{5}$; $\pi_2 = 2/5$; $\pi_3 = \frac{1}{5}$

$\pi = (\pi_1\ \pi_2\ \pi_3)$

$= \left(\frac{2}{5}\ \frac{2}{5}\ \frac{1}{5}\right)$

④ Find the Limiting - State Probabilities associated with the following tpm

$$\begin{bmatrix} 0.4 & 0.5 & 0.1 \\ 0.3 & 0.3 & 0.4 \\ 0.3 & 0.2 & 0.5 \end{bmatrix}$$

(Markov chain ① – (iv) irreducible)

Soln:

Let $P = \otimes$

Let $\pi = (\pi_1\ \pi_2\ \pi_3)$

$\pi P = \pi \longrightarrow ①$

$\pi_1 + \pi_2 + \pi_3 = 1 \longrightarrow ②$

① $\Rightarrow \pi P = \pi$

$$(\pi_1\ \pi_2\ \pi_3)\begin{bmatrix} 0.4 & 0.5 & 0.1 \\ 0.3 & 0.3 & 0.4 \\ 0.3 & 0.2 & 0.5 \end{bmatrix} = (\pi_1\ \pi_2\ \pi_3)$$

$1 + 0.4 = 1.4$; $1 - 0.6$

$0.4\pi_1 + 0.3\pi_2 + 0.3\pi_3 = \pi_1$

$0.4\pi_1 + 0.3\pi_2 + 0.3\pi_3 - \pi_1 = 0$

$-0.6\pi_1 + 0.3\pi_2 + 0.3\pi_3 = 0 \longrightarrow ③$

$0.5\pi_1 + 0.3\pi_2 + 0.2\pi_3 = \pi_2$

$0.5\pi_1 + 0.3\pi_2 + 0.2\pi_3 - \pi_2 = 0$

$0.5\pi_1 - 0.7\pi_2 + 0.2\pi_3 = 0 \longrightarrow ④$

$0.1\pi_1 + 0.4\pi_2 + 0.5\pi_3 = \pi_3 \longrightarrow ⑤$

Calculator use

② & ④ & ⑤ $\Rightarrow \pi_1 = \frac{1}{3}$; $\pi_2 = \frac{1}{3}$; $\pi_3 = \frac{1}{3}$.

(5) The transition probability matrix of the Markov Chain $\{X_n\}$, $n = 1, 2, 3 \ldots$ having 3 states 1, 2 & 3 is

$$P = \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix}$$ & the initial distribution is

$P^{(0)} = (0.7 \quad 0.2 \quad 0.1)$. Find (i) $P(X_2 = 3)$

(ii) $P[X_3 = 2, X_2 = 3, X_1 = 3, X_0 = 2]$

Soln:

Given $P = \begin{matrix} \\ 1 \\ 2 \\ 3 \end{matrix} \begin{matrix} \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix} \end{matrix}$.

Given $P^{(0)} = (0.7 \quad 0.2 \quad 0.1)$

$P[X_0 = 1] = 0.7$, $P(X_0 = 2) = 0.2$ $P(X_0 = 3) = 0.1$

$P^{(1)} = P^{(0)} P$

$= (0.7 \quad 0.2 \quad 0.1) \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix}$

$= (0.07 + 0.12 + 0.03 \quad 0.35 + 0.04 + 0.04 \quad 0.28 + 0.04 + 0.03)$

$P^{(1)} = (0.22 \quad 0.43 \quad 0.35)$

$P(X_1 = 1) = 0.22$; $P(X_1 = 2) = 0.43$; $P(X_1 = 3) = 0.35$

$P^{(2)} = P^{(1)} P$

$= (0.22 \quad 0.43 \quad 0.35) \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix}$.

$= (0.022 + 0.258 + 0.105 \quad 0.11 + 0.086 + 0.14 \quad 0.088 + 0.088 + 0.105)$

$P^{(2)} = (0.385 \quad 0.336 \quad 0.279)$

$P(X_2 = 1) = 0.385$; $P(X_2 = 2) = 0.336$; $P(X_2 = 3) = 0.279$

(i) $P(X_2=3) = 0.279$

(ii) $P[X_3=2, X_2=3, X_1=3, X_0=2]$.

$= P_{32}^1 \; P_{33}^1 \; P_{23}^1 \; P(X_0=2)$

$= (0.4)(0.3)(0.2)(0.2)$

$= 0.0048$.

$P[X_3=2, X_2=3, X_1=3, X_0=2]$ — P_{32}, P_{33}, P_{23}; $3-2=1$, $2-1=1$, $1-0=1$

---

(3) Ans

Sub (3) in (4)

$0.6\pi_1 + 0.1\pi_2 + 0.6(1-\pi_1-\pi_2) = \pi_1$

$0.6\pi_1 + 0.1\pi_2 + 0.6 - 0.6\pi_1 - 0.6\pi_2 = \pi_1$

$-\pi_1 = 0$

$-0.5\pi_2 = -0.6$

$\pi_1 + 0.5\pi_2 = 0.6$ → (7)

(4) $\Rightarrow 0.2\pi_1 + 0.8\pi_2 - \pi_2 = 0$

(2) $\times 0.2 \Rightarrow \pi_1 - 0.2\pi_2 = 0$ → (8)

$0.2\pi_1 + 0.1\pi_2 = 0.12$

(7) $\times 1 \Rightarrow$

$0.2\pi_1 - 0.2\pi_2 = 0$

(+) (+) (−)

$0.3\pi_2 = 0.12$

$\pi_2 = \frac{0.12}{0.3} \times \frac{100}{100}$

$= \frac{12}{30} = 2/5$

$\pi_2 = 2/5$

(8) $\Rightarrow$

$0.2\pi_1 - 0.2\pi_2 = 0$

$0.2\pi_1 - 0.2(2/5) = 0$

$0.2\pi_1 - 0.4 = 0$

$0.2\pi_1 = 0.4$

$\pi_1 = \frac{0.4}{0.2 \times 5}$

$\pi_1 = \frac{0.4}{5}$

$\pi_1 - \pi_2 = 0$

$\pi_1 = \pi_2$

$\pi_1 = 2/5$

$\pi_3 = 1 - \pi_1 - \pi_2$

$= 1 - 2/5 - 2/5$

$= 1 - 4/5$

$= \frac{5-4}{5} = 1/5$

$\pi_3 = 1/5$

(4) $\Rightarrow 0.6\pi_1 + 0.1\pi_2 + 0.6\pi_3 = \pi_1$

$0.6\pi_1 + 0.1\pi_2 + 0.6\pi_3 - \pi_1 = 0$

$-0.4\pi_1 + 0.1\pi_2 + 0.6\pi_3 = 0$

$-0.4 \times (2/5) + 0.1(2/5) + 0.6\pi_3 = 0$

$-0.3 \times 2/5 + 0.6\pi_3 = 0$

$-\frac{0.6}{5} + 0.6\pi_3 = 0$

$0.6\pi_3 = \frac{0.6}{5}$

$\pi_3 = \frac{0.6}{5} \times 0.6$

$\pi_3 = 1/5$

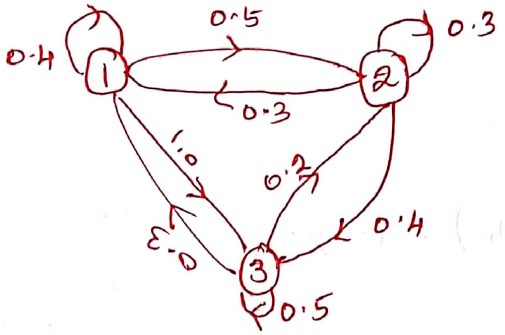
$\pi = \pi_1 \pi_2 \pi_3$

$\pi = [2/5 \;\; 2/5 \;\; 1/5]$ (X).

Markov Chain

16 marks

Find the transition probability matrix & the limiting state probabilities of the process represented by the state transition diagram. Show in the below Diagra



Sol: Transition Diagra

	1	2	3
1	0.4	0.5	0.1
2	0.3	0.3	0.4
3	0.3	0.2	0.5

(i) Given $X_n = \{1, 2, 3\}$ ⇒ finite

(ii) Irreducible :-

$1 \to 1$

$1 \to 2 \to 1$

$1 \to 2 \to 3 \to 1$

All the states are communicate with each oth

It is irreducible.

(iii) aperiodic :- ⇒ Atleast one Diagonal element > 0

$P_{11} = 0.4 > 0$

$P_{22} = 0.3 > 0$

$P_{33} = 0.5 > 0$

The states are aperiodic

(iv) persistent and non-null Persistent :-

$X_n = \{1, 2, 3\}$ ⇒ finite

It is irreducible

All the states are persistent & non- null Pers

(v) Ergodic State :-

A non-null persistent & aperiodic state is called Ergodic state.

∴ 1 & 2 & 3 are Ergodic state.

(vii) The limiting state probabilities exist & are given by ~~$\pi_1 = \pi_2 = \pi_3 = \frac{1}{3}$~~

$$\pi P = \pi \longrightarrow (1)$$

$$\pi_1 + \pi_2 + \pi_3 = 1 \longrightarrow (2)$$

$$\pi_3 = 1 - \pi_1 - \pi_2 \longrightarrow (3)$$

$$\pi P = \pi$$

$$(\pi_1\ \pi_2\ \pi_3)\begin{bmatrix} 0.4 & 0.5 & 0.1 \\ 0.3 & 0.3 & 0.4 \\ 0.3 & 0.2 & 0.5 \end{bmatrix} = [\pi_1\ \pi_2\ \pi_3]$$

$$0.4\pi_1 + 0.3\pi_2 + 0.3\pi_3 = \pi_1 \longrightarrow (4)$$

$$0.5\pi_1 + 0.3\pi_2 + 0.2\pi_3 = \pi_2 \longrightarrow (5)$$

$$0.1\pi_1 + 0.4\pi_2 + 0.5\pi_3 = \pi_3 \longrightarrow (6)$$

Sub (3) in (4)

$$0.4\pi_1 + 0.3\pi_2 + 0.3(1 - \pi_1 - \pi_2) = \pi_1$$

$$0.4\pi_1 + 0.3\pi_2 + 0.3 - 0.3\pi_1 - 0.3\pi_2 = \pi_1$$

$0.4 - 0.3 = 0.1$

$$0.1\pi_1 - \pi_1 = -0.3$$

$$-0.9\pi_1 = -0.3$$

$$\pi_1 = \frac{-0.3}{-0.9} = \frac{1}{3}$$

$$\boxed{\pi_1 = \frac{1}{3}}$$

-1.2, 0.3, 0.9

$-1 - 0.2 = -1.2$

Sub (3) in (5)

$$0.5\pi_1 + 0.3\pi_2 + 0.2(1 - \pi_1 - \pi_2) = \pi_2$$

$$0.5\pi_1 + 0.3\pi_2 + 0.2 - 0.2\pi_1 - 0.2\pi_2 - \pi_2 = 0$$

$$0.3\pi_1 - 0.9\pi_2 = -0.2$$

$$0.3\left(\frac{1}{3}\right) - 0.9\pi_2 = -0.2$$

$$0.1 - 0.9\pi_2 = -0.2$$

$$-0.9\pi_2 = -0.2 - 0.1$$

$$-0.9\pi_2 = -0.3$$

$$\pi_2 = \frac{-0.3}{-0.9} = \frac{1}{3}$$

0.3×10, 0.9×10, $3/9 = 1/3$

$$\boxed{\pi_2 = \frac{1}{3}}$$

$$\pi_1 + \pi_2 + \pi_3 = 1$$

$$\frac{1}{3} + \frac{1}{3} + \pi_3 = 1$$

$$\frac{2}{3} + \pi_3 = 1$$

$$\pi_3 = 1 - \frac{2}{3}$$

$$= \frac{3-2}{3}$$

$$\boxed{\pi_3 = \frac{1}{3}}$$

The steady state probability distribution is

$$\pi = (\pi_1 \ \pi_2 \ \pi_3)$$

$$\pi = \left(\frac{1}{3} \ \frac{1}{3} \ \frac{1}{3}\right)$$

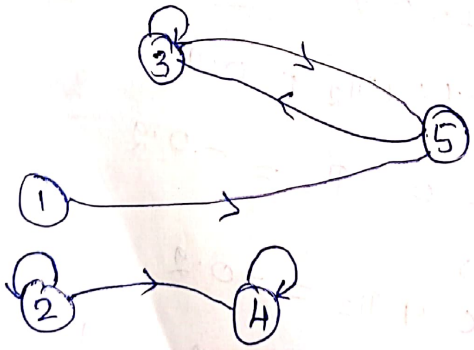
---

(2) The following is the transition probability matrix of a markov chain with state space [1, 2, 3, 4, 5]. Specify the classes and determine which classes are transient and which are recurrent

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & \frac{1}{3} & 0 & 2/3 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2/5 & 0 & 3/5 \end{bmatrix} \end{array}$$

Soln:-

Transition Diagram



(i) Given $X_n = \{1, 3, 5\}$ & $\{2, 4\}$ ⇒ finite.

(ii) Not Irreducible:-

State (1) & state (2) are not communicate with each other.

(iii) Aperiodic ⇒ Atleast one diagonal element > 0

$P_{44} = 1 > 0$

$P_{55} = 2/5 > 0$

∴ The states are aperiodic.

(iv) Not non-null persistent :-

$X_n = \{1, 3, 5\} = \{2, 4\} = \text{finite}$

It is not irreducible

The states are not non-null Persistent.

(v) Not Ergodic State :-

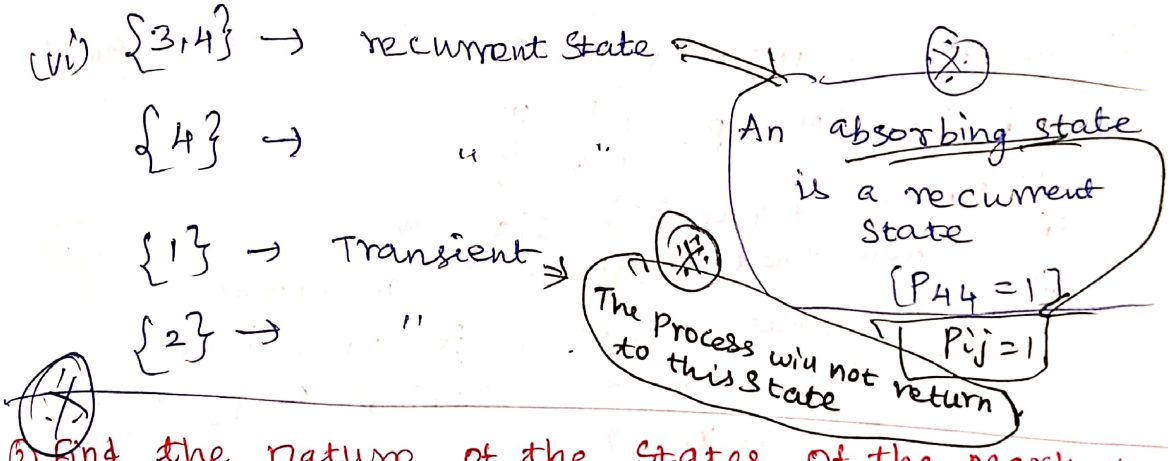
③ & ④ are not Ergodic states.

(vi) $\{3, 4\} \rightarrow$ recurrent state

$\{4\} \rightarrow$ " "

$\{1\} \rightarrow$ Transient ⇒

$\{2\} \rightarrow$ "



③ Find the nature of the States of the Markov chain with the tpm.

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

Soln:-

(i) Given $X_n = \{0, 1, 2\} \Rightarrow$ finite

(ii) Irreducible :-

All the states are 0-1-2, 0-1-1, 0-1-2-1-0 communicate with each other.

It is irreducible.

(iii) Non-null persistent :

$X_n = \{0, 1, 2\} \Rightarrow$ finite

It is irreducible.

All the states are non-null persistent.

(iv) Not aperiodic :

The state are not aperiodic

State 0 = 0 → 1 → 0 (2 steps)
" 1 = 1 → 0 → 1 (")
" 1 = 1 → 2 → 1 (")
" 2 = 2 → 1 → 2 (")

Period = 2

(v) Periodic.

(x) All the Diagonal elements = 0.

Period = 2.

(vi) Not Ergodic :

From (3) & (4) the states are not Ergodic.

---

(4) Consider a markov Chain with state space {0,

and the tpm $P = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

(i) Draw a transition Diagram.
(ii) Show that state 0 is recurrent.
(iii) Show that state 1 is transient.
(iv) Is the state 1 Periodic ? If so, what is the period ?
(v) Is the chain irreducible ?
(vi) Is the Chain Ergodic ? Explain.

Soln :

Given $P = \begin{matrix} 0 \\ 1 \end{matrix} \begin{matrix} \overset{0}{\overset{\downarrow}{}} & \overset{1}{\overset{\downarrow}{}} \\ \end{matrix} \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

Let $X_n = \{0, 1\} \Rightarrow$ finite.

(i)

(ii)

$P_{00} = 1$

State 0 is recurrent.

[All absorbing states are recurrent]

(iii) Transient => A +ve probability that the process will not return to this state.

State 1 is transient

(iv) $P_{11} = \frac{1}{2} > 0$.

State 1 is aperiodic

It is not Periodic.

(v) Not irreducible :-

State 0 & state 1 are not communicate with each other.

It is not irreducible.

(i) Not Ergodic :-

It is not irreducible.

It is not non-null persistent.

It is not Ergodic.

Unit - IV

Vector spaces

Type: 1 Bases and Dimensions

① Determine the basis and dimension of the solution space of the linear homogeneous system $x+y-z=0$; $-2x-y+2z=0$; $-x+z=0$

Soln: Given $x+y-z=0$

$-2x-y+2z=0$

$-x+z=0$

$A=\begin{bmatrix}1 & 1 & -1\\ -2 & -1 & 2\\ -1 & 0 & 1\end{bmatrix}$; $X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}$; $B=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$

$AX=B$

Augmented matrix :

$$A \sim \left[\begin{array}{ccc|c}1 & 1 & -1 & 0\\ -2 & -1 & 2 & 0\\ -1 & 0 & 1 & 0\end{array}\right]$$

$R_2 \to R_2+2R_1$

$= \begin{array}{ccc} 2 & 2 & -2 \Rightarrow 2R_1\\ -2 & -1 & 2 \Rightarrow R_2\\ \hline 0 & 1 & 0\end{array}$

$$\sim \left[\begin{array}{ccc|c}1 & 1 & -1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 1 & 0 & 0\end{array}\right]$$

$R_2 \to R_2+2R_1$

$R_3 \to R_3+R_1 \Rightarrow$

$\begin{array}{ccc} -1 & 0 & 1\\ 1 & 1 & -1\\ \hline 0 & 1 & 0\end{array}$

$$\sim \left[\begin{array}{ccc|c}1 & 1 & -1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\end{array}\right]$$

$R_3 \to R_3-R_2 \Rightarrow R_3$: $0 \quad 1 \quad 0$

R_2 : $0 \quad -1 \quad 0$

$\therefore$ one non-zero equation, 2 unknowns

$\therefore$ Dimension $= 2-1 = 1$

Basis $= \{(1,0,1)\}$

$x+y-z=0$

$x-z=0$; $y=0$

$x=z$

Put $x=z=t$; $y=0$

$$\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}t\\ 0\\ t\end{bmatrix} = t\begin{bmatrix}1\\ 0\\ 1\end{bmatrix}$$

$\alpha_1 = \begin{bmatrix}1\\ 0\\ 1\end{bmatrix}$

It is a basis ✓

Dim $= 1$

② $x+y-z=0$; $y=0$

$x-z=0$

$x=z$

$\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}1\\ 0\\ 1\end{bmatrix}$

(2) Find a basis and the dimension of the solution space of the homogeneous system

$$x_1 + 3x_2 - 2x_3 + 2x_5 = 0$$

$$2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 = 0$$

$$5x_3 + 10x_4 + 15x_6 = 0$$

$$2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 = 0$$

Soln

Given

Augmented matrix:

$$A \sim \left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & 0 \\ 0 & 0 & 5 & 10 & 0 & 15 & 0 \\ 2 & 6 & 0 & 8 & 4 & 18 & 0 \end{array}\right]$$

$R_2 \Rightarrow 2\ 6\ -5\ -2\ 4\ -3$

$-2R_1 \Rightarrow -2\ -6\ 4\ 0\ -4\ 0$

$0\ 0\ -1\ -2\ 0\ -3$

$\frac{R_3}{5} = \frac{0}{5}\ \frac{0}{5}\ \frac{5}{5}\ \frac{10}{5}\ \frac{0}{5}\ \frac{15}{5}$

$= 0\ 0\ 1\ 2\ 0\ 3$

$R_4 \Rightarrow 2\ 6\ 0\ 8\ 4\ 18$

$-2R_1 \Rightarrow -2\ -6\ 4\ 0\ -4\ 0$

$0\ 0\ 4\ 8\ 0\ 18$

$$\sim \left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 4 & 8 & 0 & 18 & 0 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$

$R_3 \to \frac{R_3}{5}$

$R_4 \to R_4 - 2R_1$

$$\sim \left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 0 \end{array}\right]$$

$R_3 \to R_3 + R_2$

$R_4 \to R_4 + 4R_2$

$R_3 \Rightarrow 0\ 0\ 1\ 2\ 0\ 3$

$R_2 \Rightarrow 0\ 0\ -1\ -2\ 0\ -3$

$0\ 0\ 0\ 0\ 0\ 0$

$R_4 \Rightarrow 0\ 0\ 4\ 8\ 0\ 18$

$4R_2 \Rightarrow 0\ 0\ -4\ -8\ 0\ -12$

$0\ 0\ 0\ 0\ 0\ 6$

$x_1 + 3x_2 - 2x_3 + 2x_5 = 0 \to (1)$

$-x_3 - 2x_4 - 3x_6 = 0 \to (2)$

$6x_6 = 0 \to (3)$

$\boxed{x_6 = 0} \to (3)$

Sub ③ in ②

$$-x_3 - 2x_4 - 3x_6 = 0$$

$$-x_3 - 2x_4 - 0 = 0$$

$$-x_3 - 2x_4 = 0$$

$$\boxed{x_3 = -2x_4} \rightarrow ⑤$$

From ①,

$$x_1 + 3x_2 - 2x_3 + 2x_5 = 0$$

$$x_1 + 3x_2 + 4x_4 + 2x_5 = 0$$

$$x_1 = -3x_2 - 4x_4 - 2x_5$$

$$x_2 = x_2$$

$$x_3 = -2x_4$$

$$x_4 = x_4$$

$$x_5 = x_5$$

$$x_6 = 0$$

Solution:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_4 + \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} x_5$$

$\therefore x_2, x_4, x_5$ = Independent ✓

x_1, x_3 = Dependent.

$\therefore$ one non-zero equation.

5 unknowns.

Basic = $\{(-3,1,0,0,0,0), (-4,0,-2,1,0,0), (-2,0,0,0,1,0)\}$

Dimension = 3.

It is a basis.

---

③ Determine the basis and the dimension of the homogeneous system

$$2x_1 + 2x_2 - x_3 + x_5 = 0$$

$$-x_1 - x_2 + 2x_3 - 3x_4 + x_5 = 0$$

$$x_1 + x_2 - 2x_3 - x_5 = 0$$

$$x_3 + x_4 + x_5 = 0$$

Sub ③ in ②

$-x_3 - 2x_4 - 3x_6 = 0$

$-x_3 - 2x_4 - 0 = 0$

$-x_3 - 2x_4 = 0$

$\boxed{x_3 = -2x_4}$ → ④

$x_1 + 3x_2 - 2x_3 + 2x_5 = 0$

$x_1 + 3x_2 + 4x_4 + 2x_5 = 0$

$x_1 = -3x_2 - 4x_4 - 2x_5$

$x_2 = x_2$

$x_3 = -2x_4$

$x_4 = x_4$

$x_5 = x_5$

$x_6 = 0$

Solution:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_4 + \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} x_5$$

$\therefore x_2, x_4, x_5$ = Independent ✓

x_1, x_3 = Dependent.

∴ one non-zero equation.

5 unknowns.

Basic = $\{(-3,1,0,0,0,0), (-4,0,-2,1,0,0), (-2,0,0,0,1,0)\}$

Dimension = 3.

It is a basis.

③ Determine the basis and the dimension of the homogeneous system

$2x_1 + 2x_2 - x_3 + x_5 = 0$

$-x_1 - x_2 + 2x_3 - 3x_4 + x_5 = 0$

$x_1 + x_2 - 2x_3 - x_5 = 0$

$x_3 + x_4 + x_5 = 0$

$$A = \left[\begin{array}{ccccc|c} & & & 0 & 1 & 0 \\ -1 & -1 & 2 & -3 & 1 & 0 \\ 0 & 0 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right]$$

$2R_2 \to$ -2 -2 4 -6 2
$R_1 \to$ 2 2 -1 0 1
0 0 3 -6 3

$2R_3 \to$ -4 0 -2
$-R_1 \to$ 1 0 -1
-3 0 -3

$$\sim \left[\begin{array}{ccccc|c} 2 & 2 & -1 & 0 & 1 & 0 \\ 0 & 0 & 3 & -6 & 3 & 0 \\ 0 & 0 & -3 & 0 & -3 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right]$$

$R_2 \to 2R_2 + R_1$
$R_3 \to 2R_3 - R_1$
$R_4 \to R_4$

$$\sim \left[\begin{array}{ccccc|c} 2 & 2 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 & +1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right]$$

$R_2 \to \frac{R_2}{3}$

$R_3 \to \frac{R_3}{-3}$

$R_3 \to$ 0 0 1
$-R_2 \to$ -1 2 -1
0 2 0

$$\sim \left[\begin{array}{ccccc|c} 2 & 2 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \end{array}\right]$$

$R_3 \to R_3 - R_2$
$R_4 \to R_4 - R_2$

$R_4 \to$ 1 1 1
$-R_2 \to$ -1 2 -1
0 3 0

$$\sim \left[\begin{array}{ccccc|c} 2 & 2 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$2R_4 \to$ 6 0
$-3R_3 \to$ -6 0
0 0

$R_4 \to 2R_4 - 3R_3$

$R_3 \to \frac{R_3}{2}$

$$2x_1 + 2x_2 - x_3 + x_5 = 0 \longrightarrow (1)$$

$$x_3 - 2x_4 + x_5 = 0 \longrightarrow (2)$$

$$x_4 = 0. \longrightarrow (3)$$

Sub ③ in ①

$$x_3 - 2x_4 + x_5 = 0$$

$$x_3 + x_5 = 0$$

$$\boxed{x_3 = -x_5} \longrightarrow ④$$

Sub ④ in ①

$$2x_1 + 2x_2 - x_3 + x_5 = 0$$

$$2x_1 + 2x_2 + x_5 + x_5 = 0$$

$$2x_1 + 2x_2 + 2x_5 = 0$$

$\div 2$

$$x_1 + x_2 + x_5 = 0.$$

$$x_1 = -x_2 - x_5$$

$x_2 = x_2$

$x_5 = x_5$

Solution

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} x_5$$

5 unknowns & One non-zero equation

Basic $= \{(-1, 1, 0, 0, 0), (1, 0, -1, 0, 1)\}$.

Dimension $= 2$.

It is a basis.

H.W

④ Find the dimension & a basis for the solution space W of the system of homogeneous equation given below

$$x_1 + 2x_2 + 2x_3 - x_4 + 3x_5 = 0$$

$$x_1 + 2x_2 + 3x_3 + x_4 + x_5 = 0$$

$$3x_1 + 6x_2 + 8x_3 + x_4 + 5x_5 = 0.$$

Type: 2 (R^3) Linearly dependent = 0 (Not Basis)
Linearly Independent ≠ 0 (Basis)

① Determine whether the following set is a bases for R^3 $\{(1,-3,-2), (-3,1,3), (-2,-10,-2)\}$

Soln: To find the set is linearly independent (or) linearly dependent.

coefficient matrix :.

$$A = \begin{vmatrix} 1 & -3 & -2 \\ -3 & 1 & 3 \\ -2 & -10 & -2 \end{vmatrix} = 1(-2+30)+3(6+6)-2(30+2)$$
$$= 1(28)+3(12)-2(32)$$
$$= 28+36-64$$
$$= 64-64$$
$$|A| = 0$$

∴ It is a linearly dependent.

∴ The given set is not a basis for R^3.

---

② The vectors $u_1 = (2,-3,1)$, $u_2 = (1,4,-2)$ $u_3 = (-8,12,-4)$, $u_4 = (1,37,-17)$ & $u_5 = (-3,-5,8)$ generate R^3. Find a subset of the set $\{u_1, u_2, u_3, u_4, u_5\}$ that is a basis for R^3.

Soln:

To find the set is linearly independent (or) linearly dependent.

Select the 3 vectors : $u_1 = (2,-3,1)$, $u_2 = (1,4,-2)$, $u_5 = (-3,-5,8)$

Coefficient matrix:

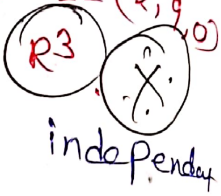
$$\begin{vmatrix} 2 & -3 & 1 \\ 1 & 4 & -2 \\ -3 & -5 & 8 \end{vmatrix} = 2(32-10)+3(8-6)+1(-5+12)$$
$$= 2(22)+3(2)+1(7)$$
$$= 44+6+7$$

∴ It is a linearly independent.

There are 3 vectors & dimension of R^3 } = 3.

(Hw) Basis = $\{(2,-3,1), (1,4,-2), (-3,-5,8)\}$

(3) Show that the vectors $v_1 = (1,2,1)$, $v_2 = (2,9,0)$ $v_3 = (3,3,4)$ form a basis for R^3



Soln:- To find the set is linearly independent or not.

Coefficient matrix,

$$|A| = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 9 & 0 \\ 3 & 3 & 4 \end{vmatrix} = 1(36-0) - 2(8-0) + 1(6-27)$$

$$= 36 - 16 - 21 \quad (-21)$$

$$= 36 - 37$$

$$= -1 \neq 0.$$

∴ It is a linearly independent.

There are 3 vectors & Dimension of R^3 } = 3.

It is a basis.

(4) Determine if (1,0,0), (0,1,0) is a basis for R^3

Soln There are 2 vectors and dimension of $R^3 = 2$.

The subset of R^2 contains only 2 vectors

∴ The given set is not a basis for R^3.

(or)

$$|A| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{vmatrix} = 1(0) + 0 + 0 = 0$$

It is a linearly dependent.

∴ It is a not basis.

① Determine whether the following set is a bases for $P_2 R$ (X)

$\{-1-x+2x^2, 2+x-2x^2, 1-2x+4x^2\}$.

Soln:

To find the Set is linearly independent or not

coefficient matrix

$$|A| = \begin{vmatrix} -1 & -1 & 2 \\ 2 & 1 & -2 \\ 1 & -2 & 4 \end{vmatrix} = -1(4-4)+1(8+2)+2(-4+1)$$

$$= 0+1(10)+2(-3)$$

$$= 10-6$$

$$= 0$$

It is a linearly dependent.

There are 3 vectors & dimension of $P_2(R)$ $\} = 2$.

∴ The given Set is basis for $P_2 R$.

---

② Determine whether or not the Set (X)

$S = \{1+2x+x^2, 3+x^2, x+x^3\}$ forms a basis for $P_2(R)$

Soln To find the Set is linearly independent (or) not

coefficient matrix:

$$|A| = \begin{vmatrix} 1 & 2 & 1 \\ 3 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 1(-1)-2(3)+1(3)$$

$$= -1-6+3$$

$$= -7+3$$

$$= -4 \neq 0.$$

It is a linearly independent.

There are 3 vectors & dimension of $P_2(R)$ $\} = 2$

∴ The given set is basis for $P_2 R$.

Unit - V

Linear Transformation And Inner Product spaces

Gram Schmidt orthogonalization process

Gram Schmidt Process :

$$v_1 = u_1$$

$$v_2 = u_2 - \frac{(u_2, v_1)}{\|v_1\|^2} = u_2 - \frac{(u_2, v_1)}{(v_1, v_1)} v_1$$

$$v_3 = u_3 - \frac{(u_3, v_1)}{\|v_1\|^2} v_1 - \frac{(u_3, v_2)}{\|v_2\|^2} v_2$$

$$v_3 = u_3 - \frac{(u_3, v_1)}{(v_1, v_1)} v_1 - \frac{(u_3, v_2)}{(v_2, v_2)} v_2$$

Orthogonal Set $= (v_1, v_2, v_3)$

Orthonormal Basis

$$q_1 = \frac{1}{\|v_1\|} v_1 = \frac{1}{\sqrt{v_1, v_1}} v_1$$

$$q_2 = \frac{1}{\|v_2\|} v_2 = \frac{1}{\sqrt{v_2, v_2}} v_2$$

$$q_3 = \frac{1}{\|v_3\|} v_3 = \frac{1}{\sqrt{v_3, v_3}} v_3$$

---

(1) Let R^3 have the Euclidean inner product u_x. Gram-Schmidt Process $\{u_1, u_2, u_3\}$ into an orthogonal basis where $u_1 = (1,1,1)$, $u_2 = (0,1,1)$, $u_3 = (0,0,1)$

Soln : Given $u_1 = (1,1,1)$, $u_2 = (0,1,1)$, $u_3 = (0,0,1)$

Let $v_1 = u_1 = (1,1,1)$

$$\|v_1\|^2 = (v_1, v_1) = (1,1,1)\,(1,1,1)$$

$$= 1+1+1$$

$$\|v_1\|^2 = 3$$

$$\|v_1\| = \sqrt{3}$$

$$v_2 = u_2 - \frac{(u_2, v_1)}{\|v_1\|^2} v_1$$

$$= (0,1,1) - \frac{2}{3}(1,1,1)$$

$$= (0,1,1) + \left(-\frac{2}{3}, -\frac{2}{3}, -\frac{2}{3}\right)$$

$$= \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

v_2

$$\|v_2\|^2 = (v_2, v_2)$$

$$= \left(\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)\left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

$$= \frac{4}{9} + \frac{1}{9} + \frac{1}{9}$$

$$= \frac{6}{9}$$

$$\boxed{\|v_2\|^2 = \frac{2}{3}}$$

$$\boxed{\|v_2\| = \sqrt{\frac{2}{3}}}.$$

$$(u_2, v_1) = (0,1,1)(1,1,1)$$
$$= 0+1+1$$
$$= 2.$$

$$(u_3, v_1) = (0,0,1)(1,1,1)$$
$$= 0+0+1$$
$$= 1$$

$$(u_3, v_2) = (0,0,1)\left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$$
$$= 0+0+\frac{1}{3}$$
$$= \frac{1}{3}$$

$$v_3 = u_3 - \frac{(u_3, v_1)}{\|v_1\|^2} v_1 - \frac{(u_3, v_2)}{\|v_2\|^2} v_2$$

$$\frac{1/3}{2/3} = \frac{1}{3} \times \frac{3}{2} = \frac{1}{2}$$

$$= (0,0,1) - \frac{1}{3}(1,1,1) - \frac{1}{3}\left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

$$= (0,0,1) + \left(-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}\right) + \left(\frac{2}{6}, -\frac{1}{6}, -\frac{1}{6}\right)$$

$$= \left(0 - \frac{1}{3} + \frac{1}{3}\right) + \left(0 - \frac{1}{3} - \frac{1}{6}\right) + \left(1 - \frac{1}{3} - \frac{1}{6}\right)$$

$$= \left(0, \frac{-6-3}{18}, \frac{6-2-1}{6}\right)$$

$$= \left(0, \frac{-9}{18}, \frac{3}{6}\right)$$

$$v_3 = \left(0, -\frac{1}{2}, \frac{1}{2}\right)$$

$$\|v_3\|^2 = (v_3, v_3) = \left(0, -\frac{1}{2}, \frac{1}{2}\right)\left(0, -\frac{1}{2}, \frac{1}{2}\right)$$

$$= 0 + \frac{1}{4} + \frac{1}{4} = \frac{2}{4} = \frac{1}{2}.$$

$\|v_3\|^2 = \frac{1}{2}$

$\|v_3\| = \frac{1}{\sqrt{2}}$

Orthonormal Basis :-

v_1 v_2 v_3

orthogonal set $= \left\{ (1,1,1), \left(\frac{-2}{3}, \frac{1}{3}, \frac{1}{3}\right), \left(0, \frac{-1}{2}, \frac{1}{2}\right) \right\}$

$\frac{v_1}{\|v_1\|}$ $\frac{v_2}{\|v_2\|}$

orthonormal Basis $= \left\{ \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(\frac{\frac{-2}{3}}{\sqrt{\frac{2}{3}}}, \frac{\frac{1}{3}}{\sqrt{\frac{2}{3}}}, \frac{\frac{1}{3}}{\sqrt{\frac{2}{\sqrt{3}}}}\right) \right.$

$\frac{v_3}{\|v_3\|}$ $\left. \left(0, \frac{\frac{-1}{2}}{\frac{1}{\sqrt{2}}}, \frac{\frac{1}{2}}{\frac{1}{\sqrt{2}}}\right) \right\}$

---

(2) Use Gram-Schmidt process find an orthonormal basis for $(1,0,1)$, $(0,1,1)$, $(1,3,3)$.

Soln :-

Given $u_1 = (1,0,1)$; $u_2 = (0,1,1)$; $u_3 = (1,3,3)$.

Let $v_1 = u_1 = (1,0,1)$.

$\|v_1\|^2 = (v_1, v_1) = (1,0,1)(1,0,1) = 1+0+1 = 2$

$\|v_1\|^2 = 2$

$\|v_1\| = \sqrt{2}$

$$v_2 = u_2 - \frac{(u_2, v_1)}{\|v_1\|^2} v_1$$

$$= (0,1,1) - \frac{1}{2}(1,0,1)$$

$$= (0,1,1) + \left(\frac{-1}{2}, 0, \frac{-1}{2}\right)$$

$$v_2 = \left(\frac{-1}{2}, 1, \frac{1}{2}\right)$$

$(u_2, v_1) = (0,1,1)(1,0,1)$

$= 0+0+1$

$= 1$

$(u_3, v_1) = (1,3,3)(1,0,1)$

$= 1+0+3$

$= 4$

$(u_3, v_2) = (1,3,3)\left(\frac{-1}{2}, 1, \frac{1}{2}\right)$

$= \frac{-1}{2} + 3 + \frac{3}{2}$

$= 3 + \frac{2}{2} = 3+1$

$\|v_2\|^2 = (v_2, v_2) = \left(-\frac{1}{2}, 1, \frac{1}{2}\right)\left(-\frac{1}{2}, 1, \frac{1}{2}\right)$

$= \frac{1}{4} + 1 + \frac{1}{4} = 1 + \frac{2}{4} = \frac{4+2}{4} = \frac{6}{4} = \frac{3}{2}$.

(or) $\frac{1+4+1}{4} = \frac{6}{4} = \frac{3}{2}$

$\|v_2\|^2 = \frac{3}{2}$

$\|v_2\| = \sqrt{\frac{3}{2}}$

$v_3 = u_3 - \frac{(u_3, v_1)}{\|v_1\|^2} v_1 - \frac{(u_3, v_2)}{\|v_2\|^2} v_2$

$= (1,3,3) - \frac{4}{2}(1,0,1) - \frac{4}{3/2}\left(-\frac{1}{2}, 1, \frac{1}{2}\right)$

$= (1,3,3) + (-2,0,-2) - \frac{8}{3}\left(-\frac{1}{2}, 1, \frac{1}{2}\right)$

$= (1,3,3) + (-2,0,-2) + \left(\frac{8}{6}, -\frac{8}{3}, -\frac{8}{6}\right)$

$= \left(1 - 2 + \frac{8}{6}\right), \left(3 + 0 - \frac{8}{3}\right), \left(3 - 2 - \frac{8}{6}\right)$

$= \left(-1 + \frac{8}{6}\right), \left(3 - \frac{8}{3}\right), \left(1 - \frac{8}{6}\right)$

$= \left(\frac{-6+8}{6}, \frac{9-8}{3}, \frac{6-8}{6}\right)$

$= \left(\frac{2}{6}, \frac{1}{3}, -\frac{2}{6}\right)$

$v_3 = \left(\frac{1}{3}, \frac{1}{3}, -\frac{1}{3}\right)$

$\|v_3\|^2 = (v_3, v_3) = \left(\frac{1}{3}, \frac{1}{3}, -\frac{1}{3}\right)\left(\frac{1}{3}, \frac{1}{3}, -\frac{1}{3}\right)$

$= \frac{1}{9} + \frac{1}{9} + \frac{1}{9} = \frac{3}{9} = \frac{1}{3}$.

$\|v_3\|^2 = \frac{1}{3}$

$\|v_3\| = \frac{1}{\sqrt{3}}$

Orthonormal Basis:

Orthogonal Set $= \{(v_1, v_2, v_3)\} = \left\{(1,0,1), \left(-\frac{1}{2}, 1, \frac{1}{2}\right), \left(\frac{1}{3}, \frac{1}{3}, -\frac{1}{3}\right)\right\}$

Orthonormal Set $= \left\{\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)\left(\frac{-1}{\sqrt{3/2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3/2}}\right)\left(\frac{\frac{1}{3}}{\frac{1}{\sqrt{3}}}, \frac{\frac{1}{3}}{\frac{1}{\sqrt{3}}}, \frac{-\frac{1}{3}}{\frac{1}{\sqrt{3}}}\right)\right\}$

(3) Apply the Gram-Schmidt process to the vectors $u_1=(1,0,1)$, $u_2=(1,0,-1)$, $u_3=(0,3,4)$ to obtain an orthonormal basis for $R^3(R)$ with the standard inner product.

Soln:-

Given $u_1=(1,0,1)$; $u_2=(1,0,-1)$; $u_3=(0,3,4)$.

Let $v_1 = u_1 = (1,0,1)$

$$\|v_1\|^2 = (v_1,v_1) = (1,0,1)(1,0,1)$$
$$= 1+0+1$$

$$\boxed{\|v_1\|^2 = 2}$$
$$\boxed{\|v_1\| = \sqrt{2}}$$

$$(u_2,v_1) = (1,0,-1)(1,0,1)$$
$$= 1+0-1$$
$$= 0$$

$$(u_3,v_1) = (0,3,4)(1,0,1)$$
$$= 0+0+4$$
$$= 4$$

$$(u_3,v_2) = (0,3,4)(1,0,-1)$$
$$= 0+0-4$$
$$= -4$$

$$v_2 = u_2 - \frac{(u_2,v_1)}{\|v_1\|^2} v_1$$
$$= (1,0,-1) - 0$$
$$\boxed{v_2 = (1,0,-1)}$$

$$\|v_2\|^2 = (v_2,v_2) = (1,0,-1)(1,0,-1)$$
$$= 1+0+1$$

$$\boxed{\|v_2\|^2 = 2}$$
$$\boxed{\|v_2\| = \sqrt{2}}$$

$$v_3 = u_3 - \frac{(u_3,v_1)}{\|v_1\|^2} v_1 - \frac{(u_3,v_2)}{\|v_2\|^2} v_2$$
$$= (0,3,4) - \frac{4}{2}(1,0,1) - \frac{(-4)}{2}(1,0,-1)$$
$$= (0,3,4) + (-2,0,-2) + (2,0,-2)$$
$$= (0-2+2,\ 3+0+0,\ 4-2-2)$$
$$v_3 = (0,3,0)$$

$\|v_3\|^2 = (v_3, v_3) = (0,3,0),(0,3,0)$

$= 0+9+0$

$\|v_3\|^2 = 9$

$\|v_3\| = 3$

orthonormal Basis

orthogonal set $= (v_1, v_2, v_3)$

$= \{(1,0,1), (1,0,-1), (0,3,0)\}$

orthonormal Basis $= \left(\frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|}, \frac{v_3}{\|v_3\|}\right)$

$= \left\{\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}, \left(0, \frac{3}{3}, 0\right)\right\}$

$\|\cdot\|$ → norm (or) length

Least Square Approximation

Let $A = \begin{bmatrix} t_1 & 1 \\ t_2 & 1 \\ \vdots & \vdots \\ t_n & 1 \end{bmatrix}$; $y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ $x_0 = \begin{bmatrix} c \\ d \end{bmatrix}$ Error: $E = \|Ax_0 - y\|^2$ $\therefore x_0 = (A^*A)^{-1} A^*y$

(1) For the set of data values, use the "least square approximation" to find the best fit and also find the errors for $\{(1,2), (3,4), (5,7), (7,9), (9,12)\}$

Soln:

$\therefore$ Given $y = ct + d$

Let $A = \begin{bmatrix} 1 & 1 \\ 3 & 1 \\ 5 & 1 \\ 7 & 1 \\ 9 & 1 \end{bmatrix}$; $y = \begin{bmatrix} 2 \\ 4 \\ 7 \\ 9 \\ 12 \end{bmatrix}$; $x_0 = \begin{bmatrix} c \\ d \end{bmatrix}$

By least Squre Approximation,

$$x_0 = (A^*A)^{-1} A^*y$$

$$A^*A = \begin{bmatrix} 1 & 3 & 5 & 7 & 9 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 3 & 1 \\ 5 & 1 \\ 7 & 1 \\ 9 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+9+25+49+81 & 1+3+5+7+9 \\ 1+3+5+7+9 & 1+1+1+1+1 \end{bmatrix}$$

$$A^*A = \begin{bmatrix} 165 & 25 \\ 25 & 5 \end{bmatrix}$$

$$(A^*A)^{-1} = \frac{1}{|A^*A|} \operatorname{adj}(A^*A).$$

$$= \frac{1}{\begin{vmatrix} 165 & 25 \\ 25 & 5 \end{vmatrix}} \begin{bmatrix} 5 & -25 \\ -25 & 165 \end{bmatrix}$$

$$(A^*A)^{-1} = \frac{1}{|A^*A|} \operatorname{adj} A^*A$$

$$= \frac{1}{(825-625)} \begin{bmatrix} 5 & -25 \\ -25 & 165 \end{bmatrix}$$

$$= \frac{1}{200} \begin{bmatrix} 5 & -25 \\ -25 & 165 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5}{200} & -\frac{25}{200} \\ -\frac{25}{200} & \frac{165}{200} \end{bmatrix}$$

$$(A^*A)^{-1} = \begin{bmatrix} \frac{1}{40} & -\frac{1}{8} \\ -\frac{1}{8} & \frac{33}{40} \end{bmatrix}$$

$$A^*y = \begin{bmatrix} 1 & 3 & 5 & 7 & 9 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 7 \\ 9 \\ 12 \end{bmatrix}$$

$$= \begin{bmatrix} 2+12+35+63+108 \\ 2+4+7+9+12 \end{bmatrix}$$

$$A^*y = \begin{bmatrix} 220 \\ 34 \end{bmatrix}$$

$$x_0 = (A^*A)^{-1} A^*y$$

$$= \begin{bmatrix} \frac{1}{40} & -\frac{1}{8} \\ -\frac{1}{8} & \frac{33}{40} \end{bmatrix} \begin{bmatrix} 220 \\ 34 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{220}{40} - \frac{34}{8} \\ -\frac{220}{8} + \frac{33 \times 34}{40} \end{bmatrix} = \begin{bmatrix} 5.5 - 4.25 \\ -27.5 + 28.05 \end{bmatrix}$$

$$= \begin{bmatrix} 1.25 \\ 0.55 \end{bmatrix} = \begin{bmatrix} c \\ d \end{bmatrix}$$

$\therefore$ $y = ct + d$

$y = 1.25t + 0.55$

Error :

$$E = \| Ax_0 - y \|^2$$

$$Ax_0 = \begin{bmatrix} 1 & 1 \\ 3 & 1 \\ 5 & 1 \\ 7 & 1 \\ 9 & 1 \end{bmatrix} \begin{bmatrix} 1.25 \\ 0.55 \end{bmatrix} = \begin{bmatrix} 1.8 \\ 4.3 \\ 6.8 \\ 9.3 \\ 11.8 \end{bmatrix}$$

$$= \begin{matrix} 1.25+0.55 \\ 3.75+0.55 \\ 6.25+0.55 \\ 8.75+0.55 \end{matrix} \quad \Big| \quad 11.25+0.55$$

$$Ax_0 - y = \begin{bmatrix} 1.8 \\ 4.3 \\ 6.8 \\ 9.3 \\ 11.8 \end{bmatrix} - \begin{bmatrix} 2 \\ 4 \\ 7 \\ 9 \\ 12 \end{bmatrix} \begin{bmatrix} -0.2 \\ 0.3 \\ -0.2 \\ 0.3 \\ -0.2 \end{bmatrix}$$

$$E = \|Ax_0 - y\|^2 = (-0.2)^2 + (0.3)^2 + (-0.2)^2 + (0.3)^2 + (0.2)^2$$

$$E = \boxed{\|Ax_0 - y\|^2 = 0.3}$$

Solution ::

$y = 1.25t + 0.55$

$E = 0.3$

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② Using least square approximation determine the best linear fit for the data

$\{(1,2), (2,3), (3,5), (4,7)\}$.

Soln::

Given $y = ct + d$

Let $A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{bmatrix}$ $y = \begin{bmatrix} 2 \\ 3 \\ 5 \\ 7 \end{bmatrix}$; $x_0 = \begin{bmatrix} c \\ d \end{bmatrix}$

By least Square Approximation,

$$x_0 = (A^* A)^{-1} A^* y$$

$$A^* A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+4+9+16 & 1+2+3+4 \\ 1+2+3+4 & 1+1+1+1 \end{bmatrix}$$

$$A^* A = \begin{bmatrix} 30 & 10 \\ 10 & 4 \end{bmatrix}$$

$$(A^*A)^{-1} = \frac{1}{|A^*A|}\,\text{adj}(A^*A)$$

$$= \frac{1}{\begin{vmatrix} 30 & 10 \\ 10 & 4 \end{vmatrix}} \begin{bmatrix} 4 & -10 \\ -10 & 30 \end{bmatrix}$$

$$= \frac{1}{(120-100)} \begin{bmatrix} 4 & -10 \\ -10 & 30 \end{bmatrix}$$

$$= \frac{1}{20} \begin{bmatrix} 4 & -10 \\ -10 & 30 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{4}{20} & -\frac{10}{20} \\ -\frac{10}{20} & \frac{30}{20} \end{bmatrix}$$

$$(A^*A)^{-1} = \begin{bmatrix} \frac{1}{5} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix}$$

$$A^*y = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 5 \\ 7 \end{bmatrix}$$

$$= \begin{bmatrix} 2+6+15+28 \\ 2+3+5+7 \end{bmatrix}$$

$$= \begin{bmatrix} 51 \\ 17 \end{bmatrix}$$

$$x_0 = (A^*A)^{-1}(A^*y)$$

$$= \begin{bmatrix} \frac{1}{5} & -\frac{1}{2} \\ -\frac{1}{2} & 3/2 \end{bmatrix} \begin{bmatrix} 51 \\ 17 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{51}{5} - \frac{17}{2} \\ -\frac{51}{2} + \frac{51}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 10.2 - 8.5 \\ 0 \\ -25.5 + 0.08 \end{bmatrix} = \begin{bmatrix} 1.7 \\ 0 \end{bmatrix} = \begin{bmatrix} c \\ d \end{bmatrix}$$

$\therefore$ $y = ct + d$

$$\boxed{y = 1.7t}$$

Error :

$$E = \| Ax_0 - y \|^2$$

$$A\hat{x}_0 = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1.7 \\ 0 \end{bmatrix} = \begin{bmatrix} 1.7 \\ 3.4 \\ 5.1 \\ 6.8 \end{bmatrix}$$

$$A\hat{x}_0 - y = \begin{bmatrix} 1.7 \\ 3.4 \\ 5.1 \\ 6.8 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \\ 5 \\ 7 \end{bmatrix} = \begin{bmatrix} -0.3 \\ 0.4 \\ 0.1 \\ -0.2 \end{bmatrix}$$

$$E = \| A\hat{x}_0 - y \|^2$$

$$= (-0.3)^2 + (0.4)^2 + (0.1)^2 + (-0.2)^2$$

$$= 0.09 + 0.16 + 0.01 + 0.04$$

$$\boxed{E = 0.30}$$

Solution :-

$$\boxed{\begin{aligned} y &= 1.7t \\ E &= 0.3 \end{aligned}}$$

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(3) Find the least square solution, the least error vector, and the least squares error of the linear system.

$x_1 - x_2 = 4$; $3x_1 + 2x_2 = 1$; $-2x_1 + 4x_2 = 3$.

Soln :-

Given $y = ct + d$

Let $A = \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ -2 & 4 \end{bmatrix}$; $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$; $y = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$

By least Square approximation,

$$x_0 = (A^*A)^{-1}(A^*y)$$

$$A^*A = \begin{bmatrix} 1 & 3 & -2 \\ -1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ -2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1+9+4 & -1+6-8 \\ -1+6-8 & 1+4+16 \end{bmatrix}$$

$$A^*A = \begin{bmatrix} 14 & -3 \\ -3 & 21 \end{bmatrix}$$

$$(A^*A)^{-1} = \frac{1}{|A^*A|} \text{adj}(A^*A)$$

$$(A^*A)^{-1} = \frac{1}{294-9} \begin{bmatrix} 21 & 3 \\ 3 & 14 \end{bmatrix}$$

$$(A^*A)^{-1} = \frac{1}{285} \begin{bmatrix} 21 & 3 \\ 3 & 14 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{21}{285} & \frac{3}{285} \\ \frac{3}{285} & \frac{14}{285} \end{bmatrix}$$

$$(A^*A)^{-1} = \begin{bmatrix} 0.07 & 0.01 \\ 0.01 & 0.049 \end{bmatrix}$$

$$A^*y = \begin{bmatrix} 1 & 3 & -2 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4+3-6 \\ -4+2+12 \end{bmatrix}$$

$$A^*y = \begin{bmatrix} 1 \\ 10 \end{bmatrix}$$

$$x_0 = (A^*A)^{-1}(A^*y)$$

$$= \begin{bmatrix} 0.07 & 0.01 \\ 0.01 & 0.049 \end{bmatrix} \begin{bmatrix} 1 \\ 10 \end{bmatrix}$$

$$= \begin{bmatrix} 0.07+0.1 \\ 0.01+0.49 \end{bmatrix}$$

$$x_0 = \begin{bmatrix} 0.17 \\ 0.50 \end{bmatrix} = \begin{bmatrix} c \\ d \end{bmatrix}$$

$$y = ct + d$$

$$\boxed{y = 0.17t + 0.5}$$

Error :-

$$E = \| Ax_0 - y \|^2$$

$$Ax_0 = \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 0.17 \\ 0.50 \end{bmatrix}$$

$$= \begin{bmatrix} 0.17 - 0.50 \\ 0.51 + 1 \\ -0.34 + 2 \end{bmatrix}$$

$$Ax_0 = \begin{bmatrix} -0.33 \\ 1.51 \\ 1.66 \end{bmatrix}$$

$$Ax_0 - y = \begin{bmatrix} -0.33 \\ 1.51 \\ 1.66 \end{bmatrix} - \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$$

$$Ax_0 - y = \begin{bmatrix} -4.3 \\ 0.51 \\ -1.34 \end{bmatrix}$$

$$E = \| Ax_0 - y \|^2 = (-4.3)^2 + (0.51)^2 + (-1.34)^2$$

$$= 18.49 + 0.2601 + 1.7956$$

$$E = \| Ax_0 - y \|^2 = 20.5457$$

Solution

$y = 0.17x + 0.5$

$E = 20.5457$

Orthogonal Projection

Formula: $x_0 = (A^*A)^{-1}(A^*y)$ (or) $(A^*A)\,x_0 = A^*y$.

$u = Ax$.

(1) Find the orthogonal projection of the vector $u=(-3,-3,8,9)$ on the subspace of R^4 spanned by the vectors $v_1=(3,1,0,1)$; $v_2=(1,2,1,1)$ $v_3=(-1,0,2,-1)$.

Soln:

Let $A=\begin{bmatrix}3&1&-1\\1&2&0\\0&1&2\\1&1&1\end{bmatrix}$; $X=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$; $u=(-3,-3,8,9)$ $y=\begin{bmatrix}-3\\-3\\8\\9\end{bmatrix}$

~~$x_0 = (A^*A)^{-1}(A^*y)$ (or)~~

$(A^*A)(x_0) = A^*y$.

~~$A^*A =$~~ $\begin{bmatrix}3&1&0&1\\1&2&1&1\\-1&0&2&1\end{bmatrix}\begin{bmatrix}3&1&-1\\1&2&0\\0&1&2\\1&1&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}3&1&0&1\\1&2&1&1\\-1&0&2&-1\end{bmatrix}\begin{bmatrix}-3\\-3\\8\\9\end{bmatrix}$

$\Rightarrow \begin{bmatrix}9+1+0+1 & 3+2+0+1 & -3+0+0-1\\3+2+0+1 & 1+4+1+1 & -1+0+2-1\\-3+0+0-1 & -1+0+2-1 & 1+0+4+1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}9-3+0+9\\-3-6+8+9\\3+0+16-9\end{bmatrix}$

~~$A^*A =$~~ $\begin{bmatrix}11&6&-4\\6&7&0\\-4&0&6\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}-3\\8\\10\end{bmatrix}$

$(A^*A)^{-1} = \dfrac{\text{adj}(A^*A)}{|A^*A|}$

$|A^*A| = \begin{vmatrix}11&6&-4\\6&7&0\\-4&0&6\end{vmatrix}$

$= 11(42)-6(36)-4(0+28)$

$= 462-216-112$

$= 462-328$

$|A^*A| = 134$

$11x_1+6x_2-4x_3=-3$

$6x_1+7x_2+0x_3=8$

$-4x_1+0x_2+6x_3=10$

$x_1=-1$

$x_2=2$

$x_3=1$

$$U = Ax$$

$$= \begin{bmatrix} 3 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -3+2-1 \\ -1+4+0 \\ 0+2+2 \\ -1+2-1 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ 4 \\ 0 \end{bmatrix}.$$

H.W

(4) Find an orthonormal basis of R^3 given that an arbitrary basis is $v_1 = (3,0,4)$, $v_2 = (-1,0,7)$, $v_3 = (2,9,11)$.