Unit I - classification of signals and systems

Unit I : Topic I - standard signals

Lesson 01 : signal concepts

Definition : Signal

signal is a physical quantity that varies with independent variable like time, space, etc.

→ signal carries information in the form of text, data, audio, video, etc.

→ Signal comprises of frequency components.

i) Low frequency components

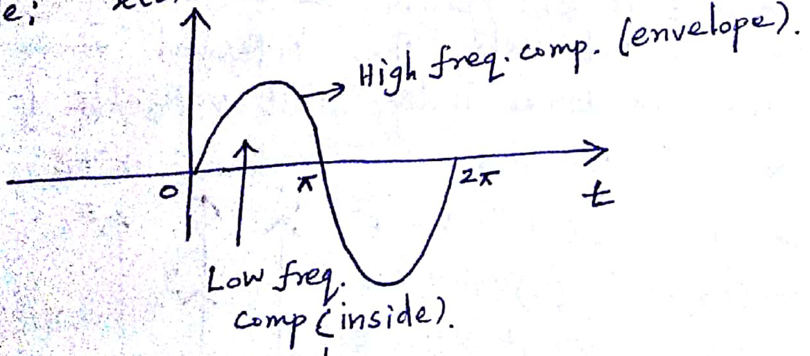
It carries required (or) relevant (or) repeated (or) wanted information.

ii) Hig frequency components

It lies at the edges of a signal. It gives security (or) protection to the low frequency components.

Some frequency components present along with low frequency components, which gives disturbance to the low frequency components. such frequency components are referred as high frequency components.

Example: $x(t) = \sin wt$



classification of signals:

***Based on number of independent variable**

a) One dimensional signal

A signal is a function of only one independent variable.

Examples: $x(t) = 5t^2 + 3t + 2$, Temperature of a Room varies w.r.t. time 't'. only.

b) Multi dimensional signal

A signal is a function of more than one independent variable.

Examples: $s(x,t) = 3x^2 + 5t^2 + 8x^2t + 2xt^2 + 8$

where, $s \to$ Temperature of a room varies w.r.t time 't' and space 'x'.

2-dimensional signal → Black and white photo

3-dimensional signal → Black and white Television signal

Multi dimensional signal → colour TV signal.

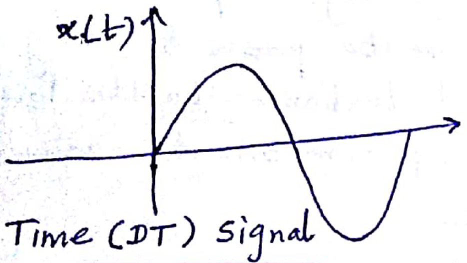
* Based on type of independent variable

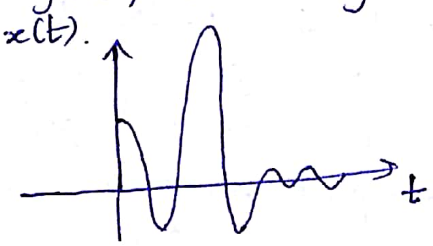
a) Continuous Time (CT) signal [Analog signal]

The signal amplitude (or) magnitude (or) value is defined for every value of time 't'.

The continuous time independent variable is denoted as 't'.

Examples: $x(t) = 5\sin\omega t$, ECG signal, EEG signal.





b) Discrete Time (DT) Signal

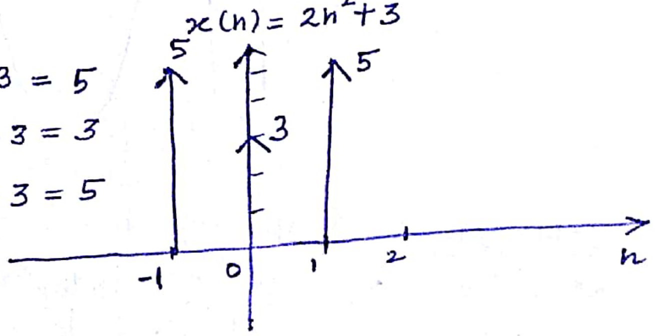
The signal amplitude (or) magnitude (or) value is defined only at specific (or) regular intervals. The interval should be an integer. The discrete time independent variable is denoted as 'n'.

Example: $x(n) = 2n^2 + 3$

$n = -1$; $x(-1) = 2(-1)^2 + 3 = 5$

$n = 0$; $x(0) = 2(0)^2 + 3 = 3$

$n = 1$; $x(1) = 2(1)^2 + 3 = 5$



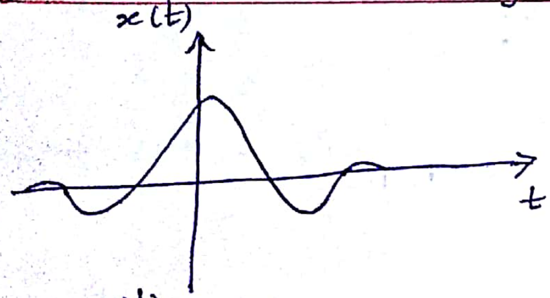
Lesson 01 : Signal concepts

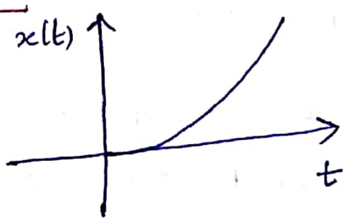
Remedial Lecture notes :

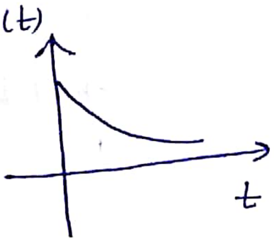
i) Always continuous time independent variable is denoted as time 't'. continuous time (CT) signal is denoted by x(t). 't' varies from '0' to '∞'. (or) '0' to finite interval (or) $-\infty$ to ∞.

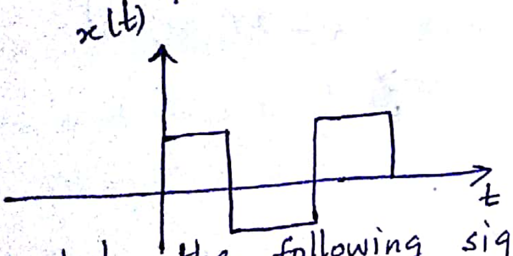
ii) Always Discrete time independent variable is denoted as time 'n'. Discrete Time (DT) signal is denoted by x(n). 'n' should be an integer [positive, negative].

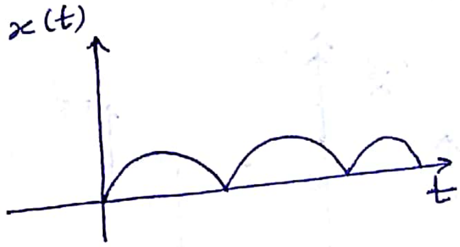
Continuous Time (CT) signal :











***sketch the following signal:**

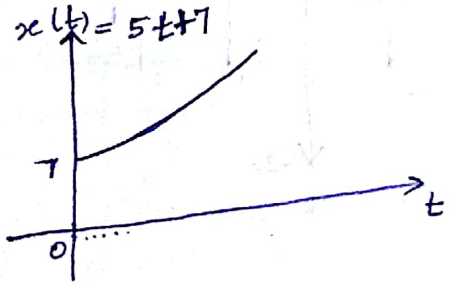
$x(t) = 5t + 7 \leftarrow$ CT signal

$t = 0 \rightarrow x(0) = 5(0) + 7 = 7$

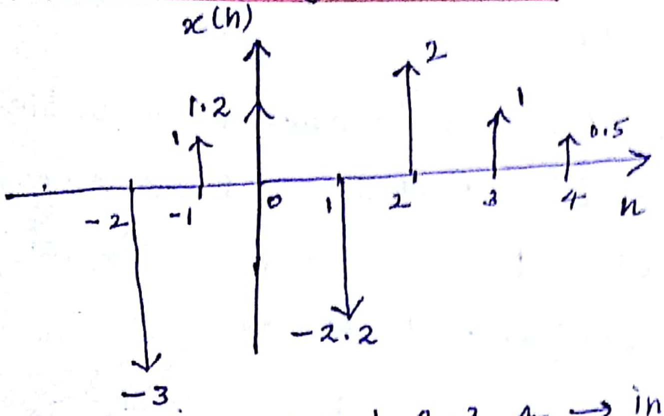
$t = 0.1 \rightarrow x(0.1) = 5(0.1) + 7 = 7.5$

$t = 0.2 \rightarrow x(0.2) = 5(0.2) + 7 = 8$

Here, t = 0, 0.1, 0.2 ..... are continuous values, not are integers.



Discrete Time Signal (DT) :



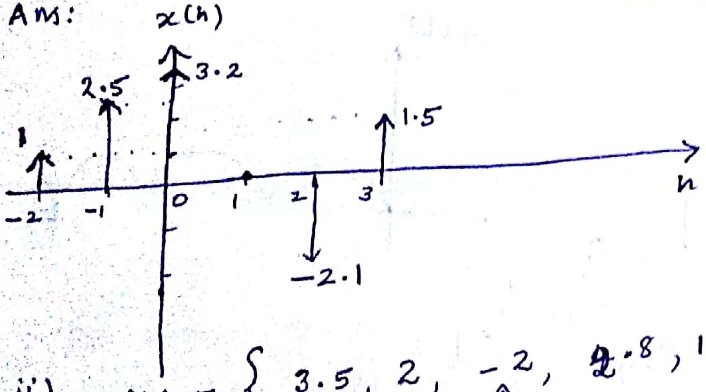
Here n = -2, -1, 0, 1, 2, 3, 4 → integers. So, the given signal is Discrete Time signal. Signal values are x(1) = -2.2, x(2) = 2, x(0) = 1.2, x(-2) = -3, x(-1) = 1, x(3) = 1, x(4) = 0.5.

Sketch the following signal:

i)

n	-2	-1	0	1	2	3
x(n)	1	2.5	3.2	0	-2.1	1.5

Ans:

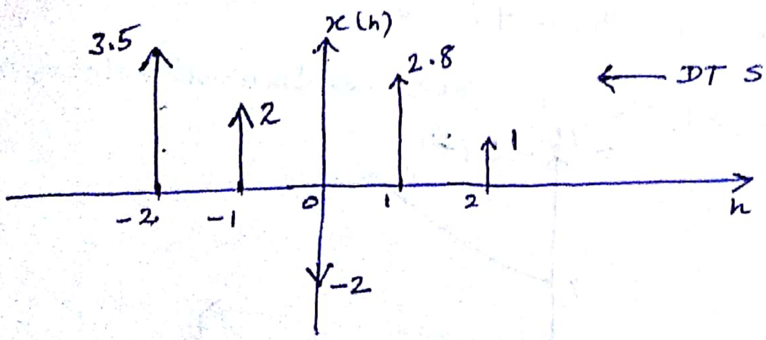


← DT signal

ii) $x(n) = \{ 3.5, 2, \underset{\uparrow}{-2}, 2.8, 1 \}$

n=0

Ans:

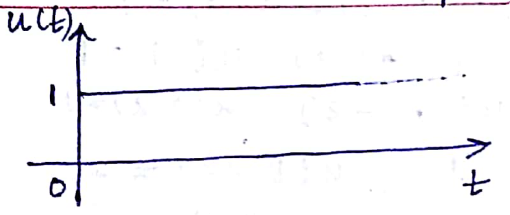


← DT signal

Unit 1 - Topic-1 : Lesson 02 (a) - Basic Signals.

i) **Unit step signal:** [Unit → The signal has an amplitude of unity].

a) **Continuous Time (CT) step signal :**



From the graph,

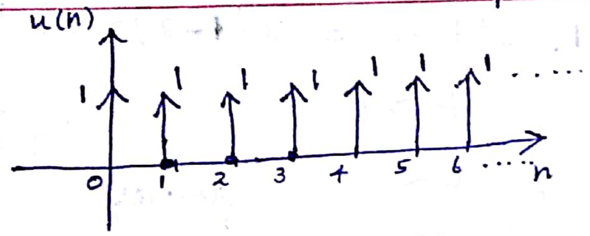
$$u(t) = \begin{cases} 1 & ; t \geq 0 \\ 0 & ; t < 0 \end{cases}$$

We understand that $u(0) = u(0.1) = u(0.2) \ldots\ldots = u(\infty) = 1$. Also, $u(-\infty) = \ldots\ldots = u(-0.2) = u(-0.1) = 0$.

't' → CT independent variable

't' → 0 to ∞

b) **Discrete Time (DT) step signal :**



From the graph

$$u(n) = \begin{cases} 1 & ; n \geq 0 \\ 0 & ; n < 0 \end{cases}$$

n → DT independent variable

'n' → 0 to ∞ [it should be an integer : 0, 1, 2, 3, ... ∞]

We understand that $u(0) = u(1) = u(2) \ldots = u(\infty) = 1$. Also, $u(-\infty) = \ldots\ldots = u(-2) = u(-1) = 0$.

Understanding purpose:

Sketch the following signals.

1. $u(t-2)$:

Soln:

Identify at what value of 't', $t-2 = 0$? we know that

At $t = 2$, $t - 2 = 2 - 2 = 0$. ∴ The given signal

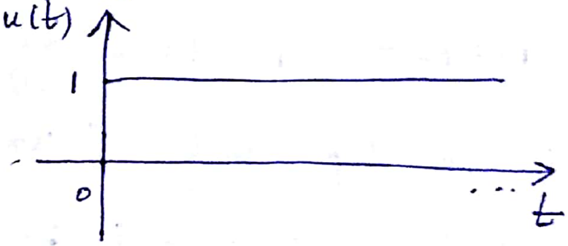
step signal starts at

starts at $t = 2$. Because of

$t = 0$. i.e, $u(t) = u(0) = 1$.

The given signal is CT step signal.

$u(t) = \begin{cases} 1 & ; t \geq 0 \\ 0 & ; t < 0 \end{cases}$



When $t = 2$; $u(t-2) = u(2-2) = u(0) = 1$

When $t = 2.1$; $u(t-2) = u(2.1-2) = u(0.1) = 1$

when $t = 2.2$; $u(t-2) = u(2.2-2) = u(0.2) = 1$

For negative values of 't', $u(t-2) = 0$.

Because, W.K.T, $u(-\infty) = \dots = u(-1) = 0$.

For example:

when $t = -2$; $u(t-2) = u(-2-2) = u(-4) = 0$.

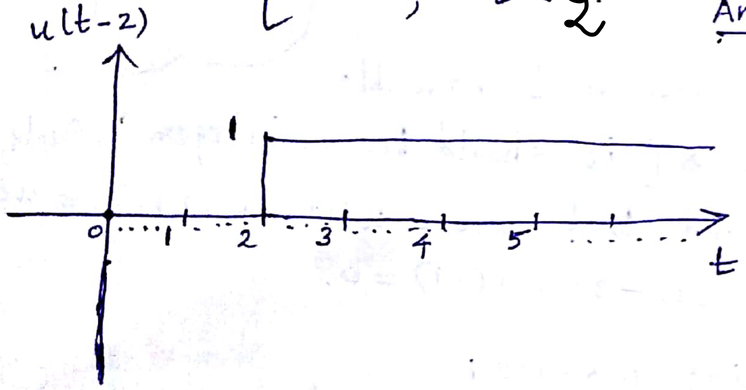
More over, $t < 2$, $u(t-2) = 0$.

Example: when $t = +1$, $u(t-2) = u(1-2) = u(-1) = 0$.

Now,

$u(t-2) = \begin{cases} 1 & ; t \geq 2 \\ 0 & ; t < 2 \end{cases}$

Answer:

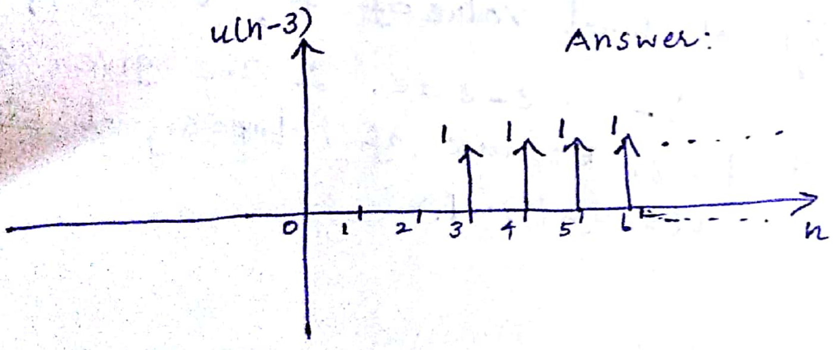


2) .... $u(n-3)$:

Soln:

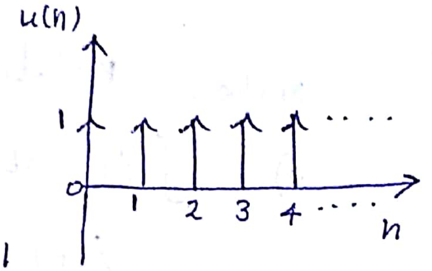
$u(n-3) = \begin{cases} 1 & ; n \geq 3 \\ 0 & ; n < 3. \end{cases}$

Answer:



Explanation:

W.K.T $u(n) = \begin{cases} 1 & ; n \geq 0 \\ 0 & ; n < 0 \end{cases}$



$u(0) = u(1) = u(2) = \dots = u(\infty) = 1$

n → should be an integer [DT signal].

When $n = 3$, $u(n-3) = u(3-3) = u(0) = 1$

When $n = 4$, $u(n-3) = u(4-3) = u(1) = 1$

So, starts at $n = 3$.

When $n < 3$, $u(n-3) = 0$.

Example : $n = 2$, $u(n-3) = u(2-3) = u(-1) = 0$.

$n = 1$, $u(n-3) = u(1-3) = u(-2) = 0$

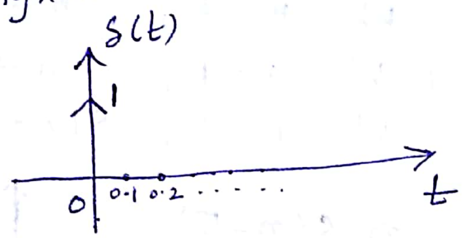
$n = 0$; $u(n-3) = u(0-3) = u(-3) = 0$

$n = -1$; $u(n-3) = u(-1-3) = u(-4) = 0$.

ii) Unit Impulse Signal (or) Impulse function

a) Continuous Time (CT) impulse signal :

$\delta(t) = \begin{cases} 1 & ; t = 0 \\ 0 & ; t \neq 0 \end{cases}$



From the graph, $\delta(0) = 1$,

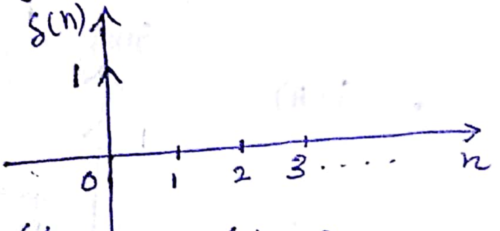
$\delta(0.1) = \delta(0.2) = \dots = \delta(\infty) = 0$. Also,

$\delta(-\infty) = \dots = \delta(-0.2) = \delta(-0.1) = 0$.

δ → del

b) Discrete Time (DT) impulse signal :

$\delta(n) = \begin{cases} 1 & ; n = 0 \\ 0 & ; n \neq 0 \end{cases}$



From the graph, $\delta(0) = 1$,

$\delta(-\infty) = \dots = \delta(-2) = \delta(-1) = \delta(1) = \delta(2) = \dots = \delta(+\infty) = 0$

Note: Relation between $\delta(t)$ & $u(t)$ and $\delta(n)$ & $u(n)$:

1) $\int \delta(t)\, dt = u(t)$

$$\delta(t) = \frac{d}{dt} u(t)$$

2) $\int_{-\infty}^{\infty} \delta(t)\, dt = 1$

3) $\delta(n) = u(n) - u(n-1)$

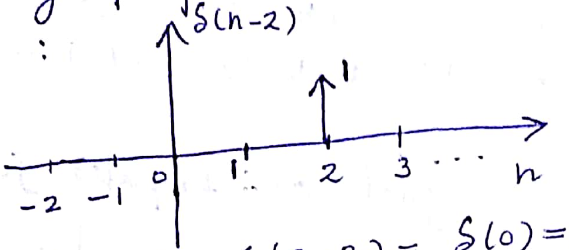
4) $u(n) = \sum_{k=0}^{\infty} \delta(n-k)$

$\Sigma \rightarrow$ Summation

$\delta(n-k) \rightarrow$ delayed unit impulse.

Understanding purpose:

1) $\delta(n-2)$:



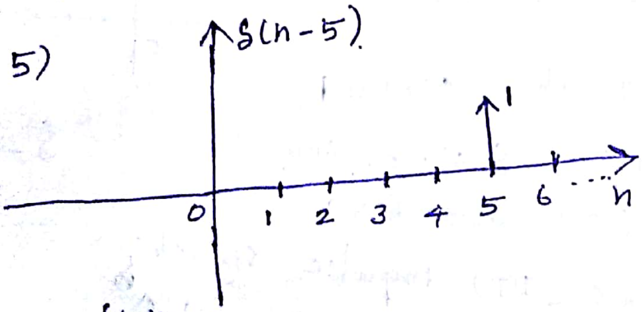
Explanation:

When $n = 2$, $\delta(n-2) = \delta(2-2) = \delta(0) = 1$

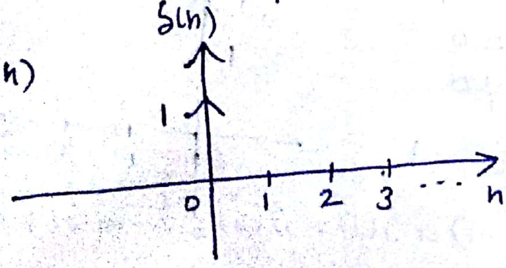
When $n = 1$, $\delta(n-2) = \delta(1-2) = \delta(-1) = 0$

When $n = 3$, $\delta(n-3) = \delta(3-2) = \delta(1) = 0$.

2) $\delta(n-5)$



3) $\delta(n)$

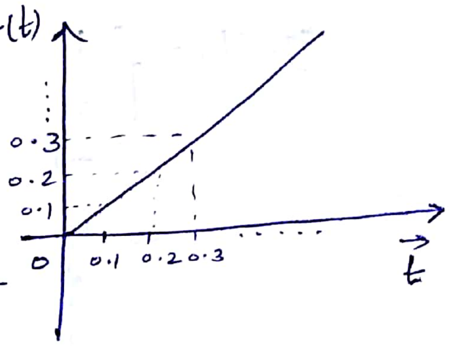


Unit 1 - Topic 1 : Lesson 02 (b) - Basic signals.

iii) Unit Ramp signal

a) Continuous Time (CT) Ramp signal :

$$r(t) = \begin{cases} t & ; \ t \geq 0 \\ 0 & ; \ t < 0 \end{cases}$$



Explanation : understanding purpose

$r(t) = t$

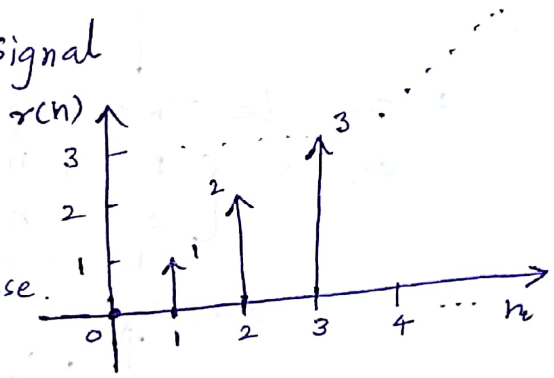
$t=0, \ r(t) = r(0) = 0.$

$t=0.1, \ r(0.1) = 0.1$

$t=0.2, \ r(0.2) = 0.2$

b) Discrete Time (DT) Ramp Signal

$$r(n) = \begin{cases} n & ; \ n \geq 0 \\ 0 & ; \ n < 0 \end{cases}$$



Explanation : Understanding purpose.

$r(n) = n$

$n=0, \ r(0) = 0$

$n=1, \ r(1) = 1$

$n=2, \ r(2) = 2.$

Note : Relation between $\delta(t)$, $u(t)$ and $r(t)$.

1) $u(t) = \frac{d}{dt} r(t)$

2) $r(t) = \int_{-\infty}^{\infty} u(t)\,dt$

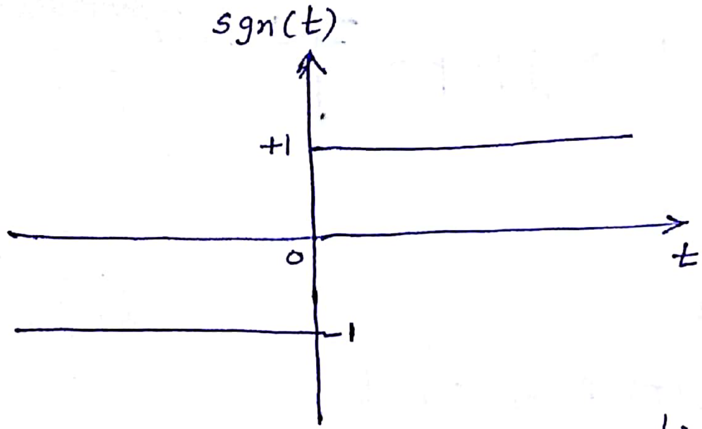
3) $r(t) = \int u(t)\,dt = \int\int \delta(t)\,dt \qquad \left[\because u(t) = \int \delta(t)\,dt \right]$

4) $\delta(t) = \frac{d}{dt} u(t) = \frac{d^2}{dt^2} r(t) \qquad \left[\because u(t) = \frac{d}{dt} r(t) \right]$

Note : Signum function:

a) Continuous time signum function

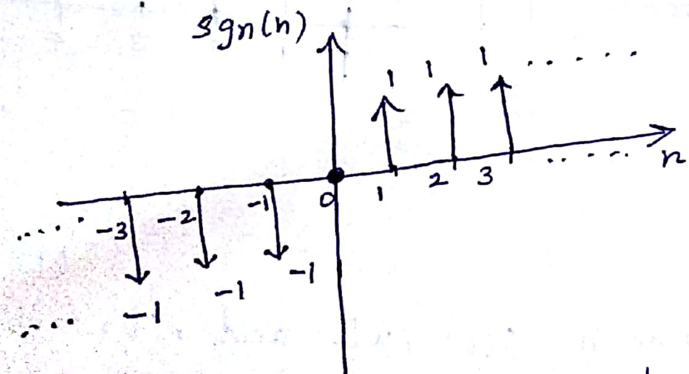
$$sgn(t) = \begin{cases} 1 & ; t > 0 \\ 0 & ; t = 0 \\ -1 & ; t < 0 \end{cases}$$



[At time $t = 0$, Magnitude of the given signal is average of +1 and -1] i.e, $\frac{1-1}{2} = 0$.

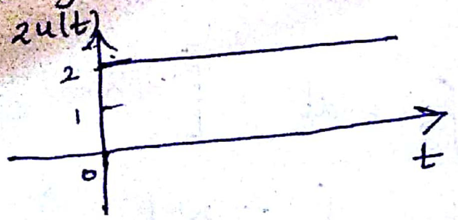
b) Discrete Time signum function

$$sgn(n) = \begin{cases} 1 & ; n > 0 \\ 0 & ; n = 0 \\ -1 & ; n < 0 \end{cases}$$

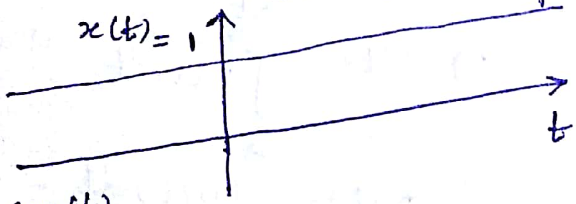


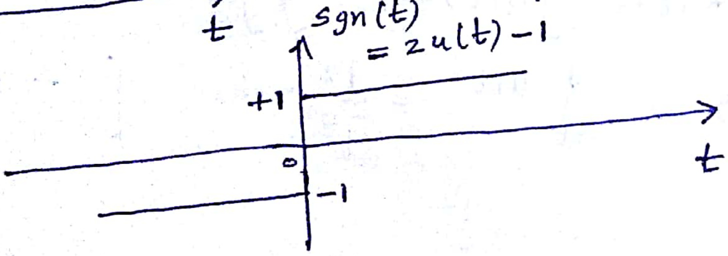
* Relation between u(t) and sgn(t) :

$$sgn(t) = 2u(t) - 1$$



DC signal [constant amplitude]





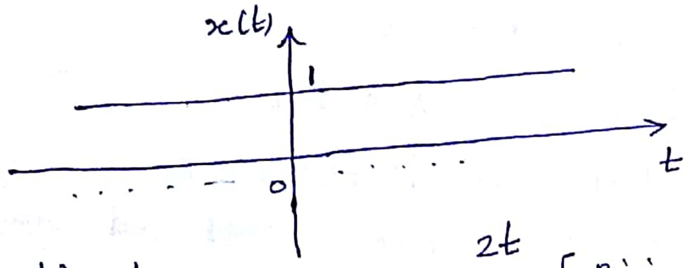
Unit 1 - Topic : 1 - Lesson 03 Basic signals

iv) Real Exponential Signal

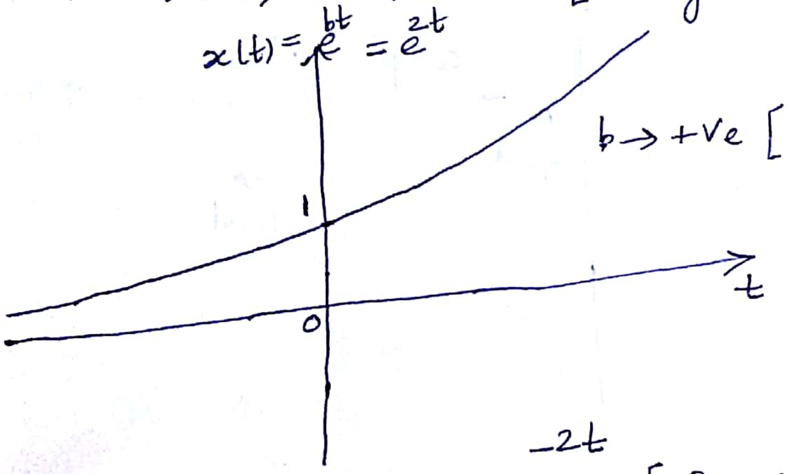
The signal is in the form $x(t) = A e^{bt}$. The shape of exponential signal depends upon 'b'.

A → amplitude (or) magnitude. [Assume : A = 1]

case : a) $b = 0 \Rightarrow x(t) = A \cdot e^{0} = A \cdot 1 = A = 1$



case : b) $b > 1 \Rightarrow x(t) = e^{2t}$ [Rising exponential]

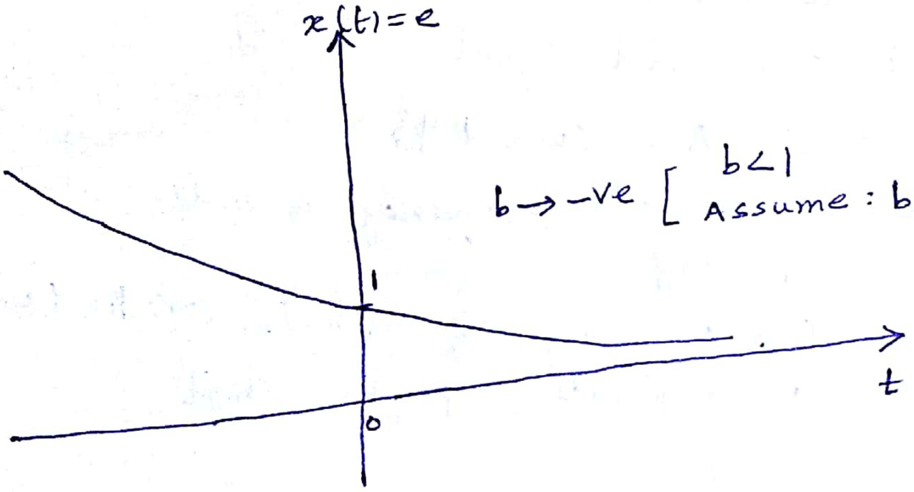


$x(t) = e^{bt} = e^{2t}$

b > 1 Assume

b → +ve [b = 2]

case : c) $b < 1 \Rightarrow x(t) = e^{-2t}$ [Decaying exponential]



$x(t) = e^{-2t}$

b → -ve [b < 1 Assume : b = -2]

v) Complex Exponential signal :

$$x(t) = A\,e^{j\omega_0 t}$$

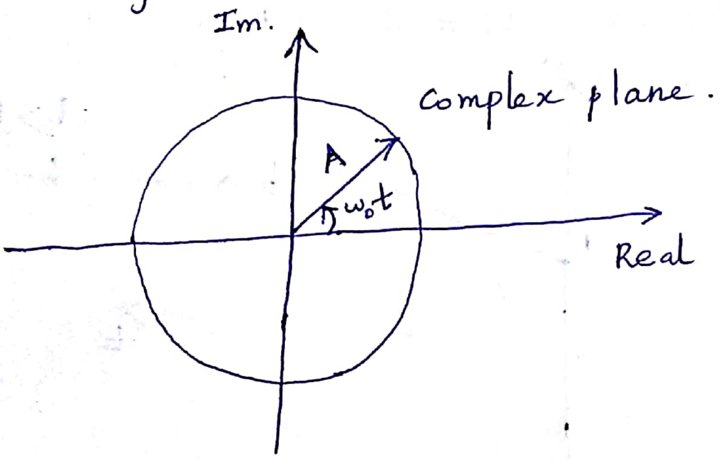
$$x(t) = A\left[\cos\omega_0 t + j\sin\omega_0 t\right]$$

$$x(t) = A\cos\omega_0 t + jA\sin\omega_0 t$$

Real part of $x(t) = A\cos\omega_0 t$

Imaginary part of $x(t) = A\sin\omega_0 t$

Complex exponential signal is the vector sum of two sinusoidal signals of the form $\cos\omega_0 t$ and $\sin\omega_0 t$.



vi) Sinusoidal signal :

case (i) cosinusoidal signal

$$x(t) = A\cos(\omega_0 t \pm \phi)$$

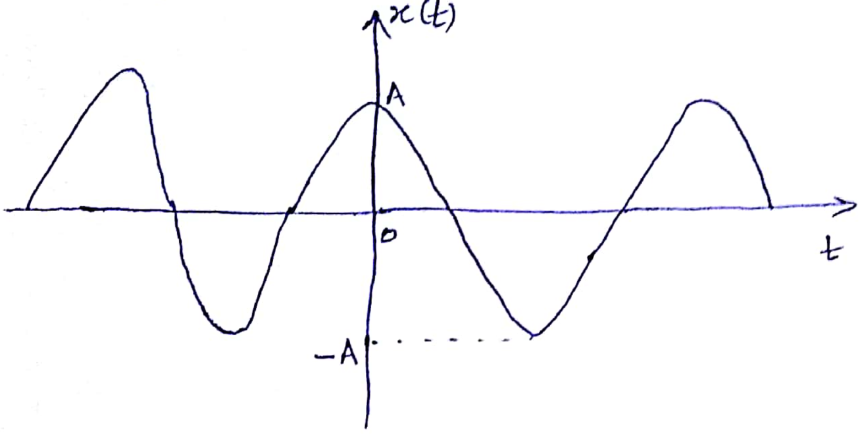
where, ω_0 → angular frequency in rad/sec

$\omega_0 = 2\pi f_0$

f_0 → frequency in cycles/sec (or) Hz. (Hertz)

A → amplitude, ϕ = phase shift.

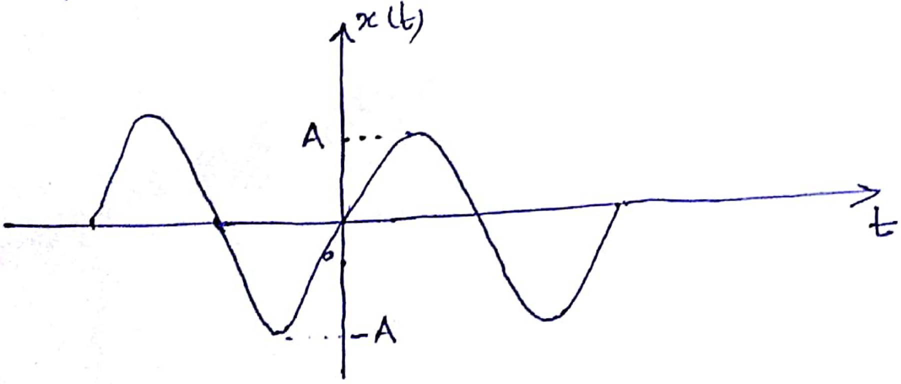
when $\phi = 0$, $x(t) = A \cos \omega_0 t$



case ii) sine wave

$x(t) = A \sin(\omega_0 t \pm \phi)$.

when $\phi = 0$, $x(t) = A \sin \omega_0 t$



Note: $\omega_0 = 2\pi f_0$

$f_0 \rightarrow$ fundamental Frequency.

$$f_0 = \frac{\omega_0}{2\pi}$$

Fundamental Period : $T = \frac{1}{f_0} = \frac{2\pi}{\omega_0}$.

Note : 1) $x(t) = a + jb$

magnitude of $x(t) = |x(t)| = |a+jb|$

$$|x(t)| = \sqrt{a^2+b^2}.$$

Example : $|2+5j| = \sqrt{2^2+5^2} = \sqrt{4+25} = \sqrt{29}.$

2) $x(t) = \exp[a-jb] = e^{a-jb}.$

$$|x(t)| = |e^{a-jb}| = |e^{a}| \cdot |e^{-jb}|.$$

$$= e^{a} \cdot |\cos b - j\sin b|$$

$$= e^{a} \cdot \sqrt{(\cos b)^2 + (-\sin b)^2}$$

$$= e^{a} \cdot \sqrt{\cos^2 b + \sin^2 b} = e^{a} \cdot \sqrt{1}$$

$$= e^{a}.$$

Topic : 2

Continuous Time (CT) and Discrete Time (DT)

classification of signals:

a) Deterministic and non-deterministic (random) signals.
b) Periodic and non-periodic (aperiodic) signals
c) odd and Even signals.
d) Power and energy signals.

a) Deterministic and Random signals:

A signal is said to be deterministic, if it can be characterized (or) specified (or) represented by mathematical equation at any instant of time (or) specified signal values at any time. No uncertainty to define

Example: $x(t) = 5$, $x(t) = 2\cos wt$

A signal is said to be random, if it cannot be characterized (or) specified (or) represented by mathemical equation at any time.

Example: Noise in an electronic circuit, seismic signal.

b) Periodic and Aperiodic signals:

If the signal value repeats at regular intervals, it is called as periodic signal.

Example: $x(t) = \cos wt$, $x(t) = \sin wt$, $x(t) = e^{jwt}$

If the signal value does not repeat at regular intervals, it is called as aperiodic signal.

Example : $x(t) = e^{3t}$, $x(t) = 5r(t)$

condition for periodicity:

→ CT signal:

$$\boxed{x(t+T) = x(t)} \longrightarrow \text{periodic signal}$$

Where, T = fundamental Period

Fundamental frequency : $f_0 = \frac{1}{T}$

$$T = \frac{1}{f_0}$$

$$\left[\omega_0 = 2\pi f_0 \right]$$
$$f_0 = \frac{\omega_0}{2\pi}$$

$$\boxed{T = \frac{2\pi}{\omega_0}}$$

& $[x(t+T)] \neq x(t) \longrightarrow$ Aperiodic signal.

→ DT signal:

$\boxed{x(n+N) = x(n)} \rightarrow$ For Periodic signal

where, $N \rightarrow$ fundamental period

$f_0 \rightarrow$ fundamental frequency

$$N = \frac{1}{f_0}$$

$$\left[\omega_0 = 2\pi f_0 \right.$$
$$\left. f_0 = \frac{\omega_0}{2\pi} \right]$$

$$\boxed{N = \frac{2\pi}{\omega_0}}$$

$\boxed{x(n+N) \neq x(n)} \rightarrow$ For Aperiodic signal.

Proof: a) $x(t) = 5 \cos \omega_0 t$.

$$x(t+T) = 5 \cos \omega_0 (t+T)$$
$$= 5 \cos [\omega_0 t + \omega_0 T]$$
$$= 5 \cos [\omega_0 t + \omega_0 \times \frac{2\pi}{\omega_0}]$$
$$= 5 \cos [\omega_0 t + 2\pi]$$
$$= 5 \cos \omega_0 t$$

$$\cos [2\pi + \theta] = \cos\theta$$

$$\boxed{x(t+T) = x(t)}$$

Hence, $x(t) = 5 \cos \omega_0 t$ is periodic signal.

b) Complex Exponential signal:

$$x(t) = 2e^{j\omega_0 t}$$

$$x(t+T) = 2e^{j\omega_0 (t+T)} = 2e^{j\omega_0 t} \cdot e^{j\omega_0 T} = 2e^{j\omega_0 t} \cdot e^{j\omega_0 \times \frac{2\pi}{\omega_0}}$$

$$x(t+T) = 2e^{j\omega_0 t} \cdot e^{j2\pi} = 2e^{j\omega_0 t} [\cancelto{1}{\cos 2\pi} + j \cancelto{0}{\sin 2\pi}]$$

$$x(t+T) = 2e^{j\omega_0 t} \Rightarrow \boxed{x(t+T) = x(t)}$$

Hence, $x(t) = 2e^{j\omega_0 t}$ is periodic signal.

Exercise : 1.25 TB→1 Page No: 61

Problems:

I Determine whether or not each of the following continuous-time signal is periodic. If the signal is periodic, determine, its fundamental period.

a) $x(t) = 3\cos\left(4t + \frac{\pi}{3}\right)$

Here, $\omega_0 = 4$

$x(t+T) = 3\cos\left[4(t+T) + \frac{\pi}{3}\right]$

$= 3\cos\left[4t + 4T + \frac{\pi}{3}\right]$

$= 3\cos\left[4t + 4 \times \frac{2\pi}{\omega_0} + \frac{\pi}{3}\right]$

$= 3\cos\left[4t + \cancel{4} \times \frac{2\pi}{\cancel{4}} + \frac{\pi}{3}\right] = 3\cos\left[2\pi + 4t + \frac{\pi}{3}\right]$

$= 3\cos\left[4t + \frac{\pi}{3}\right] = x(t).$ $\left[\because \cos(360+\theta) = \cos\theta\right]$ 4^{th} quadrant

Hence, the given signal is periodic.

Fundamental period : $T = \frac{2\pi}{\omega_0} = \frac{2\pi}{4} = \frac{\pi}{2}$

b) $x(t) = e^{j(\pi t - 1)}$

$x(t) = e^{j\pi t} \times e^{-j}$

$e^{-j} \rightarrow$ constant. & Here, $\omega_0 = \pi$

$x(t+T) = e^{j\pi(t+T)}$

$= e^{j\pi\left(t + \frac{2\pi}{\omega_0}\right)} = e^{j\pi\left[t + \frac{2\pi}{\pi}\right]}$

$= e^{j\pi t} \cdot \left[e^{j2\pi}\right] = e^{j\pi t} = x(t)$

Hence, the given signal is periodic signal.

Fundamental period: $T = \frac{2\pi}{\omega_0} = \frac{2\pi}{\pi} = 2$

c) $x(t) = e^{(j\pi - 2)t}$

$x(t) = e^{j\pi t} \times e^{-2t}$

e^{-2t} ↓ Real exponential [Aperiodic]

$e^{j\pi t}$ ↓ complex exponential [Peridic] ⇒ Fundamental Period:

$$T_1 = \frac{2\pi}{\omega_0} = \frac{2\pi}{\pi} = 2.$$

Composite signal = Periodic × Aperiodic = Aperiodic signal.

Note:

$x_1(t) + x_2(t) = x(t)$

$x(t)$ is periodic if and only if $x_1(t)$ and $x_2(t)$ is periodic and ratio of T_1 and T_2 is a rational number.

$\frac{T_1}{T_2}$ = rational number.

Fundamental Period is obtained by following concept:

To find LCM: First convert T_1 and T_2 to integers by multiplying by a common number. Find LCM of Integer values of T_1 and T_2. Then divide this LCM by the common number.

d) $x(t) = 2\cos\frac{2\pi t}{3} + 3\cos\frac{2\pi t}{7}$

$$T_1 = \frac{2\pi}{\omega_{01}} = \frac{2\pi}{2\pi/3} = 3$$

$$T_2 = \frac{2\pi}{\omega_{02}} = \frac{2\pi}{2\pi/7} = 7$$

$\frac{T_1}{T_2} = \frac{3}{7}$ [rational]

∴ The given signal $x(t)$ is periodic.

Fundamental Period = LCM of T_1 & T_2

= LCM of 3 and 7

$$\boxed{T = 21}$$

e) $x(t) = 2\cos 3t + 3\sin 7t$.

$$T_1 = \frac{2\pi}{3}$$

$$T_2 = \frac{2\pi}{7}$$

$$\frac{T_1}{T_2} = \frac{\frac{2\pi}{3}}{\frac{2\pi}{7}} = \frac{7}{3} \text{ [rational]}$$

∴ The given signal $x(t)$ is periodic.

$$T_1 = \frac{2\pi}{3} \times \frac{21}{2\pi} = 7$$

$$T_2 = \frac{2\pi}{7} \times \frac{21}{2\pi} = 3$$

LCM of 7 and 3 is 21.

$$\text{Fundamental period} = \frac{21}{\frac{21}{2\pi}} = 2\pi$$

f) $x(t) = 2\cos 3\pi t + 7\cos 9t$.

$$T_1 = \frac{2\pi}{3\pi} = \frac{2}{3}$$

$$T_2 = \frac{2\pi}{9}$$

$$\frac{T_1}{T_2} = \frac{\frac{2}{3}}{\frac{2\pi}{9}} = \frac{2}{3} \times \frac{9}{2\pi} = \frac{3}{\pi} \text{ [not rational]}$$

∴ The given signal is aperiodic.

g) $x(t) = 5\cos 4\pi t + 3\sin 8\pi t$

$$T_1 = \frac{2\pi}{4\pi} = \frac{1}{2}$$

$$T_2 = \frac{2\pi}{8\pi} = \frac{1}{4}$$

$$\frac{T_1}{T_2} = \frac{\frac{1}{2}}{\frac{1}{4}} = 2 \text{ [rational]}$$

∴ The given signal periodic signal.

$$T_1 = \frac{1}{2} \times 4 = 2$$

$$T_2 = \frac{1}{4} \times 4 = 1$$

LCM of 2 and 1 is 2.

Fundamental Period $= \frac{2}{4} = \frac{1}{2}$.

e). $x(n) = \cos\frac{n}{8} \cdot \cos\frac{n\pi}{8}$

$x(n) = x_1(n) \cdot x_2(n)$ → composite signal.

$$N = \left(\frac{2\pi}{\omega_0}\right)m$$

$$N_1 = \left(\frac{2\pi}{\omega_{01}}\right)m = \left(\frac{2\pi}{\frac{1}{8}}\right)m = (16\pi)m$$

For any value of m; 'N₁' should not be an integer

$$N_2 = \left(\frac{2\pi}{\omega_{02}}\right)m = \left(\frac{2\pi}{\frac{\pi}{8}}\right)1 = 16$$

Now, $\frac{N_1}{N_2} = \frac{16\pi}{16} = \pi$ [not rational].

∴ The given composite signal is aperiodic.

f) $x(n) = \cos\left(\frac{\pi}{2}n\right) \cdot \cos\left(\frac{\pi}{4}n\right)$

$x(n) = x_1(n) \cdot x_2(n)$ → composite signal.

$$N_1 = \left(\frac{2\pi}{\omega_{01}}\right)m = \left(\frac{2\pi}{\pi/2}\right)1 = 4$$

$$N_2 = \left(\frac{2\pi}{\omega_{02}}\right)m = \left(\frac{2\pi}{\pi/4}\right)1 = 8$$

Then,

$$\frac{N_1}{N_2} = \frac{4}{8} = \frac{1}{2} \text{ [rational]}$$

∴ The given signal $x(n)$ is periodic.

Fundamental Period = LCM of 4 and 8

$= 8$

2 | 4, 8
2 | 2, 4
2 | 1, 2
   1, 1

g) $x(n) = \sin\left(\frac{6\pi}{7}n + 1\right)$

Soln:

W.K.T sinusoidal signal is periodic.

Fundamental period

$$N = \left(\frac{2\pi}{\omega_0}\right)m = \left(\frac{2\pi}{6\pi/7}\right)m = \left(2\pi \times \frac{7}{6\pi}\right)m = \left(\frac{7}{3}\right)m.$$

The smallest value of 'm' for which 'N' is an integer is '3'. i.e, $N = \left(\frac{7}{3}\right) \times 3 = 7$.

∴ $\boxed{N = 7}$

h) $x(n) = \cos\left(\frac{n}{8} - \pi\right)$

Soln:

W.K.T $\omega_0 = \frac{1}{8}$

$$N = \left(\frac{2\pi}{\omega_0}\right)m = \left(\frac{2\pi}{1/8}\right)m = \cancel{\left(\frac{\pi}{4}\right)m}. \quad 16\pi m$$

For all values of 'm', N is not an integer.

∴ The given signal is aperiodic.

i) $x(n) = \cos(\frac{\pi}{8} n^2)$.

$\omega_0 = \frac{n\pi}{8}$

$N = \left(\frac{2\pi}{\omega_0}\right) m = \left(\frac{2\pi}{\pi n/8}\right) m = 16 \frac{m}{n}$

minimum value of 'n' = 2 & $N = \left(\frac{16}{2}\right) m = 8m$.

The smallest value of 'm' for which 'N' is an integer is '1'.

$N = 8(1) = 8$

$\boxed{N = 8}$

j) $x(n) = \cos(\frac{\pi}{2} n) \cdot \cos(\frac{\pi}{4}) n$.

$x(n) = x_1(n) \cdot x_2(n) \longrightarrow$ composite signal.

$N_1 = \left(\frac{2\pi}{\omega_{01}}\right) m = \left(\frac{2\pi}{\frac{\pi}{2}}\right) m = 4m$

$N_1 = 4(1) = 4$

$N_2 = \left(\frac{2\pi}{\omega_{02}}\right) m = \left(\frac{2\pi}{\pi/4}\right) m = 8m$

$N_2 = 8(1) = 8$

$\frac{N_1}{N_2} = \frac{4}{8} = \frac{1}{2}$ (rational)

The given composite signal is periodic.

Fundamental period = LCM of 4 and 8 = 8.

k) $x(n) = 2\cos\left(\frac{\pi}{4}n\right) + \sin\left(\frac{\pi}{8}n\right) - 2\cos\left(\frac{\pi}{2}n + \frac{\pi}{6}\right).$

Soln:

$$N_1 = \left(\frac{2\pi}{\omega_{01}}\right)m = \left(\frac{2\pi}{\pi/4}\right)m = 8m = 8(1) = 8$$

$$N_2 = \left(\frac{2\pi}{\omega_{02}}\right)m = \left(\frac{2\pi}{\pi/8}\right)m = 16m = 16(1) = 16$$

$$N_3 = \left(\frac{2\pi}{\omega_{03}}\right)m = \left(\frac{2\pi}{\pi/2}\right)m = 4m = 4(1) = 4.$$

$$\frac{N_1}{N_2} = \frac{8}{16} = \frac{1}{2} \text{ (rational)}$$

$$\frac{N_1}{N_3} = \frac{8}{4} = 2 \text{ (rational)}$$

$$\begin{array}{r|l} 2 & 8, 16, 4 \\ 2 & 4, 8, 2 \\ 2 & 2, 4, 1 \\ 2 & 1, 2, 1 \\ & 1, 1, 1. \end{array}$$

∴ The given signal is periodic.

Fundamental period = LCM of 8, 16, 4 = 16.

l) $x(t) = \left[\cos\left(2t - \frac{\pi}{3}\right)\right]^2$

Soln:

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$x(t) = \frac{1 + \cos 2\left(2t - \frac{\pi}{3}\right)}{2}$$

$$x(t) = \underbrace{\frac{1}{2}}_{\text{DC signal}} + \underbrace{\frac{1}{2}\cos\left(4t - \frac{2\pi}{3}\right)}_{\text{ac signal}}.$$

ac signal → cosine signal

∴ Periodic signal.

$$T = \frac{2\pi}{\omega_0} = \frac{2\pi}{4} = \frac{\pi}{2}$$

Fundamental Period $= T = \frac{\pi}{2}$

s) $x(n) = 3\,e^{j\frac{3\pi(n+\frac{1}{2})}{5}}$

Soln:

$x(n) = 3 \times e^{j\frac{3\pi}{5}n} \times e^{j\frac{3\pi}{10}}$

↳ constant

↳ Complex exponential (Periodic)

Fundamental Period = $N = \left(\frac{2\pi}{\omega_0}\right)m = \left(\frac{2\pi}{\frac{3\pi}{5}}\right)m$

$N = \left(\frac{10}{3}\right)m.$

The smallest value of 'm' for which 'N' is an integer is '3'.

$N = \left(\frac{10}{3}\right)3 = 10.$

t) $x(n) = 3\,e^{j\frac{3}{5}(n+\frac{1}{2})}$

Soln:

$x(n) = 3\,e^{j\frac{3}{5}n} \times e^{j\frac{3}{10}}$

↳ constant.

↳ complex exponential.

$N = \left(\frac{2\pi}{\omega_0}\right)m = \left(\frac{2\pi}{3/5}\right)m = \left(\frac{10}{3}\pi\right)m.$

We cannot find any integer 'm' such that $\left(\frac{10}{3}\pi\right)m$ is also an integer.

∴ The given signal $x(n)$ is aperiodic.

iv) Determine the fundamental period of the signal $x(t) = 2\cos(10t+1) - \sin(4t-1)$.

soln:

$$x(t) = 2\cos(10t+1) - \sin(4t-1)$$

$$T_1 = \frac{2\pi}{\omega_{01}} = \frac{2\pi}{10} = \frac{\pi}{5}$$

$$T_2 = \frac{2\pi}{\omega_{02}} = \frac{2\pi}{4} = \frac{\pi}{2}$$

$$\frac{T_1}{T_2} = \frac{\frac{\pi}{5}}{\frac{\pi}{2}} = \frac{\pi}{5} \times \frac{2}{\pi} = \frac{2}{5} \text{ (rational)}$$

∴ The given signal $x(t)$ is periodic signal.

$$T_1 = \frac{\pi}{5} \times \frac{10}{\pi} = 2$$

$$T_2 = \frac{\pi}{2} \times \frac{10}{\pi} = 5$$

LCM of T_1 and T_2 = $2 \times 5 = 10$

Fundamental Period = $\frac{10}{10/\pi} = \pi$.

v) Determine the fundamental period of the signal $x(n) = 1 + e^{j\frac{4\pi n}{7}} - e^{j\frac{2\pi n}{5}}$.

soln:

$$x(n) = \underset{(DC)}{1} + e^{j\frac{4\pi n}{7}} - e^{j\frac{2\pi n}{5}}$$

$$T_1 = \left(\frac{2\pi}{\omega_{01}}\right)m = \left(\frac{2\pi}{\frac{4\pi}{7}}\right)m = \left(\frac{14}{4}\right)m = \left(\frac{7}{2}\right)m = \left(\frac{7}{2}\right)\times 2 = 7$$

$$T_2 = \frac{2\pi}{\omega_{02}} = \left(\frac{2\pi}{\frac{2\pi}{5}}\right) = \left(\frac{10}{2}\right)m = 5 \times 1 = 5$$

$$\frac{T_1}{T_2} = \frac{7}{5} \text{ (rational)}$$

The given signal is periodic.

Fundamental period = $7 \times 5 = 35$.

Previous year university Questions with Answers:

1) $x(t) = \sin^2 t$

$$x(t) = \frac{1-\cos 2t}{2} = \underset{(dc)}{\frac{1}{2}} - \frac{1}{2}\cos 2t.$$

cosine signal is periodic.
Fundamental period $= T = \frac{2\pi}{\omega_0} = \frac{2\pi}{2} = \pi$

2) Show that the complex exponential signal $x(t) = e^{j\omega_0 t}$ is periodic and that the Fundamental period is $\frac{2\pi}{\omega_0}$.

Soln:

$$x(t) = e^{j\omega_0 t}.$$

condition for periodicity. $x(t+T) = x(t)$

$$x(t+T) = e^{j\omega_0(t+T)}$$

$$= e^{j\omega_0 t} \times e^{j\omega_0 T}$$

$$= e^{j\omega_0 t} \times e^{j\omega_0 \times \frac{2\pi}{\omega_0}}$$

$$= e^{j\omega_0 t} \times e^{j2\pi}$$

$$= e^{j\omega_0 t}\left[\overset{1}{\cancel{\cos 2\pi}} + j\overset{0}{\cancel{\sin 2\pi}}\right]$$

$$= e^{j\omega_0 t}$$

$$\boxed{x(t+T) = x(t)}$$

Hence proved. Thus the fundamental period of $e^{j\omega_0 t}$ is $\frac{2\pi}{\omega_0}$.

Energy and Power Signals:

The terms signal energy and power are used to characterize a signal.

→ **Energy signal:**

A signal is said to be energy signal, if it has finite energy and zero power. i.e, $0 < E < \infty$ & $P = 0$.

→ **Power signal:**

A signal is said to be power signal, if it has finite power and infinite energy. i.e., $0 < P < \infty$ & $E = \infty$.

Formulas:

* **Energy:**

For continuous Time signal, $E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$.

For Discrete Time signal, $E = \sum_{n=-\infty}^{\infty} |x(n)|^2$

* **Power:**

For continuous Time signal, $P = \lim_{T \to \infty} \left[\frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt \right]$

For Discrete Time signal, $P = \lim_{N \to \infty} \left[\frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2 \right]$

Note:

$E = 0$ & Power = finite
E = finite & $P = \infty$
$E = \infty$ & $P = \infty$
} Neither Energy nor Power signal.

Problems:

1) Find the whether the signal is an energy signal or power signal:

$x(t) = e^{-2t} u(t).$

Soln:

$$x(t) = e^{-2t} u(t)$$

$$\text{Energy } (E) = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

$$= \int_{0}^{\infty} |e^{-2t}|^2 \, dt \qquad \left[u(t) = \begin{cases} 1 & ; t \geq 0 \\ 0 & ; t < 0 \end{cases} \right]$$

$$= \int_{0}^{\infty} e^{-4t} \, dt = \left[\frac{e^{-4t}}{-4} \right]_0^{\infty}$$

$$= \frac{-1}{4} \left[\cancelto{0}{e^{-\infty}} - e^{0} \right] = -\frac{1}{4}(-1)$$

$$E = \frac{1}{4} J \, [\text{Finite}]$$

J → Joules.

$$\text{Power } (P) = \underset{T \to \infty}{Lt} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt$$

$$= \underset{T \to \infty}{Lt} \frac{1}{2T} \int_{0}^{T} |e^{-2t}|^2 \, dt = \underset{T \to \infty}{Lt} \frac{1}{2T} \left[\frac{e^{-4t}}{-4} \right]_0^{T}$$

$$= \underset{T \to \infty}{Lt} \frac{1}{2T} \times \left(\frac{1}{-4} \right) \left[e^{-4T} - e^{0} \right]$$

$$= \frac{1}{2(\infty)} \left(\frac{1}{-4} \right) \left[\cancelto{0}{e^{-\infty}} - e^{0} \right]$$

$$\boxed{P = 0} \qquad \left[\because \frac{1}{\infty} = 0 \right]$$

∴ The given signal is energy signal.

2) $x(t) = t\ u(t)$

Soln:

$$E = \int_{-\infty}^{\infty} |x(t)|^2\, dt$$

$$= \int_{0}^{\infty} t^2\, dt \qquad \left[u(t) = \begin{cases} 1 & ; t \geq 0 \\ 0 & ; t < 0 \end{cases} \right]$$

$$= \left[\frac{t^3}{3}\right]_0^{\infty} = \frac{1}{3}[\infty - 0]$$

$$\boxed{E = \infty}$$

$$P = \underset{T\to\infty}{Lt} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2\, dt$$

$$= \underset{T\to\infty}{Lt} \frac{1}{2T} \int_{0}^{T} t^2\, dt = \underset{T\to\infty}{Lt} \frac{1}{2T} \left[\frac{t^3}{3}\right]_0^T$$

$$P = \underset{T\to\infty}{Lt} \frac{1}{2T} \times \frac{1}{3}[T^3 - 0]$$

$$P = \underset{T\to\infty}{Lt} \frac{1}{6T} \times T^3 = \underset{T\to\infty}{Lt} \frac{T^2}{6} = \frac{\infty}{6}$$

$$\boxed{P = \infty}$$

The given signal is neither energy nor power signal

3) $x(t) = e^{j(2t+\frac{\pi}{4})}$

Soln:

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$= \int_{-\infty}^{\infty} \left| e^{j(2t+\frac{\pi}{4})} \right|^2 dt \qquad \left[\because \left| e^{\pm j\theta} \right| = 1 \right]$$

$$= \int_{-\infty}^{\infty} dt = [t]_{-\infty}^{\infty} = \infty + \infty$$

$$\boxed{E = \infty}$$

$$P = \underset{T\to\infty}{Lt} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$$

$$P = \underset{T\to\infty}{Lt} \frac{1}{2T} \int_{-T}^{T} \left| e^{j(2t+\frac{\pi}{4})} \right|^2 dt$$

$$P = \underset{T\to\infty}{Lt} \frac{1}{2T} \int_{-T}^{T} dt = \underset{T\to\infty}{Lt} \frac{1}{2T} [t]_{-T}^{T}$$

$$P = \underset{T\to\infty}{Lt} \frac{1}{2T} \times [T+T] = \underset{T\to\infty}{Lt} \frac{1}{2T} \times 2T = \underset{T\to\infty}{Lt} 1$$

$$\boxed{P = 1W \text{ (finite)}}$$

The given signal is Power signal.

4) Verify whether $x(t) = Ae^{-at}u(t)$, $a>0$ is an energy signal (or) not.

Soln:

$$x(t) = Ae^{-at}u(t)$$

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

$$E = A^2\int_{0}^{\infty} e^{-2at} \, dt \qquad \left[u(t) = \begin{cases} 1 ; & t \geq 0 \\ 0 ; & t < 0 \end{cases} \right]$$

$$E = A^2\left[\frac{e^{-2at}}{-2a}\right]_0^{\infty}$$

$$E = -\frac{A^2}{2a}\left[\overset{0}{\cancel{e^{-\infty}}} - e^0\right] = \frac{1}{2a}$$

$$\boxed{E = \frac{A^2}{2a} \text{ Joules (finite).}} \qquad [a>1].$$

$$\text{Power}(P) = \underset{T\to\infty}{Lt} \frac{1}{2T}\int_{-T}^{T} |x(t)|^2 \, dt$$

$$= \underset{T\to\infty}{Lt} \frac{1}{2T} \cdot A^2 \int_{0}^{T} e^{-2at} \, dt$$

$$= \underset{T\to\infty}{Lt} \frac{1}{2T} \cdot A^2\left[\frac{e^{-2at}}{-2a}\right]_0^{T}$$

$$= \underset{T\to\infty}{Lt} \frac{1}{2T} \cdot A^2\left[\frac{-1}{2a}\right]\left[e^{-2aT} - e^0\right]$$

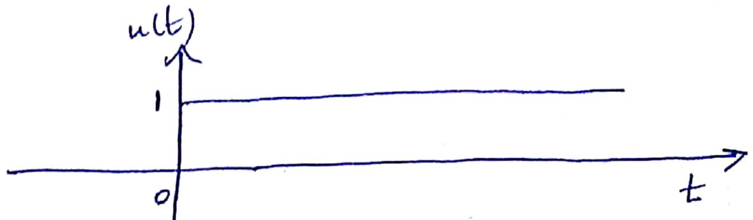
$$= \underset{T\to\infty}{Lt} \frac{1}{2T} \cdot A^2\left[-\frac{1}{2a}\right]\left[e^{-2aT} - 1\right].$$

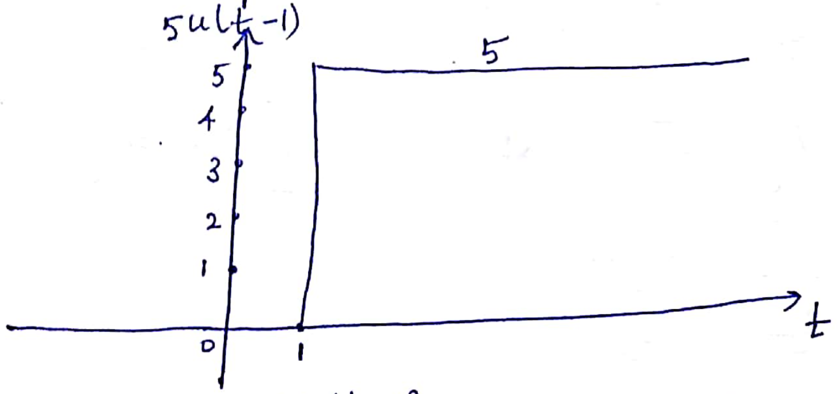
$$= \left(-\frac{A^2}{2a}\right) \underset{T\to\infty}{Lt} \left[\frac{e^{-2aT}}{2T} - \frac{1}{2T}\right]$$

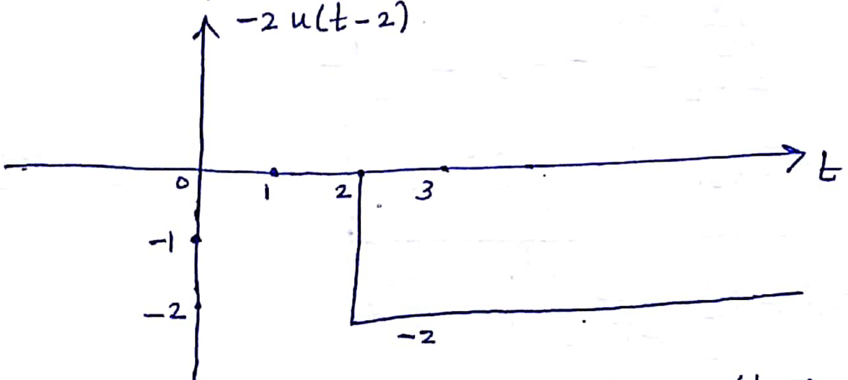
$$= \left(-\frac{A^2}{2a}\right)\left[\frac{e^{-\infty}}{2(\infty)} - \frac{1}{2(\infty)}\right] = \left(-\frac{A^2}{2a}\right)\left[\frac{0}{\infty} - \frac{1}{\infty}\right] = \left(-\frac{A^2}{2a}\right)(0-0) = 0$$

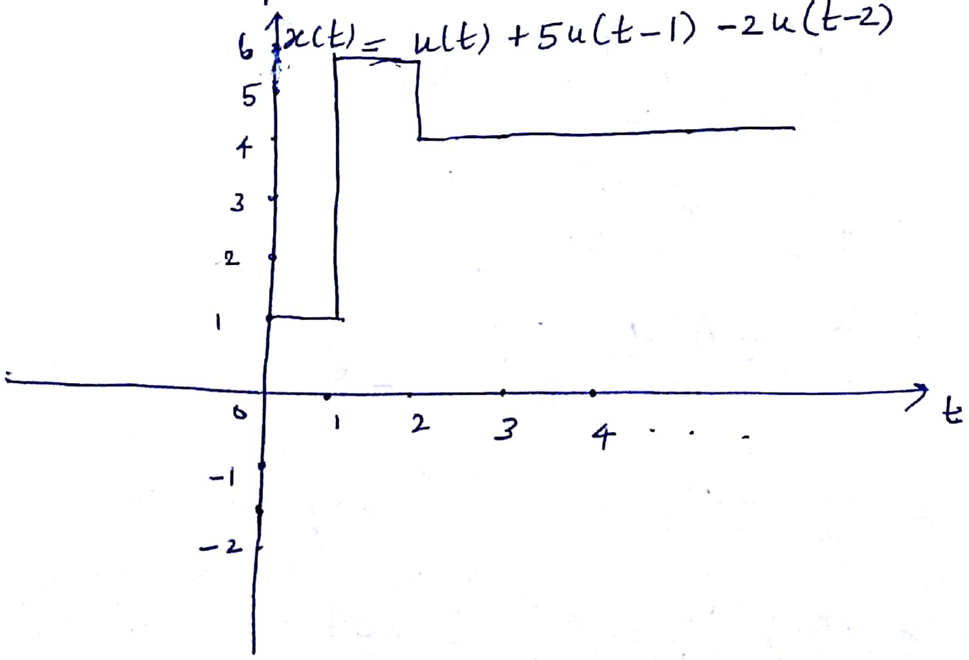
$\boxed{P = 0}$ Thus the given signal is energy signal.

5) $x(t) = u(t) + 5u(t-1) - 2u(t-2)$









$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$= \int_0^{\infty} 1^2 . dt + \int_1^{\infty} 5^2 dt + \int_2^{\infty} |(-2)|^2 dt$$

$$= [t]_0^{\infty} + 25[t]_1^{\infty} + 4[t]_2^{\infty}$$

$$= (\infty - 0) + 25(\infty - 1) + 4[\infty - 2]$$

$$\boxed{E = \infty}$$

$$P = \underset{T \to \infty}{Lt} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$$

$$= \underset{T \to \infty}{Lt} \frac{1}{2T} \left[\int_0^T dt + \int_1^T 25\, dt + \int_2^T 4\, dt \right]$$

$$= \underset{T \to \infty}{Lt} \frac{1}{2T} \left[(t)_0^T + 25(t)_1^T + 4(t)_2^T \right]$$

$$= \underset{T \to \infty}{Lt} \frac{1}{2T} \left[(T-0) + 25(T-1) + 4(T-2) \right]$$

$$= \underset{T \to \infty}{Lt} \frac{1}{2T} \left[T + 25T - 25 + 4T - 8 \right]$$

$$= \underset{T \to \infty}{Lt} \frac{1}{2T} \left[30T - 33 \right]$$

$$= \underset{T \to \infty}{Lt} \left[\frac{30T}{2T} - \frac{33}{2T} \right]$$

$$= \underset{T \to \infty}{Lt} \left[15 - \frac{33}{2T} \right] = 15 - \frac{33}{2(\infty)} = 15 - 0$$

$\boxed{P = 15W(\text{finite})}$ ∴ The given signal is power signal.

6) Determine the power and RMS value of the signal $x(t) = e^{jat} \cos w_0 t$.

soln:

$$x(t) = e^{jat} \cos w_0 t$$

$$\text{Power } (P) = \underset{T\to\infty}{Lt} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$$

$$= \underset{T\to\infty}{Lt} \frac{1}{2T} \int_{-T}^{T} \left| e^{jat} \cos w_0 t \right|^2 dt$$

$$\boxed{\begin{array}{l} |e^{jat}| = 1 \\ \cos^2\theta = \dfrac{1+\cos 2\theta}{2} \end{array}}$$

$$= \underset{T\to\infty}{Lt} \frac{1}{2T} \int_{-T}^{T} \left[\frac{1+\cos 2w_0 t}{2} \right] dt$$

$$= \underset{T\to\infty}{Lt} \frac{1}{2T} \left[\frac{1}{2} \int_{-T}^{T} dt + \frac{1}{2} \int_{-T}^{T} \cos 2w_0 t \, dt \right]$$

$$= \underset{T\to\infty}{Lt} \frac{1}{4T} \int_{-T}^{T} dt + \underbrace{\underset{T\to\infty}{Lt} \frac{1}{4T} \int_{-T}^{T} \cos 2w_0 t \, dt}_{=0}$$

$$= \underset{T\to\infty}{Lt} \frac{1}{4T} [t]_{-T}^{T}$$

$$= \underset{T\to\infty}{Lt} \frac{1}{4T} [T+T] = \underset{T\to\infty}{Lt} \frac{1}{4T} \times 2T$$

$$\boxed{P = \frac{1}{2} \text{ watts}}$$

The R.M.S value is $\sqrt{\frac{1}{2}} = 0.707$.

7) Verify whether $x(n) = \left(\frac{1}{2}\right)^n u(n)$ is energy signal or not.

Soln:

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

$$= \sum_{n=-\infty}^{\infty} \left|\left(\frac{1}{2}\right)^n u(n)\right|^2$$

$$\boxed{u(n) = \begin{cases} 1 & ; n \geqslant 0 \\ 0 & ; n < 0 \end{cases}}$$

$$= \sum_{n=0}^{\infty} \left|\left(\frac{1}{2}\right)^n\right|^2 = \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n$$

Geometrical Summation Formula (G.S.F)

$$\boxed{\sum_{n=0}^{\infty} a^n = \frac{1}{1-a} \quad ; \quad a < 1}$$

Here, $a = \frac{1}{4}$.

$$E = \frac{1}{1 - \frac{1}{4}} = \frac{1}{\frac{4-1}{4}} = \frac{4}{3} \text{ Joules (finite)}$$

$$\text{Power} = P = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

$$P = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=-N}^{N} \left|\left(\frac{1}{2}\right)^n u(n)\right|^2$$

$$\boxed{u(n) = \begin{cases} 1 & ; n \geqslant 0 \\ 0 & ; n < 0 \end{cases}}$$

$$P = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=0}^{N} \left(\frac{1}{4}\right)^n$$

$$\boxed{\begin{array}{l}\text{Geometrical summation Formula} \\ \text{(G.S.F)} \\ \displaystyle\sum_{n=0}^{N} a^n = \frac{1-a^{N+1}}{1-a} \;;\; a<1\end{array}}$$

Here, $a = \frac{1}{4}$

$$= \underset{N\to\infty}{Lt} \frac{1}{2N+1} \times \frac{1-\left(\frac{1}{4}\right)^{N+1}}{1-\frac{1}{4}}$$

$$= \frac{1}{2.\infty+1} \times \frac{1-\left(\frac{1}{4}\right)^{\infty+1}}{1-\frac{1}{4}}$$

$$\left[\because \frac{\text{any number}}{\infty} = 0\right]$$

$$\boxed{P=0}$$

From the above calculations, we concluded that given signal is <u>Energy signal</u>, since, $E = \frac{4}{3}$ (finite) and $P = 0$.

8) verify whether the given signal $x(n) = e^{j[\frac{\pi}{2}n+\frac{\pi}{8}]}$ is power signal (or) not.

Soln:

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

$$= \sum_{n=-\infty}^{\infty} \left| e^{j[\frac{\pi}{2}n + \frac{\pi}{8}} \right|^2$$

$$\boxed{\begin{array}{l} \text{w.k.T,} \\ |e^{\pm j\theta}| = 1 \end{array}}$$

$$= \sum_{n=-\infty}^{\infty} 1^2 = \sum_{n=-\infty}^{\infty} 1 \qquad [\because 1+1+1+1+\cdots\cdots = \infty]$$

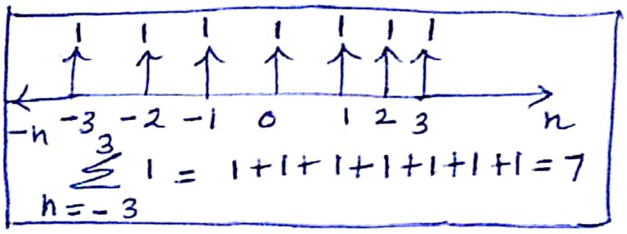
$$\boxed{E = \infty}$$

$$P = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

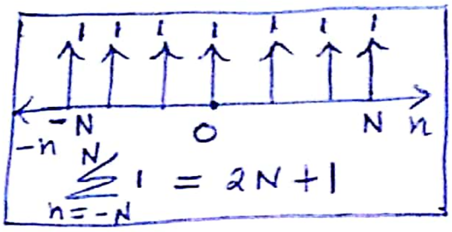
$$= \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=-N}^{N} \left| e^{j[\frac{\pi}{2}n + \frac{\pi}{8}]} \right|^2$$

$$\boxed{\begin{array}{l} \text{Wk.T} \\ |e^{+j\theta}| = 1 \end{array}}$$

$$= \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=-N}^{N} 1^2 = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=-N}^{N} 1$$



$$\sum_{n=-3}^{3} 1 = 1+1+1+1+1+1+1 = 7$$



$$\sum_{n=-N}^{N} 1 = 2N+1$$

$$P = \underset{N\to\infty}{Lt} \frac{1}{\cancel{2N+1}} \times \cancel{2N+1}$$

$$P = \underset{N\to\infty}{Lt} 1$$

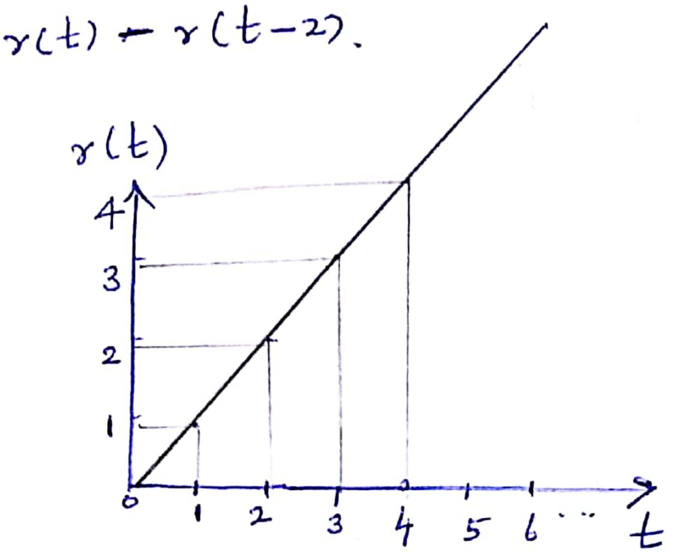
$$\boxed{P = 1 \text{ Watts (finite)}}$$

∴ The given signal is power signal.

9) check whether the given signal is energy or power.

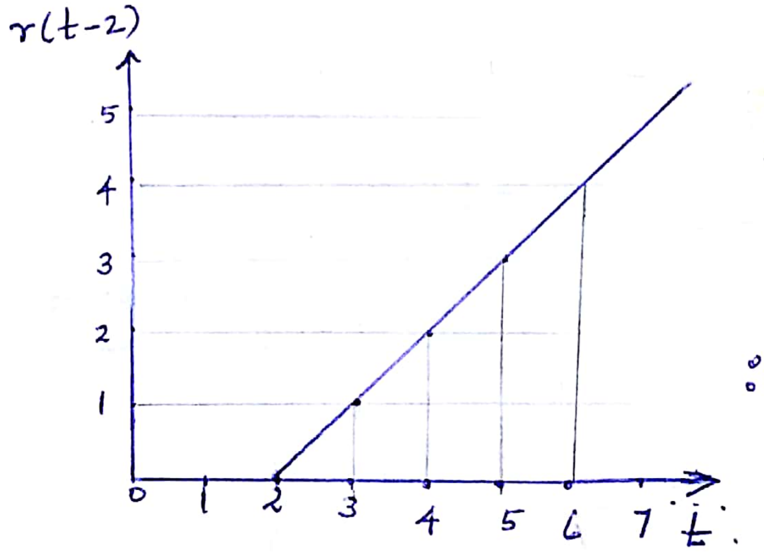
$x(t) = r(t) - r(t-2)$.

Soln:



$r(t) = t \;;\; t \geq 0$

$\quad\;\; 0 \;;\; t < 0$



$t-2=0;\; t=2$

$t-2=1;\; t=3$

$t-3=2;\; t=4$.

$\therefore$ At $t=2$, $r(t-2)$

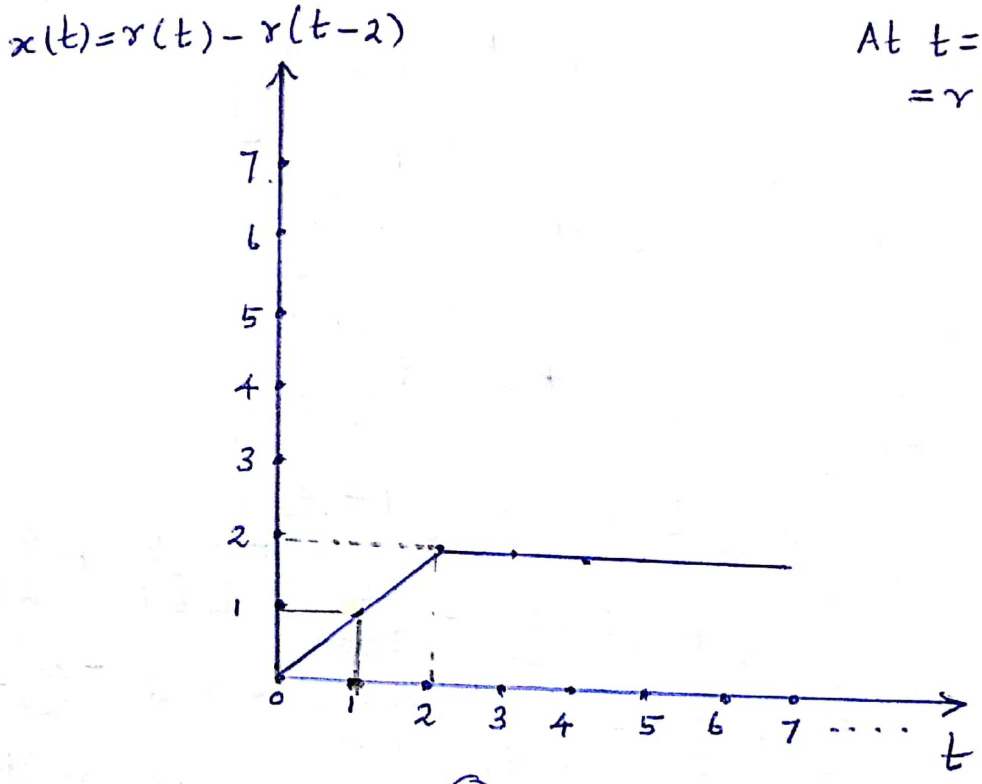
$= r(2-2) = r(0) = 0$

At $t=3$, $r(t-2)$

$r(3-2) = r(1) = 1$

At $t=4$, $r(t-2)$

$= r(4-2) = r(2) = 2$.



$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$= \int_{0}^{2} |t|^2 dt + \int_{2}^{\infty} |2|^2 dt$$

$$= \int_{0}^{2} t^2 dt + \int_{2}^{\infty} 4\, dt$$

$$= \left[\frac{t^3}{3}\right]_0^2 + 4\,[t]_2^{\infty}$$

$$= \frac{1}{3}[2^3 - 0^3] + 4[\infty - 2]$$

$$= \frac{1}{3}[8] + 4\infty - 8$$

$$\boxed{E = \infty} \qquad [\text{any number} + \infty = \infty.]$$

$$P = \underset{T\to\infty}{Lt} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$$

$$= \underset{T\to\infty}{Lt} \frac{1}{2T} \left[\int_{0}^{2} |t|^2 dt + \int_{2}^{T} |2|^2 dt\right]$$

$$= \underset{T\to\infty}{Lt} \frac{1}{2T} \left[\int_{0}^{2} t^2 dt + \int_{2}^{T} 4\, dt\right]$$

$$= \underset{T\to\infty}{Lt} \frac{1}{2T} \left[\left[\frac{t^3}{3}\right]_0^2 + 4\,[t]_2^T\right]$$

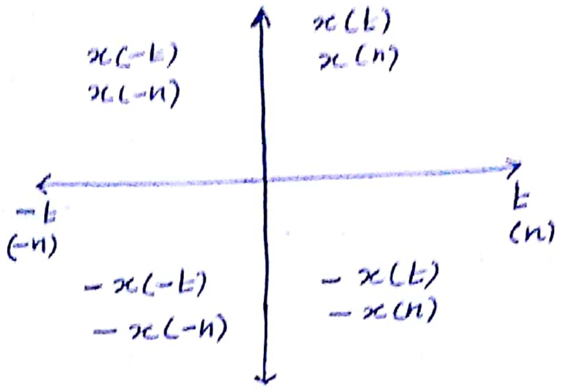
$$= \underset{T\to\infty}{Lt} \frac{1}{2T} \left[\frac{1}{3}[2^3 - 0^3] + 4[T - 2]\right]$$

$$= \underset{T\to\infty}{Lt} \frac{1}{2T} \left[\frac{8}{3} + 4T - 8\right] = \underset{T\to\infty}{Lt} \left[\frac{8}{6T} + \frac{4T}{2T} - \frac{8}{2T}\right]$$

$$= \underset{T\to\infty}{Lt} \left[\frac{8}{6.\infty} + \frac{4}{2} - \frac{8}{2.\infty}\right] = \underset{T\to\infty}{Lt} [0 + 2 - 0] = \underset{T\to\infty}{Lt}\, 2$$

$\boxed{P = 2 \text{ Watts (finite)}}$ ∴ The given signal is Power Signal

Even and odd signal:



* Even (symmetric) signal

A signal is said to be even if it satisfies the following condition :

CT signal : $x(t) = x(-t)$

DT signal : $x(n) = x(-n)$

* odd (Antisymmetric) signal

A signal is said to be odd if it satisfies the following condition:

CT signal : $x(t) = -x(-t)$

DT signal : $x(n) = -x(-n)$

Note :

The signal comprises of odd and even components. (or) parts)

i.e., $x(t) = x_e(t) + x_o(t) \rightarrow$ CT signal

$x(n) = x_e(n) + x_o(n) \longrightarrow$ DT signal

Even part

$$x_e(t) = \frac{1}{2}[x(t) + x(-t)]$$

odd Part $x_o(t) = \frac{1}{2}[x(t) - x(-t)]$

} CT signal

Even component

$$x_e(n) = \frac{1}{2}[x(n) + x(-n)]$$

odd component

$$x_o(n) = \frac{1}{2}[x(n) - x(-n)]$$

} DT signal.

Problems:

1) Find the even part and odd part of the following signal. $x(t) = \cos t + \sin t + \cos t \sin t$.

Soln:

$$x(t) = \cos t + \sin t + \cos t \sin t$$

$$x(-t) = \cos(-t) + \sin(-t) + \cos(-t)\sin(-t)$$

$$\boxed{x(-t) = \cos t - \sin t - \cos t \sin t}$$

$$\boxed{-x(-t) = -\cos t + \sin t + \cos t \sin t}$$

Even part: $x_e(t) = \frac{1}{2}[x(t) + x(-t)]$

$$= \frac{1}{2}\left[\cos t + \cancel{\sin t} + \cancel{\cos t \sin t} + \cos t - \cancel{\sin t} - \cancel{\cos t \sin t}\right]$$

$$= \frac{1}{2} \times 2\cos t$$

$$\boxed{x_e(t) = \cos t}$$

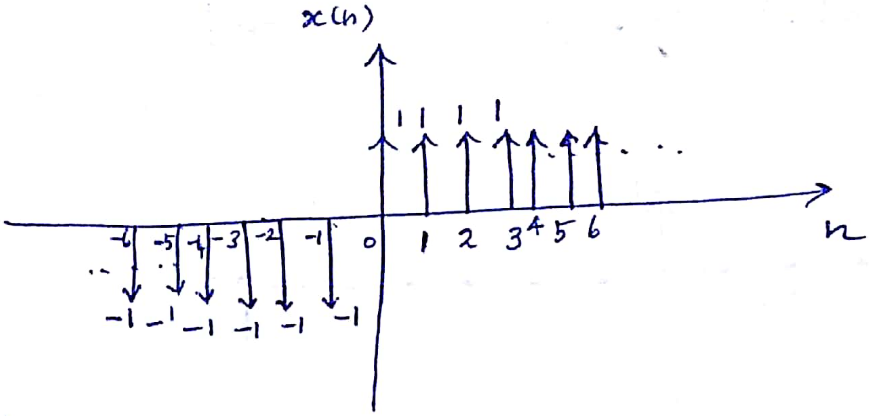
Odd part: $x_o(t) = \frac{1}{2}[x(t) - x(-t)]$

$$= \frac{1}{2}\left[\cancel{\cos t} + \sin t + \cos t \sin t - \cancel{\cos t} + \sin t + \cos t \sin t\right]$$

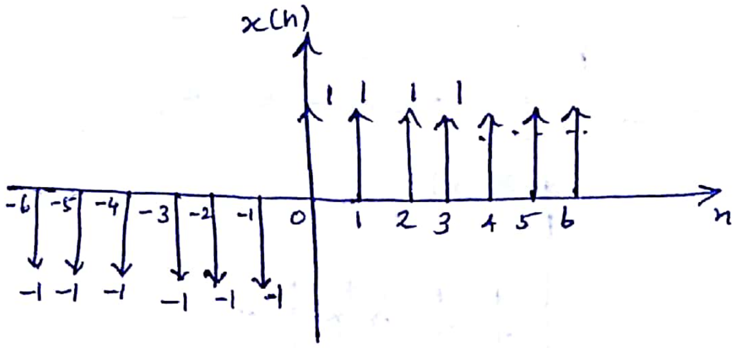
$$= \frac{1}{2}[2\sin t + 2\cos t \sin t]$$

$$\boxed{x_o(t) = \sin t + \cos t \sin t}$$

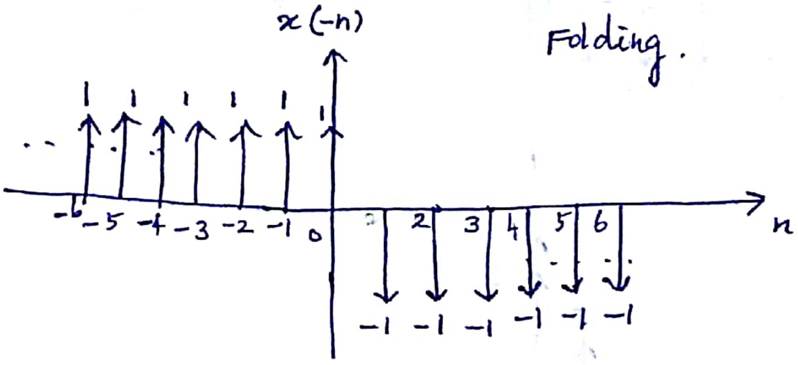
2) x(n).



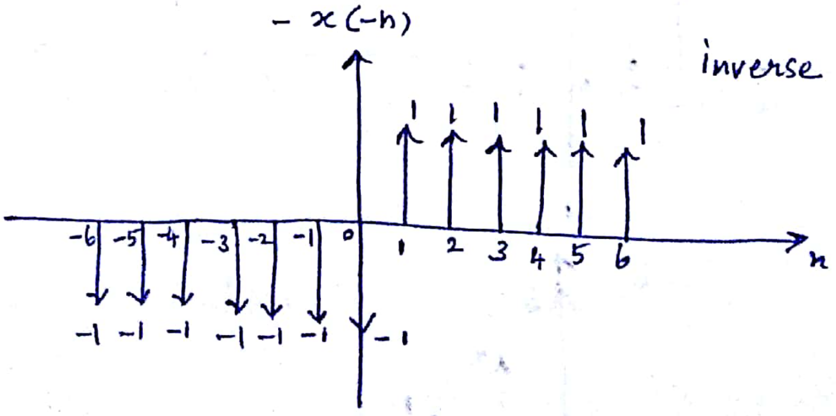
soln:



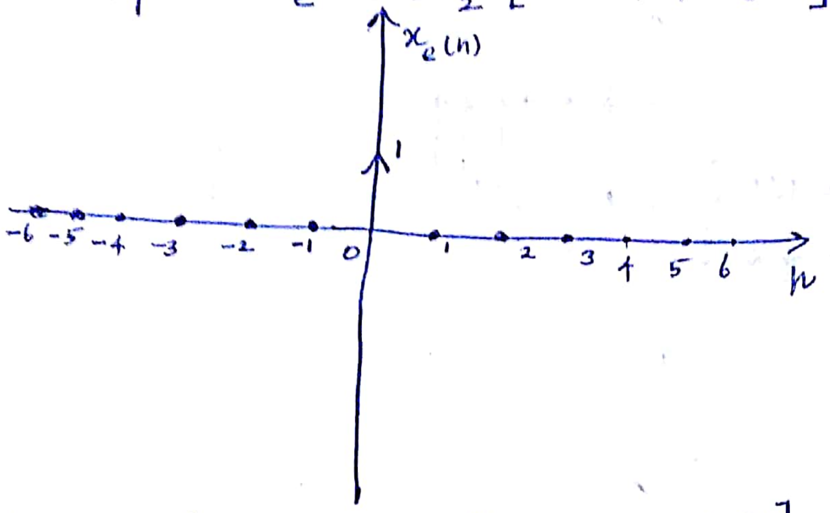
Folding.



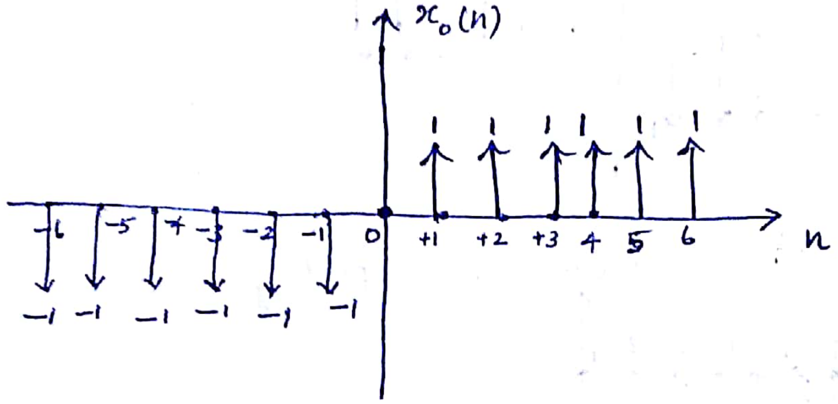
inverse



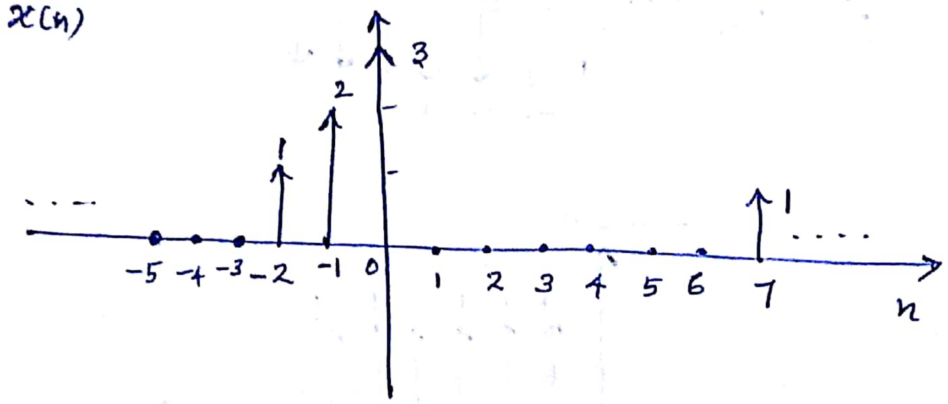
Even component : $x_e(n) = \frac{1}{2}[x(n) + x(-n)]$

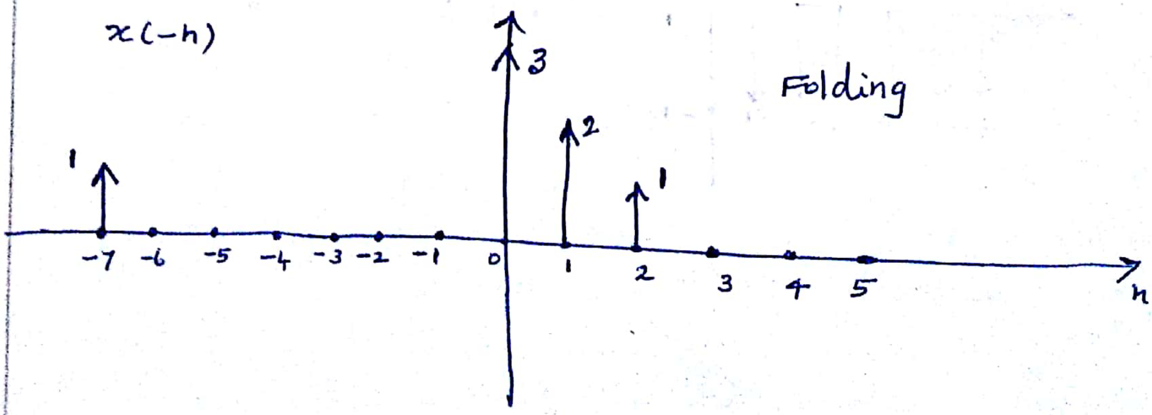


Odd component $x_o(n) = \frac{1}{2}[x(n) - x(-n)]$



3)

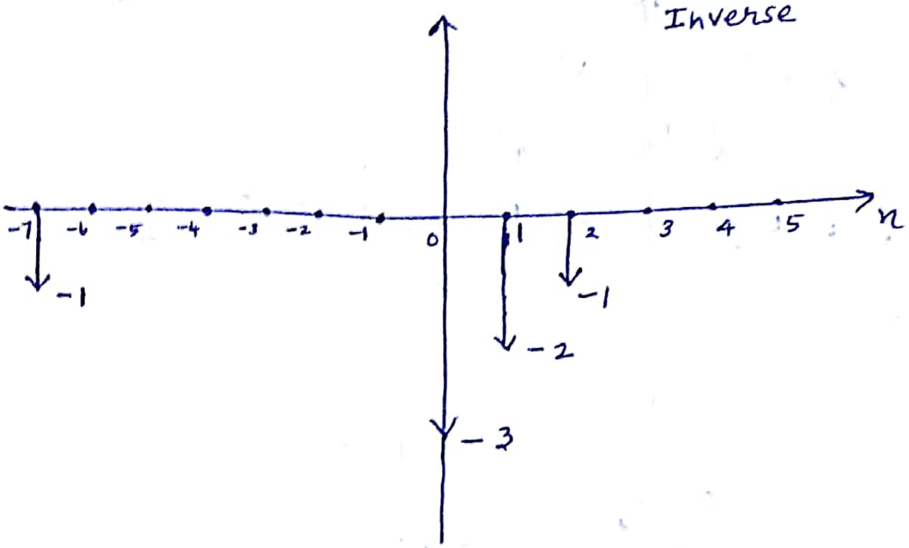




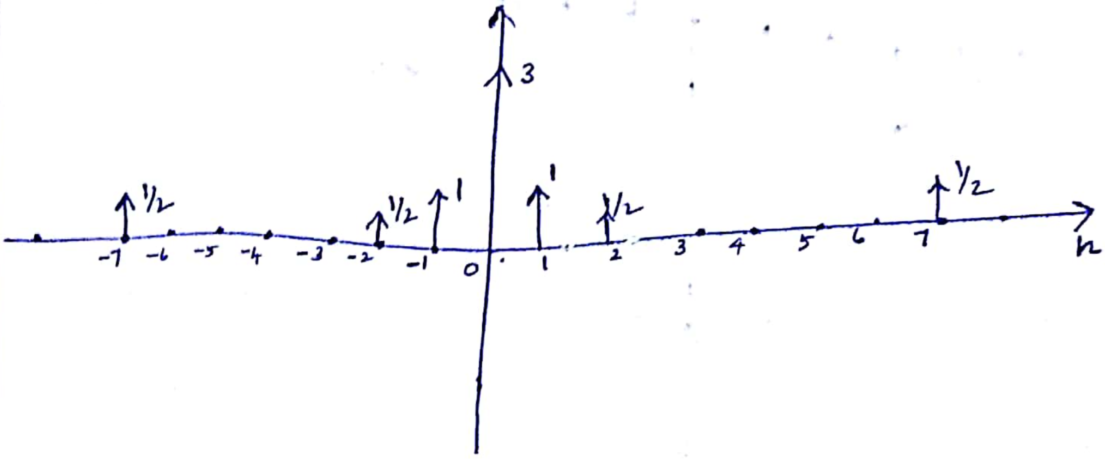
Folding

$-x(-n)$

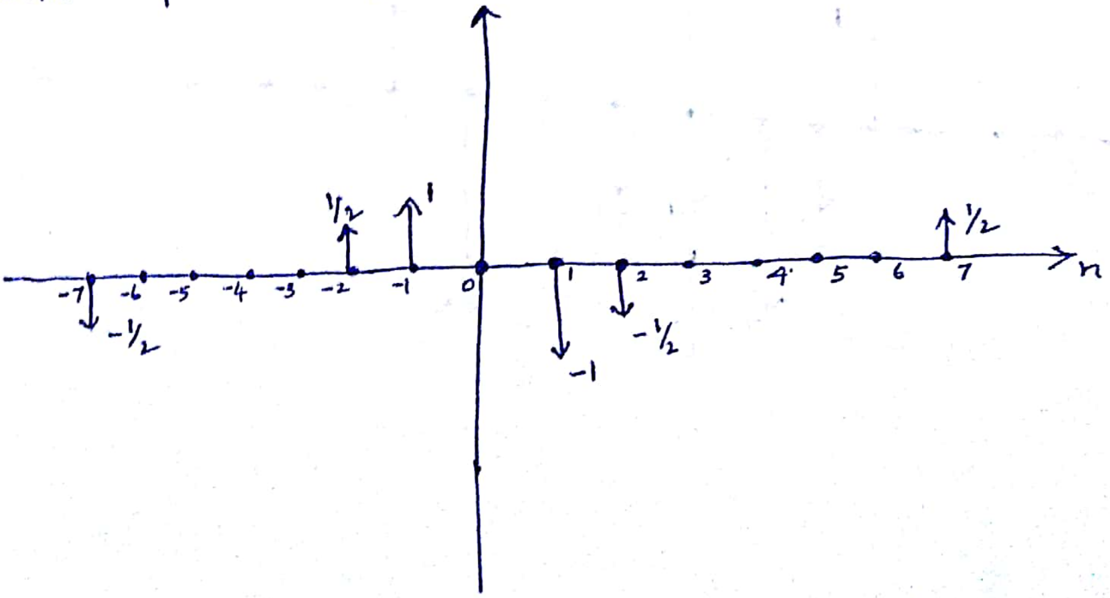
Inverse



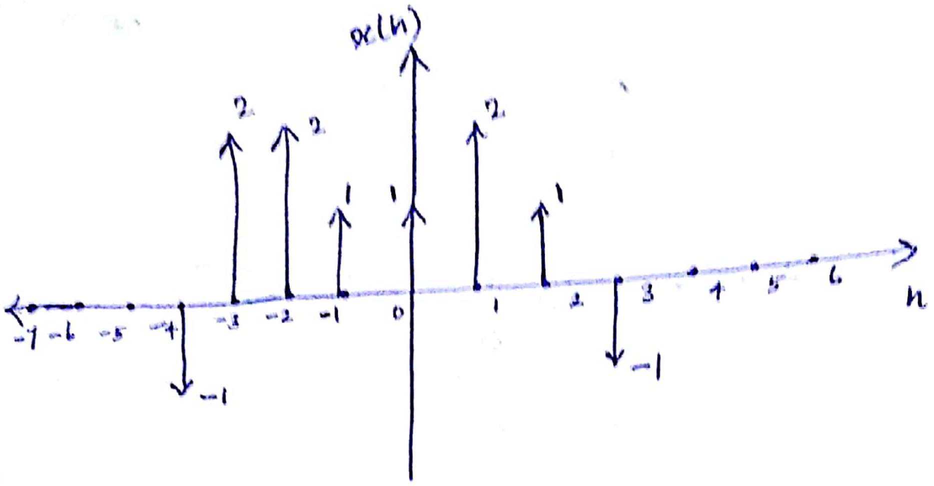
Even component: $x_e(n) = \frac{1}{2}\left[x(n) + x(-n)\right]$



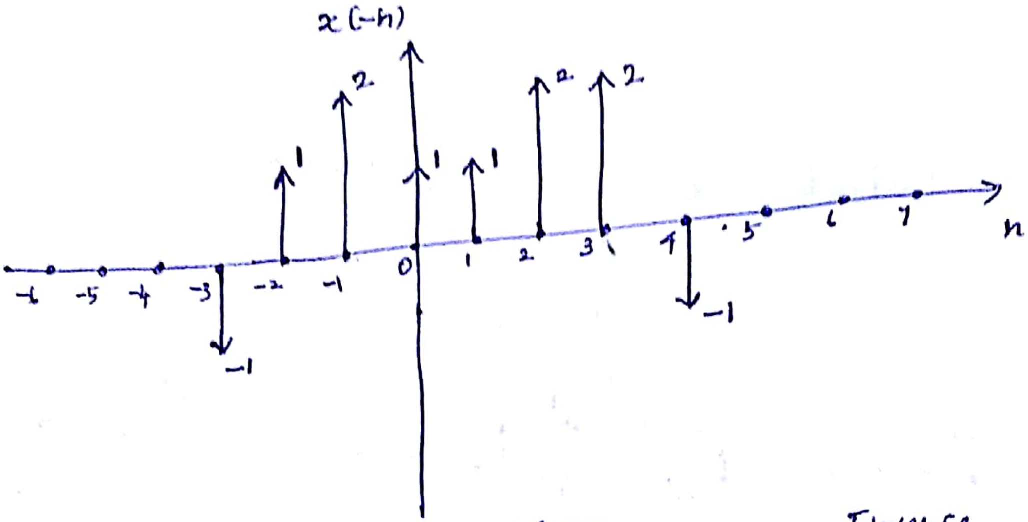
Odd component: $x_o(n) = \frac{1}{2}\left[x(n) - x(-n)\right]$



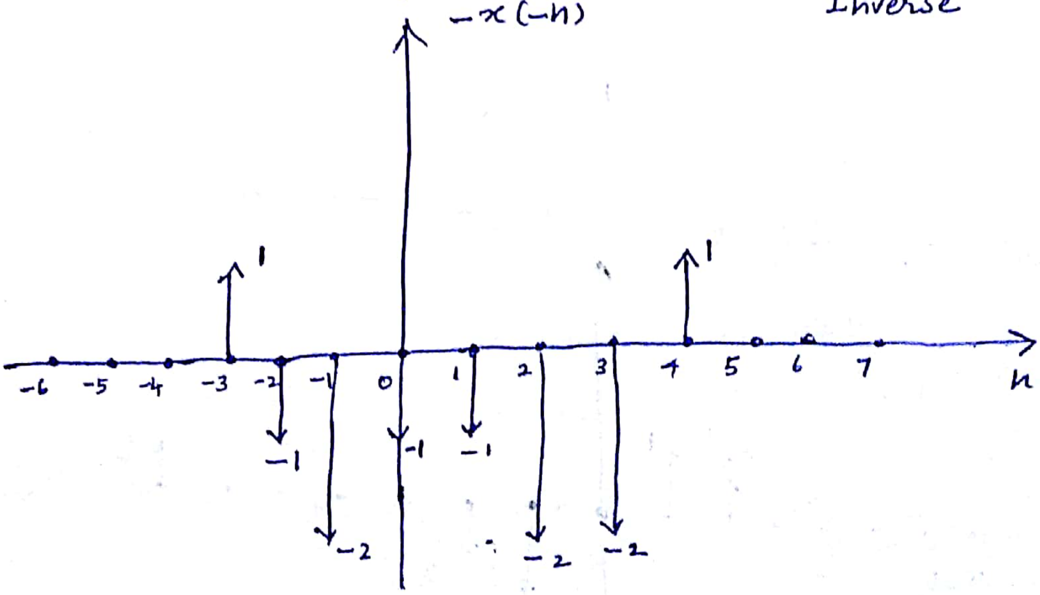
4)



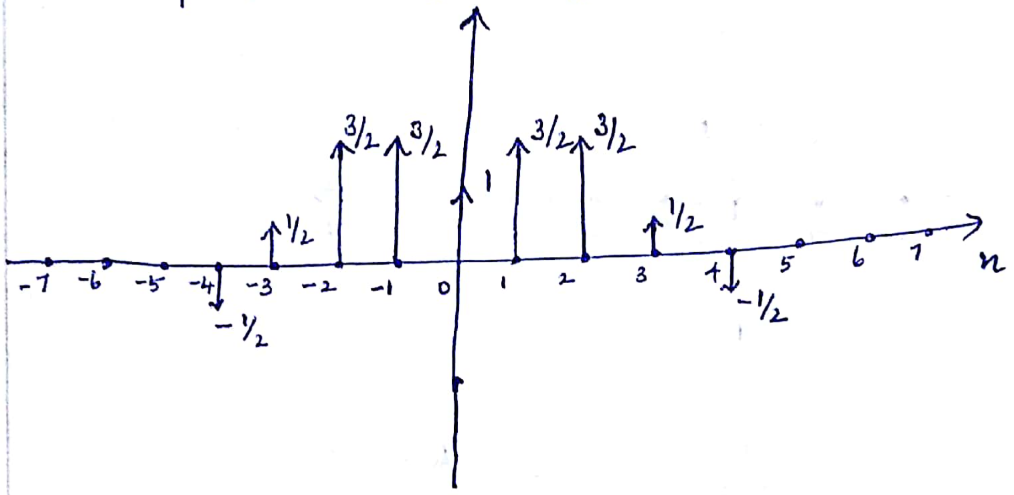
Folding



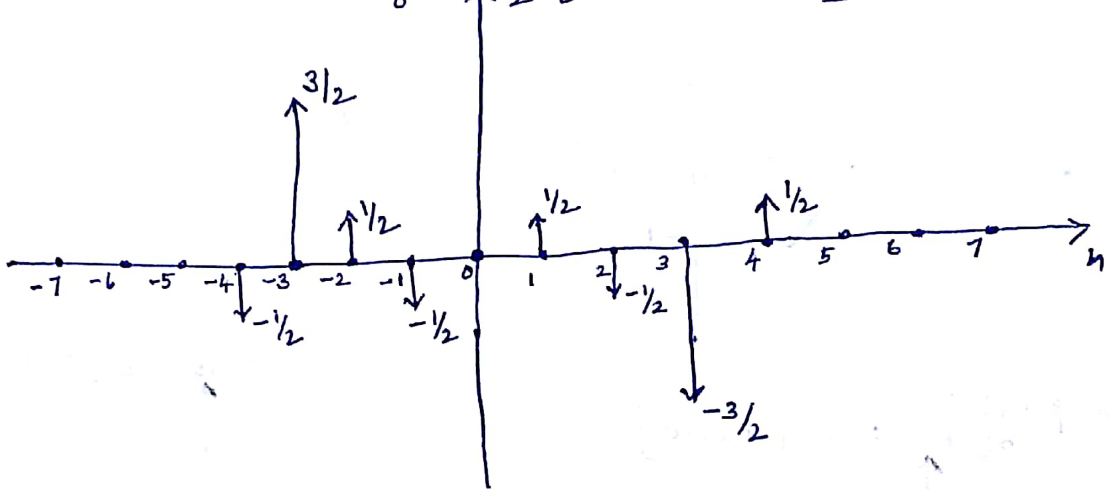
Inverse



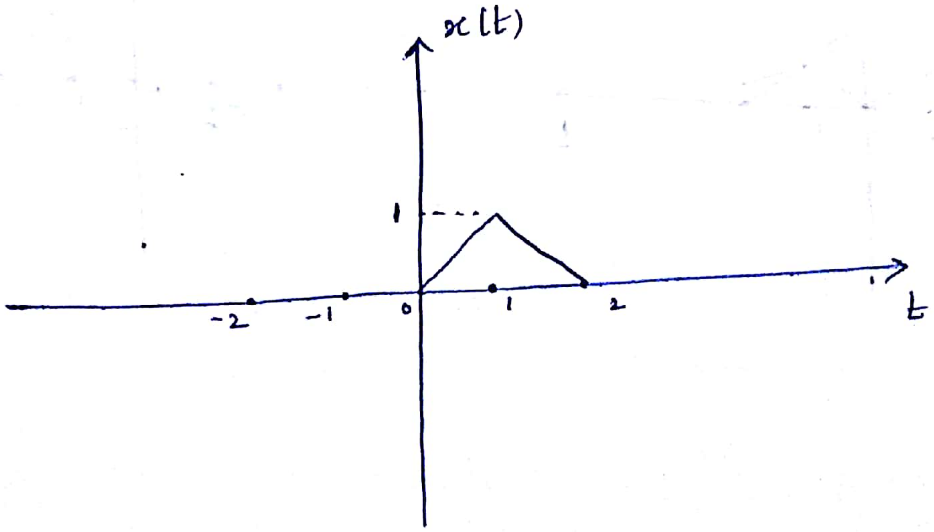
Even component : $x_e(n) = \frac{1}{2}[x(n) + x(-n)]$



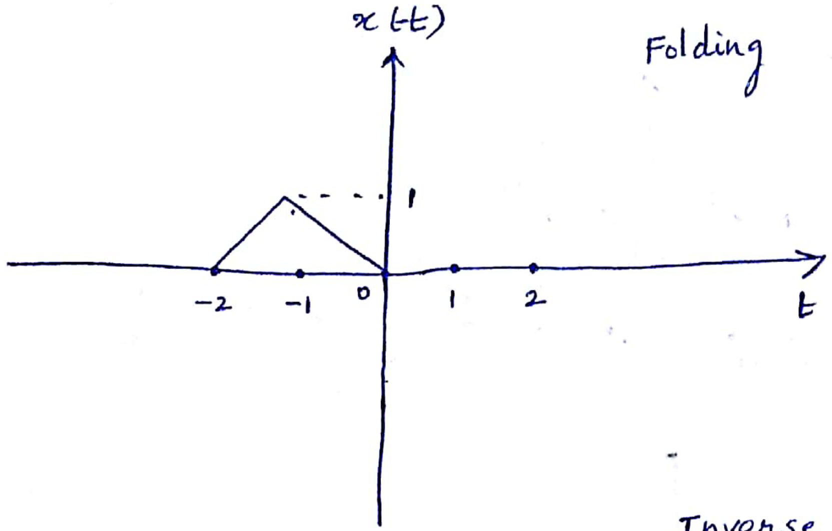
Odd component : $x_o(n) = \frac{1}{2}[x(n) - x(-n)]$



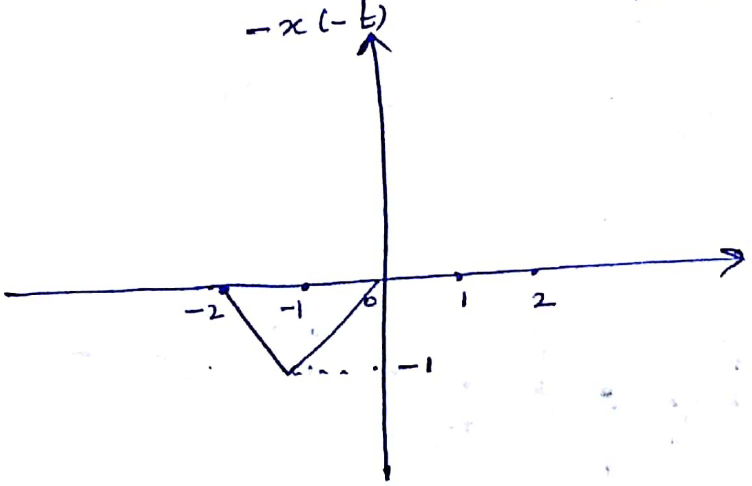
5)



Folding

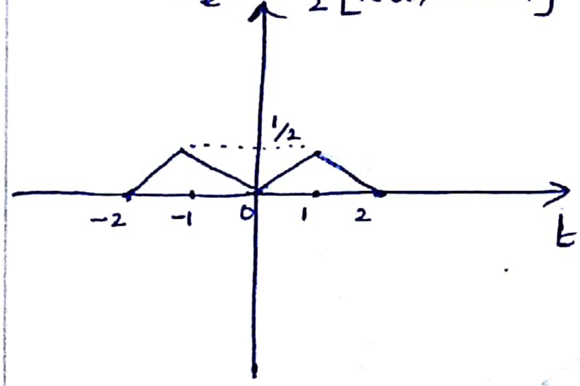


Inverse

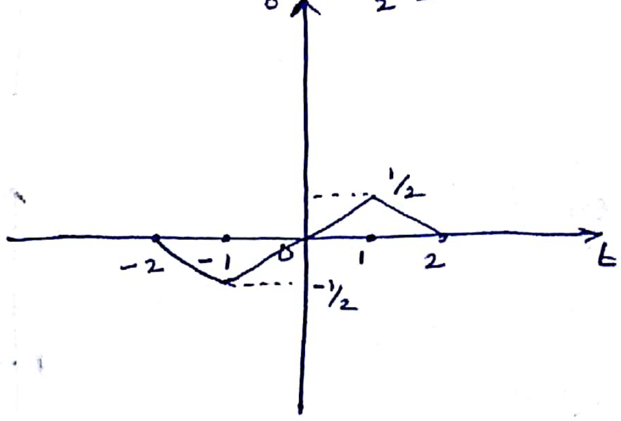


Even part:

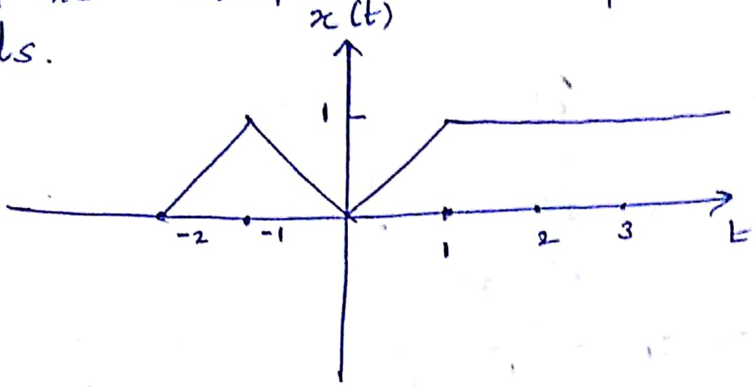
$x_e(t) = \frac{1}{2}[x(t)+x(-t)]$

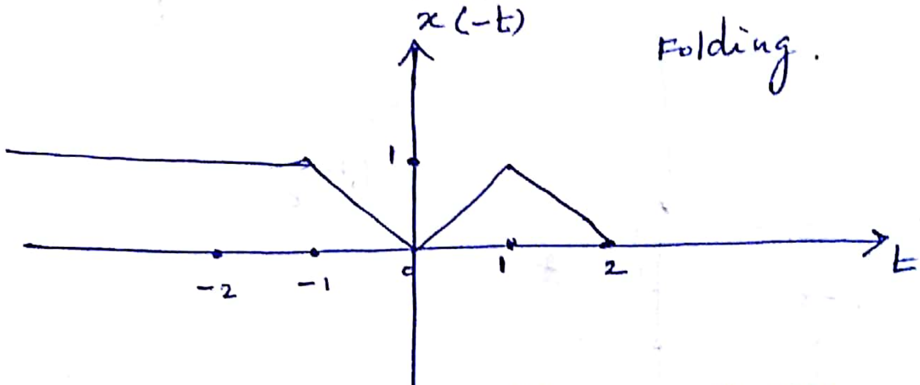


$x_o(t) = \frac{1}{2}[x(t)-x(-t)]$

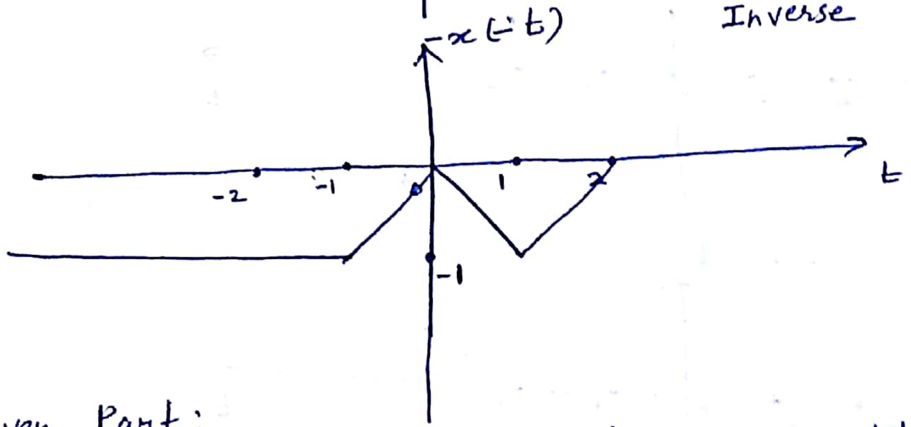


6) Find the odd part and even part of the following signals.





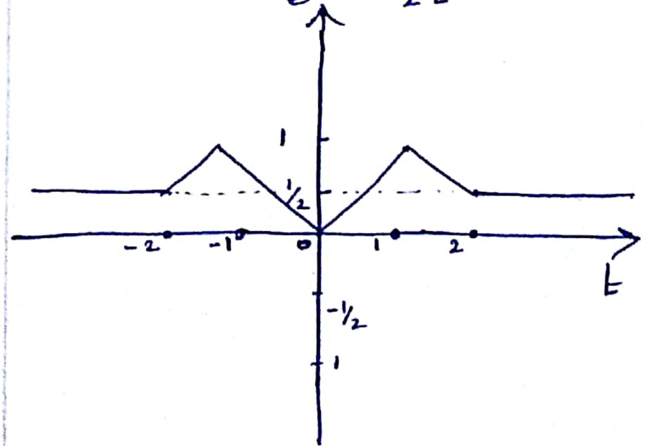
Folding.



Inverse

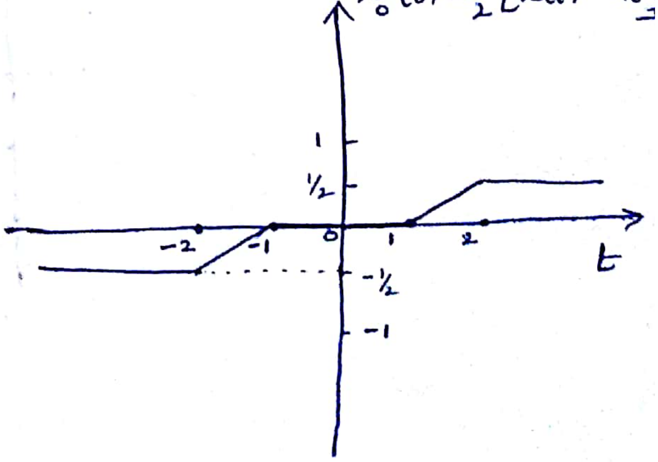
Even Part:

$x_e(t)=\frac{1}{2}[x(t)+x(-t)]$

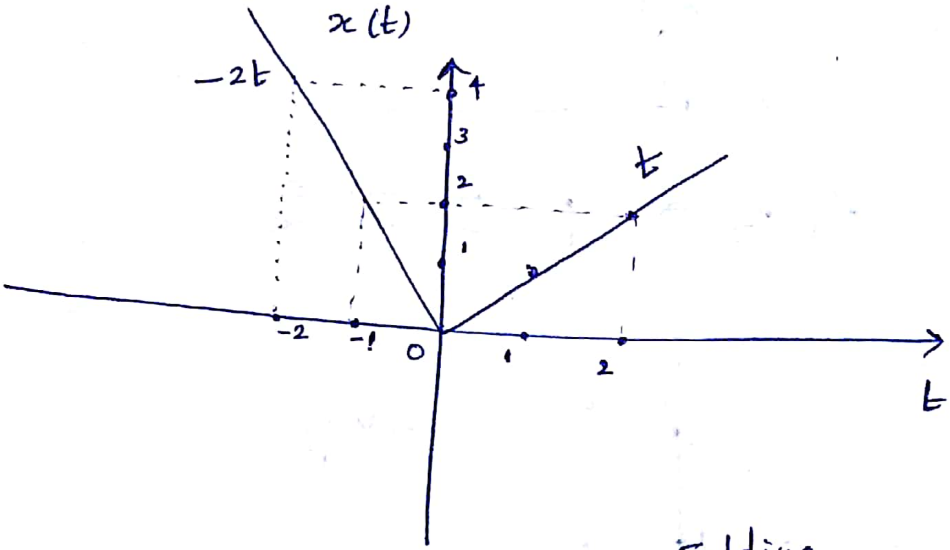


Odd Part

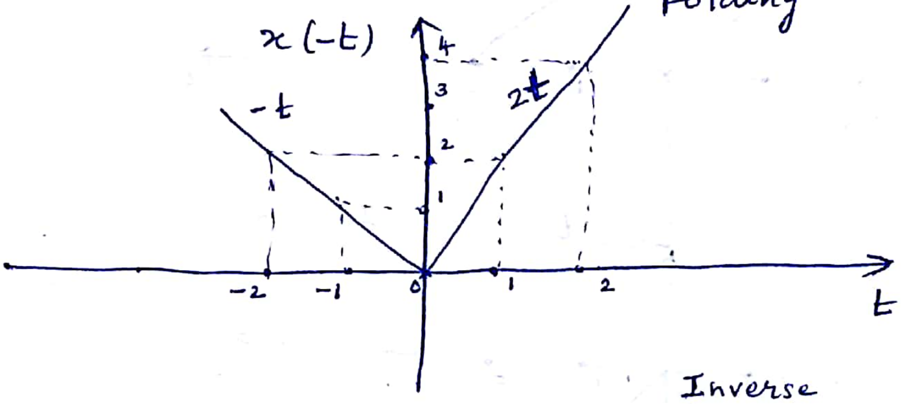
$x_o(t)=\frac{1}{2}[x(t)-x($



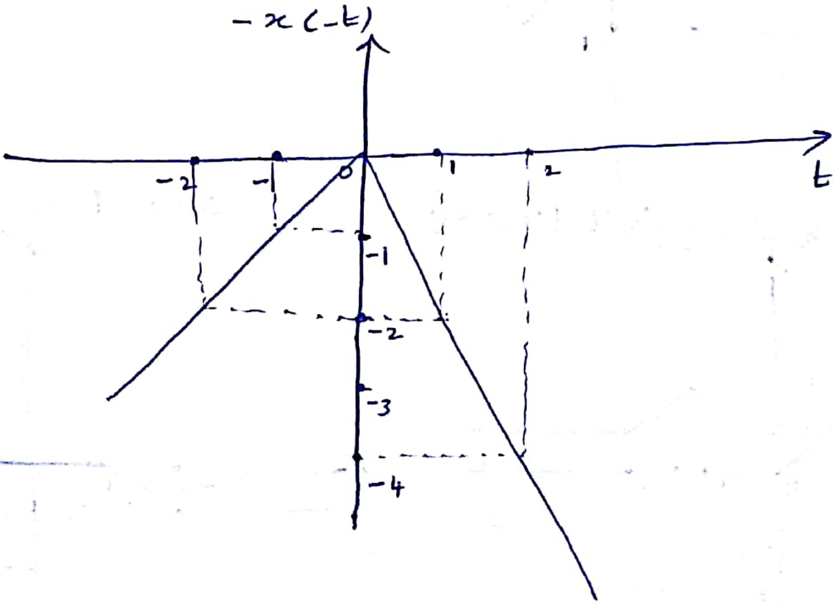
7)



Folding

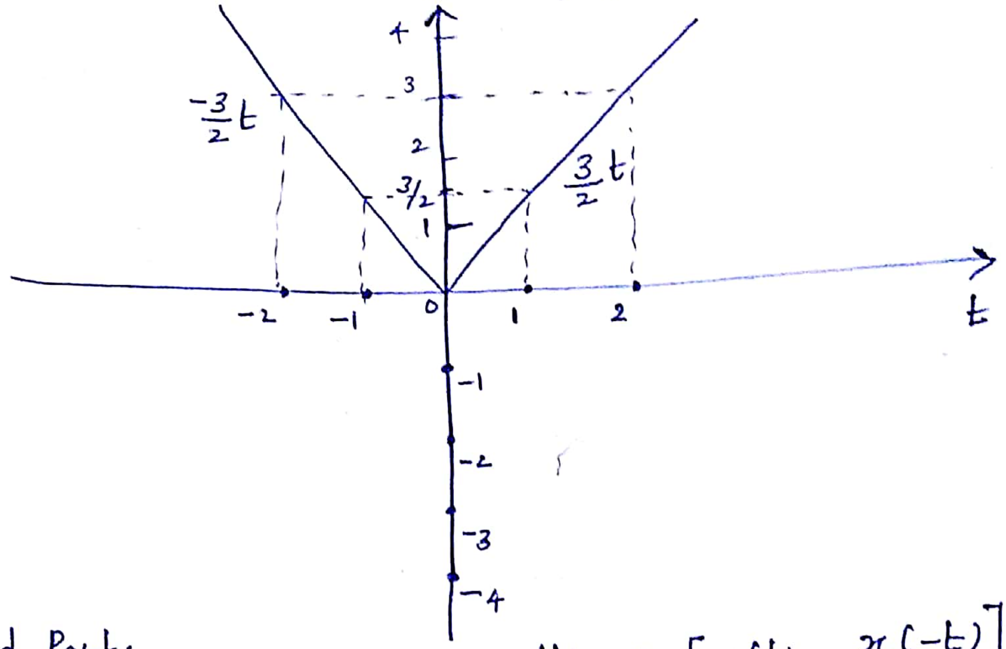


Inverse



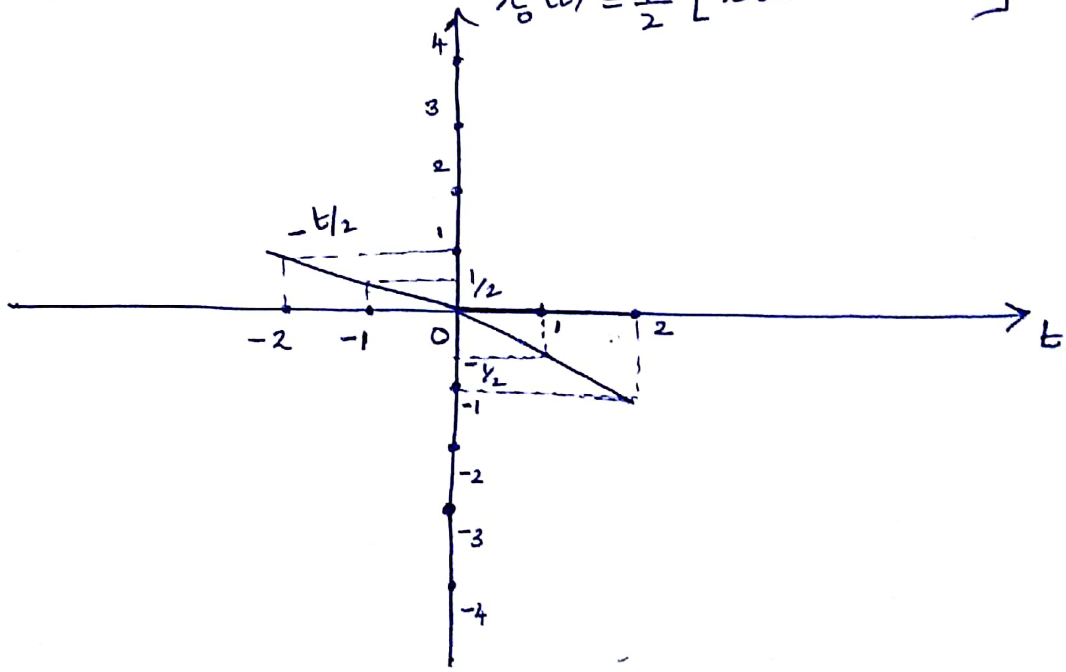
Even Part:

$x_e(t) = \frac{1}{2}\left[x(t) + x(-t)\right]$



Odd Part:

$x_o(t) = \frac{1}{2}\left[x(t) - x(-t)\right]$



Previous year university questions with answers:

1) Find the even and odd part of the signal $x(t) = e^{jt}$.

Soln:

$$x(t) = e^{jt}$$

$$x(-t) = e^{-jt}$$

$$-x(-t) = -e^{-jt}$$

Even part : $x_e(t) = \frac{1}{2}[x(t) + x(-t)]$

$$= \frac{1}{2}[e^{jt} + e^{-jt}]$$

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$\boxed{x_e(t) = \cos t}$$

odd Part : $x_o(t) = \frac{1}{2}[x(t) - x(-t)]$

$$= \frac{1}{2}[e^{jt} - e^{-jt}]$$

$$[\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}]$$

$$\boxed{x_o(t) = j\sin t}$$

2) $x(t) = e^{j2t}$

$$x(-t) = e^{-j2t}$$

$$-x(-t) = -e^{-j2t}$$

Even Part : $x_e(t) = \frac{1}{2}[x(t) + x(-t)]$

$$= \frac{1}{2}[e^{j2t} + e^{-j2t}]$$

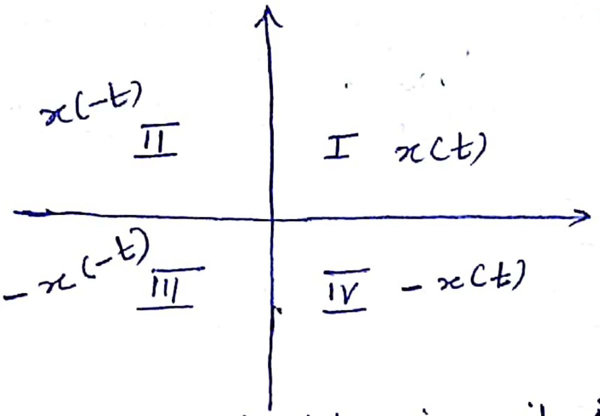
$$\boxed{x_e(t) = \cos 2t}$$

odd Part : $x_o(t) = \frac{1}{2}[x(t) - x(-t)]$

$$= \frac{1}{2}[e^{j2t} - e^{-j2t}]$$

$$\boxed{x_o(t) = j\sin 2t}$$

Note:

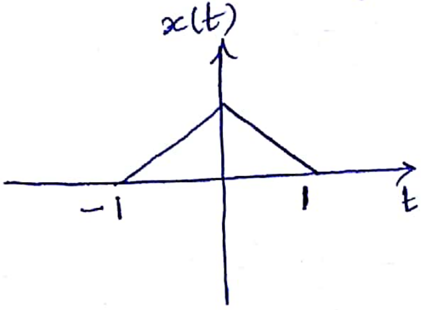


Even (Symmetry)
I = II (or) III = IV

Odd (Antisymmetry)
I = III (or) II = IV

③ For the given signal determine it is even, odd or neither even nor odd.

a)

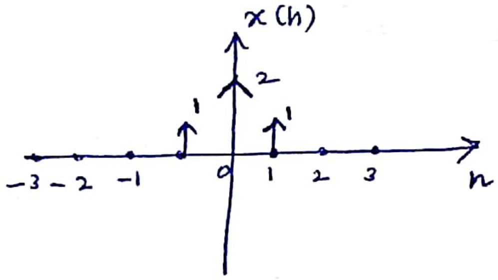


I = II

$x(t) = x(-t)$

The given signal is even

b)

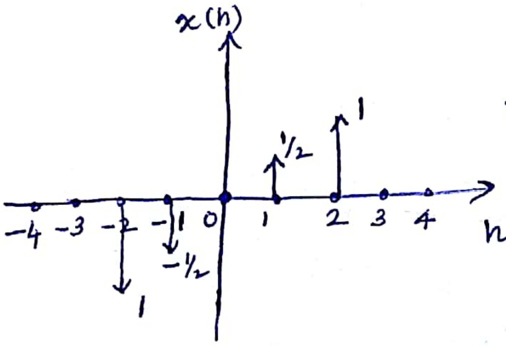


I = II

$x(n) = x(-n)$

The given signal is even

c)

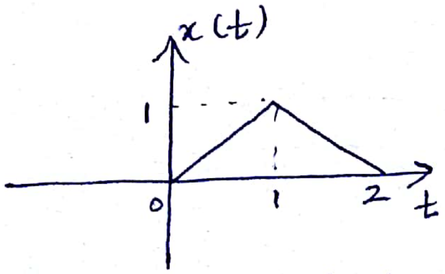


I = III

$x(n) = -x(-n)$

The given signal is odd

d)



I ≠ II

I ≠ III

$x(t) \neq x(-t)$

$x(t) \neq -x(-t)$

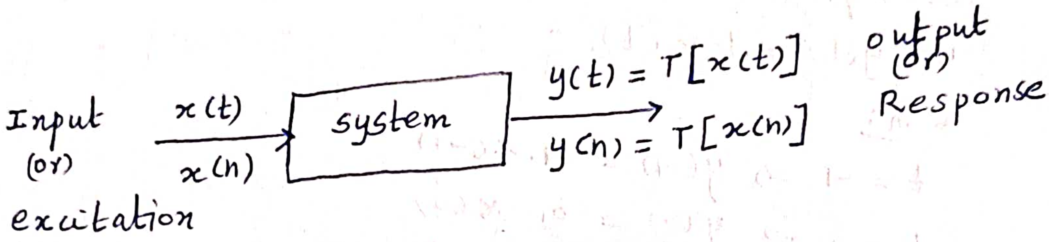
∴ The given signal is neither even nor odd.

Systems classification:

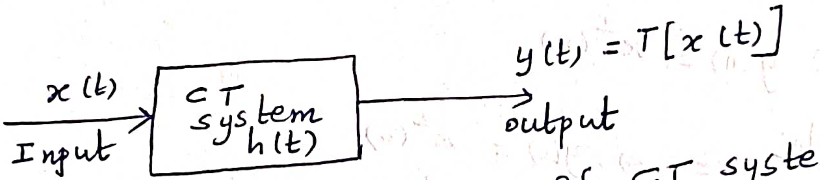
* System : It is a physical device that performs certain operations on the given input signal.

Examples : Amplifier, Modulator, Filter, Equalizer

* System Model :



* Continuous Time(CT) system :

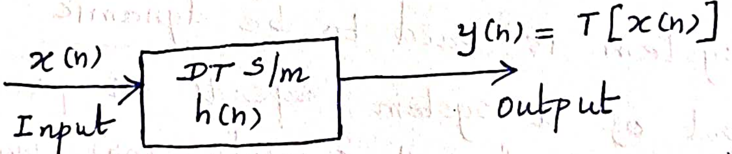


where, $h(t) \rightarrow$ impulse response of CT system

Continuous Time system is one which performs certain operation on the given continuous time input signal.

output : $y(t) = T[x(t)]$

* Discrete Time (DT) system :



where, $h(n) \rightarrow$ impulse response of DT system

Discrete Time system is one which performs certain operation on the given Discrete time input signal.

output : $y(n) = T[x(n)]$

a) Static and Dynamic System :

A system is said to be static if the present output of a system depends on present input only, not on past and future inputs. static system is also called as memoryless system.

Examples:

i) $y(t) = t\,x(t)$

Soln:

$t = -1 \Rightarrow y(-1) = -1 \cdot x(-1)$

$t = 0 \Rightarrow y(0) = 0 \cdot x(0)$

$t = 1 \Rightarrow y(1) = 1 \cdot x(1)$

ii) $y(n) = x^2(n)$

Soln:

$n = 0 \Rightarrow y(0) = x^2(0)$

$n = -1 \Rightarrow y(-1) = x^2(-1)$

$n = 1 \Rightarrow y(1) = x^2(1)$

More examples of static (or) memoryless system

$y(n) = 2x(n) + 3$, $y(n) = n\,x(n)$, $y(t) = \frac{d\,x(t)}{dt}$, $y(t) = e^{x(t)}$,

$y(t) = \cos 3t \cdot x(t)$, $\frac{dy(t)}{dt} + 3t\,y(t) = t^2 x(t)$.

A system is said to be dynamic if the present out-put of a system depends on present input & past inputs (or) present and future inputs (or) present, past and future inputs. Dynamic system is also called as system with memory.

Examples :

i) $y(t) = 2x(t^2)$

Soln:

$t = -1 \Rightarrow y(\underset{\substack{\uparrow \\ \text{present}}}{-1}) = 2x((-1)^2) = 2x(\underset{\substack{\uparrow \\ \text{future}}}{1})$

$t = 0 \Rightarrow y(0) = 2x(0)$

$t = 1 \Rightarrow y(1) = 2(1^2) = x(1)$.

ii) $y(n) = x(-n)$

Soln:

$n = -1 \Rightarrow y(\underset{\substack{\uparrow \\ \text{present}}}{-1}) = x(-(-1)) = x(\underset{\substack{\uparrow \\ \text{future}}}{1})$

$n = 0 \Rightarrow y(0) \overset{\text{present}}{=} x(0)$

$n = 1 \Rightarrow y(\underset{\substack{\uparrow \\ \text{present}}}{1}) = x(\underset{\substack{\uparrow \\ \text{past}}}{-1})$

More examples:

$y(n) = x(n-2) + 2x(n+3)$, $y(t) = \int_{-\infty}^{2t} x(\tau) d\tau$.

Note: Representation of inputs & output

$x(n) \Rightarrow$ present input (or) current input

$x(n-k) \rightarrow$ past input

$x(n+k) \rightarrow$ future input.

$y(n) \rightarrow$ current output (or) present output

More understanding:

i) $y(t) = \int_{-\infty}^{2t} x(\tau) d\tau$

Soln:

Here, 'τ' Varies from $-\infty$ to '$2t$'. i.e, Input signal $x(\tau)$ is represented as, $x(-\infty)$.....,$x(-1)$, $x(0)$, ..$x(2t)$ When, $t=-1$, $y(-1) \Rightarrow$ depends on $x(-\infty)$.... $x(2\times(-1))$ i.e, $y(-1)$ depends on $x(-\infty)$,.... $x(-3)$ $x(-2)$. So, current output depends on past outputs. So, it is dynamic system.

ii) $y(t) = \int_{-\infty}^{t} x(\tau) d\tau$

Soln: Here, 'τ' varies from $-\infty$ to 't'. i.e, Input signal is represented as, $x(-\infty)$, ... $x(0)$, $x(1)$,.. $x(t)$. When, $t=-1$, $y(-1)$ depends on $x(-\infty)$... $x(-2)$, $x(-1)$, i.e, the present output depends on present input and past outputs.

$\therefore$ The given system is dynamic system.

b) Causal and Non causal system :

A system is said to be causal, if the present output depends on only present input and past inputs. It doesn't depend on future inputs.

If the system present output depends on future inputs, then it is called as Non-causal.

Examples :

i) $y(n) = x(n^2)$

Soln :

$n=0 \Rightarrow \quad y(0) = x(0^2) = x(0)$

$n=-1 \Rightarrow \quad \underset{\uparrow \text{present}}{y(-1)} = x((-1)^2) = \underset{\hookrightarrow \text{future}}{x(1)}$

$n=1 \Rightarrow \quad y(1) = x(1^2) = x(1)$

Here, the present output $y(-1)$ depends on future input $x(1)$.

∴ The given system is Non causal.

ii) $y(t) = x^2(t)$.

Soln :

$n=-1 \Rightarrow \quad y(\underline{-1}) = x^2(\underline{-1})$

$n=0 \Rightarrow \quad y(\underline{0}) = x^2(\underline{0})$

$n=1 \Rightarrow \quad y(\underline{1}) = x^2(\underline{1})$

Here, the present output depends on present inputs. ∴ The given system is causal system.

More examples :

$y(n) = x(-n) \rightarrow$ Non causal system

$y(n) = x(n) + x(n-3) \rightarrow$ causal system

$y(n) = x(n) + x(n+2) \rightarrow$ Noncausal system.

$y(t) = \int_{-\infty}^{2t} x(\tau)\,d\tau \rightarrow$ Noncausal s/m [∵ $\underset{\uparrow \text{pre}}{y(2)} = \int_{-\infty}^{4} x(\tau)\,d\tau$

$x(\tau)$ varies from

$x(-\infty) \ldots\ldots \underset{\uparrow \text{fut}}{x(4)}$]

$y(t) = \int_{-\infty}^{t} x(\tau)\,d\tau \rightarrow$ causal system

[$y(2) = \int_{-\infty}^{2} x(\tau)\,d\tau$.

$x(\tau)$ varies from

$x(-\infty) \ldots x(2)$.]

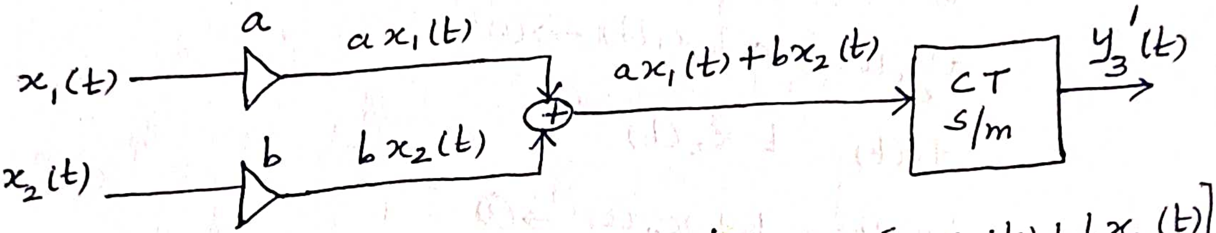
c) Linear and Non-linear system :

A system is said to be linear if it satisfies the superposition principle. Superposition principle has the following properties :

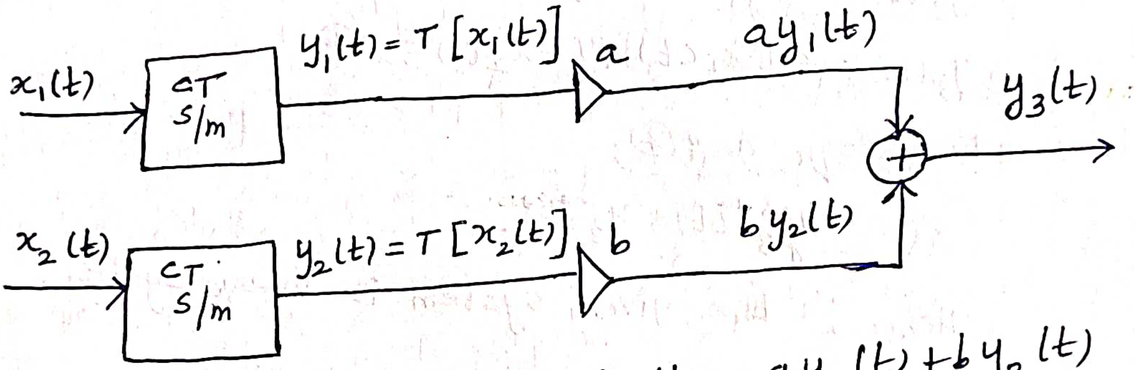
i) Additivity.

ii) scaling (or) homogenity.

Block diagram representation :



$$y_3'(t) = T[a x_1(t) + b x_2(t)]$$



$$y_3(t) = a y_1(t) + b y_2(t)$$

For Linear system,

$$y_3'(t) = y_3(t)$$

For Non Linear system,

$$y_3'(t) \neq y_3(t).$$

Problems :

1. $y(t) = t\,x(t)$

Soln:

$y_3'(t) = t\,[a x_1(t) + b x_2(t)]$

$y_3'(t) = t\,a x_1(t) + t\,b x_2(t) \rightarrow$ ①

$y_1(t) = t\,x_1(t)$

$a y_1(t) = a t x_1(t) \rightarrow$ ②

$y_2(t) = t\,x_2(t)$

$b y_2(t) = b t x_2(t) \rightarrow$ ③

$y_3(t) = a y_1(t) + b y_2(t)$

$y_3(t) = a t x_1(t) + b t x_2(t) \rightarrow$ ④

From eqn ① & ④,

$y_3'(t) = y_3(t)$.

Hence, the given system is linear system.

2. $y(t) = x^2(t)$

Soln:

$y_3'(t) = [a x_1(t) + b x_2(t)]^2$

$y_3'(t) = a^2 x_1^2(t) + b^2 x_2^2(t) + 2ab\,x_1(t)\,x_2(t) \rightarrow$ ①

$y_1(t) = x_1^2(t)$

$a y_1(t) = a x_1^2(t) \rightarrow$ ②

$$y_2(t) = x_2^2(t)$$

$$by_2(t) = bx_2^2(t) \rightarrow$$ ③

$$y_3(t) = ay_1(t)+by_2(t)$$

$$y_3(t) = ax_1^2(t)+bx_2^2(t) \rightarrow$$ ④

From equation ① & ④,

$$y_3'(t) \neq y_3(t).$$

Hence, the given system is Non-linear system.

3. $\frac{dy(t)}{dt} + 3ty(t) = t^2x(t)$

Soln:

$$\frac{dy_3'(t)}{dt} + 3ty_3'(t) = t^2[ax_1(t)+bx_2(t)]$$

Say, $y_3'(t) = t^2ax_1(t)+t^2bx_2(t) \rightarrow$ ①

$$\frac{dy_1(t)}{dt} + 3ty_1(t) = t^2x_1(t)$$

$$a\left[\frac{dy_1(t)}{dt} + 3ty_1(t)\right] = at^2x_1(t) \rightarrow$$ ②

$$\frac{dy_2(t)}{dt} + 3ty_2(t) = t^2x_2(t)$$

$$b\left[\frac{dy_2(t)}{dt} + 3ty_2(t)\right] = bt^2x_2(t) \rightarrow$$ ③

Adding ② and ③, $y_3(t) = at^2x_1(t)+bt^2x_2(t) \rightarrow$ ④

From ① & ④,

$y_3'(t) = y_3(t)$. Hence, the given system is linear system.

4. $y(t) = \int_{-\infty}^{2t} x(\tau)\, d\tau.$

Soln:

$$y_3'(t) = \int_{-\infty}^{2t} [a x_1(\tau) + b.x_2(\tau)]\, d\tau$$

$$= a\int_{-\infty}^{2t} x_1(\tau) d\tau + b\int_{-\infty}^{2t} x_2(\tau)\, d\tau \rightarrow ①$$

$$y_1(t) = \int_{-\infty}^{2t} x_1(\tau)\, d\tau$$

$$a y_1(t) = a\left[\int_{-\infty}^{2t} x_1(\tau) d\tau\right] = a\int_{-\infty}^{2t} x_1(\tau) d\tau \rightarrow ②$$

$$y_2(t) = \int_{-\infty}^{2t} x_2(\tau) d\tau$$

$$b y_2(t) = b\left[\int_{-\infty}^{2t} x_2(\tau)\, d\tau\right] = b\int_{-\infty}^{2t} x_2(\tau) d\tau \rightarrow ③$$

$$y_3(t) = a y_1(t) + b y_2(t)$$

$$y_3(t) = a\int_{-\infty}^{2t} x_1(\tau) d\tau + b\int_{-\infty}^{2t} x_2(\tau)\, d\tau \longrightarrow ④$$

From eqn ① & ④,

$$y_3'(t) = y_3(t)$$

Hence, the given system is linear system.

5. $y(t) = e^{x(t)}$

Soln:

$$y_3'(t) = e^{[a x_1(t) + b x_2(t)]}$$

$$= e^{a x_1(t)} \cdot e^{b x_2(t)} \longrightarrow ①$$

$y_1(t) = e^{x_1(t)}$

$a y_1(t) = a e^{x_1(t)} \rightarrow ②$

$y_2(t) = e^{x_2(t)}$

$b y_2(t) = b e^{x_2(t)} \rightarrow ③$

$\therefore$

$y_3(t) = a y_1(t) + b y_2(t)$

$y_3(t) = a e^{x_1(t)} + b e^{x_2(t)} \rightarrow ④$

From ① & ④,

$y_3'(t) \neq y_3(t)$

Hence, the given system is Non-linear system.

6. $y(t) = \cos 3t \cdot x(t)$.

Soln:

$y_3'(t) = \cos 3t \left[a x_1(t) + b x_2(t) \right]$

$y_3'(t) = a . \cos 3t \, x_1(t) + b . \cos 3t \, x_2(t) \longrightarrow ①$

$y_1(t) = \cos 3t \quad x_1(t)$

$a y_1(t) = a \cos 3t \, x_1(t) \rightarrow ②$

$y_2(t) = \cos 3t \; x_2(t)$

$b y_2(t) = b \cos 3t \; x_2(t) \rightarrow ③$

$y_3(t) = a \cos 3t \; x_1(t) + b \cos 3t \; x_2(t) \longrightarrow ④$

From ① and ④,

$y_3'(t) = y_3(t)$.

Hence, the given system is linear system.

Discrete Time Linear and Nonlinear system :

Problems :

1. $y(n) = x(n-2) - 2x(n-8).$

<u>soln:</u>

$$y_3'(n) = a x_1(n-2) + b x_2(n-2) - 2[a x_1(n-8) + b x_2(n-8)]$$

$$= a x_1(n-2) + b x_2(n-2) - 2a x_1(n-8) - 2b x_2(n-8) \rightarrow ①$$

$$y_1(n) = x_1(n-2) - 2x_1(n-8)$$

$$a y_1(n) = a[x_1(n-2) - 2x_1(n-8)] = a x_1(n-2) - 2a x_1(n-8) \rightarrow ②$$

$$y_2(n) = x_2(n-2) - 2x_2(n-8)$$

$$b y_2(n) = b[x_2(n-2) - 2x_2(n-8)] = b x_2(n-2) - 2b x_2(n-8) \rightarrow ③$$

$$y_3(n) = a x_1(n-2) - 2a x_1(n-8) + b x_2(n-2) - 2b x_2(n-8) \rightarrow ④$$

From ① & ④,

$$y_3'(n) = y_3(n)$$

Hence, the given system is linear system.

2. $y(n) = A x(n) + B$

<u>Soln:</u>

$$y_3'(n) = A[x_1(n) + b x_2(n)] + B$$

$$y_3'(n) = A a x_1(n) + A b x_2(n) + B \rightarrow ①$$

$$y_1(n) = A x_1(n) + B$$

$$a y_1(n) = a[A x_1(n) + B] = A a x_1(n) + Ba \rightarrow ②$$

$$y_2(n) = A x_2(n) + B$$

$$b y_2(n) = b[A x_2(n) + B] = A b x_2(n) + Bb \rightarrow ③$$

$$b y_2(n) = \rightarrow ④$$

$$y_3(n) = A a x_1(n) + Ba + A b x_2(n) + Bb$$

From ① & ④, $y_3'(n) \neq y_3(n)$

∴ The given system is Non-linear system.

3. $y(n) = x(-n)$

Soln:

$y_3'(n) = ax_1(-n) + bx_2(-n) \rightarrow ①$

$y_1(n) = x_1(-n)$

$ay_1(n) = ax_1(-n) \rightarrow ②$

$y_2(n) = x_2(-n)$

$by_2(n) = bx_2(-n) \rightarrow ③$

$y_3(n) = ax_1(-n) + bx_2(-n) \rightarrow ④$

From ① & ④

$y_3'(n) = y_3(n).$

Hence, the given system is linear system.

4. $y(n) = n\,x(n)$

Soln:

$y_3'(n) = n[ax_1(n) + bx_2(n)] = nax_1(n) + nbx_2(n) \rightarrow ①$

$y_1(n) = nx_1(n)$

$ay_1(n) = anx_1(n) \rightarrow ②$

$y_2(n) = nx_2(n)$

$b.y_2(n) = bnx_2(n) \rightarrow ③$

$y_3(n) = ay_1(n) + by_2(n)$

$y_3(n) = anx_1(n) + bnx_2(n) \rightarrow ④$

From ① & ④,

$y_3'(n) = y_3(n)$

Hence, the given system is linear system.

d) Time Invariant and Time variant system :

Input and output characteristics of a system do not change with time. Hence, it is called as Time Invariant system.

Concept :

For continuous Time system,

$y(t, T) = y(t-T) \rightarrow$ For Time Invariant system

where, $y(t, T) \rightarrow$ output due to delayed input

$y(t-T) \rightarrow$ output due to system delay.

For Discrete Time system,

$y(n, k) = y(n-k) \rightarrow$ For Invariant system

where, $y(n,k) \rightarrow$ output due to delayed input

$y(n-k) \rightarrow$ output due to system delay.

$y(t, T) \neq y(t-T) \rightarrow$ Time variant CT s/m

$y(n,k) \neq y(n-k) \rightarrow$ Time variant DT s/m.

Problems:

1. $y(t) = \cos 3t \; x(t)$.

Soln:

$y(t, T) = \cos 3t \; x(t-T) \rightarrow ①$

$y(t-T) = \cos 3(t-T) \; x(t-T) \rightarrow ②$

From ① & ②, $y(t, T) \neq y(t-T)$.

∴ The given system is Time Variant system.

2. $y(t) = x(t-2) + x(2-t)$

Soln:

$$y(t) = x(t-2) + x(2-t)$$

$$y(t,T) = x(t-2-T) + x(2-t-T) \rightarrow ①$$

$$y(t-T) = x(t-T-2) + x(2-(t-T))$$

$$y(t-T) = x(t-2-T) + x(2-t+T) \rightarrow ②$$

From ① and ②, $y(t,T) \neq y(t-T)$

∴ The given system is Time Variant System.

3. $y(t) = \begin{cases} x(t) + x(t-2) & ; t \geq 0 \\ 0 & ; t < 0 \end{cases}$

Soln:

$$y(t) = x(t) + x(t-2)$$

$$y(t,T) = x(t-T) + x(t-2-T) \rightarrow ①$$

$$y(t-T) = x(t-T) + x(t-T-2) \rightarrow ②$$

From ① and ②,

$$y(t,T) = y(t-T)$$

∴ The given system is Time Invariant System.

4. $y(t) = \int_{-\infty}^{2t} x(\tau)\,d\tau$

Soln: $y(t) = \int_{-\infty}^{2t} x(\tau)\,d\tau$

$$y(t,T) = \int_{-\infty}^{2t-T} x(\tau)\,d\tau \rightarrow ①$$

$$y(t-T) = \int_{-\infty}^{2(t-T)} x(\tau)\,d\tau \rightarrow ②$$

From ① and ②, ∴ The given system is Time variant System

$y(t,T) \neq y(t-T)$.

5. $y(t) = x(\frac{t}{3})$.

Soln:

$$y(t) = x(\frac{t}{3})$$

$$y(t,T) = x(\frac{t}{3} - T) \rightarrow ①$$

$$y(t-T) = x(\frac{t-T}{3}) \rightarrow ②$$

From ① and ②,

$$y(t,T) \neq y(t-T)$$

∴ The given system is Time Variant system.

6. $y(t) = \frac{dx(t)}{dt}$

Soln:

$$y(t) = \frac{dx(t)}{dt}$$

$$y(t,T) = \frac{d}{dt} x(t-T) \rightarrow ①$$

$$y(t-T) = \frac{d}{dt} x(t-T) \rightarrow ②$$

From ① and ②,

$$y(t,T) = y(t-T)$$

∴ The given system is Time Invariant system.

7. $y(t) = \sin[x(t)]$

Soln:

$$y(t) = \sin[x(t)]$$

$$y(t,T) = \sin[x(t-T)] \rightarrow ①$$

$$y(t-T) = \sin[x(t-T)] \rightarrow ②$$

From ① and ②, $y(t,T) = y(t-T)$

∴ The given system is Time Invariant system.

8. $y(t) = x(2t)$.

Soln:

$y(t) = x(2t)$.

$y(t,T) = x(2t - T) \rightarrow ①$

$y(t-T) = x(2(t-T))$

$y(t-T) = x(2t - 2T) \rightarrow ②$

$y(t-T) = x(2t - 2T)$

From ① & ②, $y(t,T) \neq y(t-T)$

∴ The given system is Time ... Variant system.

9. $y(n) = n x(n)$.

Soln:

$y(n) = n x(n)$

$y(n,k) = n x(n-k) \rightarrow ①$

$y(n-k) = (n-k) x(n-k) \rightarrow ②$

From ① and ②, $y(n,k) \neq y(n-k)$

∴ The given system is Time Variant system.

10. $y(n) = Ev\{x(n-1)\}$

Soln:

$y(n) = Ev\{x(n-1)\}$

$$y(n) = \frac{x(n-1) + x(-n-1)}{2}$$

$$y(n,k) = \frac{x(n-1-k) + x(-n-1-k)}{2} \rightarrow ①$$

$$y(n-k) = \frac{x(n-k-1) + x(-(n-k)-1)}{2}$$

$$y(n-k) = \frac{x(n-1-k) + x(-n-1+k)}{2} \rightarrow ②$$

From ① & ②,

$y(n,k) \neq y(n-k)$.

∴ The given system is Time Variant system.

11. $y(n) = x(-n)$.

Soln:

$y(n) = x(-n)$.

$y(n,k) = x(-n-k) \rightarrow ①$

$y(n-k) = x(-(n-k))$.

$y(n-k) = x(-n+k) \rightarrow ②$

From ① and ②, $y(n,k) \neq y(n-k)$

∴ The given system is Time variant system.

12. $y(n) = x(n^2)$

Soln:

$y(n) = x(n^2)$

$y(n,k) = x(n^2 - k) \rightarrow ①$

$y(n-k) = x(\cdot (n-k)^2)$

$y(n-k) = x(n^2 - 2kn + k^2) \rightarrow ②$

From ① & ②, $y(n,k) \neq y(n-k)$.

∴ The given system is Time variant system.

13. $y(n) = x^2(n)$

Soln:

$y(n) = x^2(n)$

$y(n,k) = x^2(n-k) \rightarrow ①$

$y(n-k) = x^2(n-k) \rightarrow ②$

From ① and ②, $y(n,k) = y(n-k)$.

∴ The given system is Time Invariant system.

e) Stable system and Unstable System.

The system is said to be stable, if the system produces bounded output for every bounded input.

BIBO (Bounded Input and Bounded output) stable conditions:

Input : $|x(t)| < \infty \rightarrow$ Bounded Input

output : $|y(t)| < \infty \rightarrow$ Bounded output

Problems:

I. check the following systems are stable or not.

1. $y(t) = 10x(t) + 5$.

Soln:

$$y(t) = 10\underset{\displaystyle u(t)}{\underset{\downarrow}{x(t)}} + 5 \qquad \left[u(t) = \begin{matrix} 1 & ; & t \geq 0 \\ 0 & ; & t < 0 \end{matrix}\right]$$

As long as $x(t)$ is bounded, the output $y(t)$ is also bounded.

∴ The given system is stable.

2. $$y(t) = \underset{\displaystyle u(t).}{\underset{\downarrow}{x\left(\frac{t}{3}\right)}}$$

As long as $x\left(\frac{t}{3}\right)$ is bounded, the output $y(t)$ is also bounded.

∴ The given system is stable.

3. $$y(t) = \cos 3t \cdot \underset{\displaystyle u(t).}{\underset{\downarrow}{x(t)}}$$

The maximum value of cosine signal is unity. "1".

As long as $x(t)$ is bounded $y(t)$ is also bounded.

∴ The given system is stable.

4. $y(n) = \sum_{k=-\infty}^{n} u(k)$.

Soln:

$$y(n) = \sum_{k=-\infty}^{n} u(k)$$

$$= u(-\infty) + \cdots + u(-1) + u(0) + u(1) + \cdots + u(n).$$

$$= 0 + \cdots\cdots + 0 + 1 + 1 + \cdots \quad + u(n). \rightarrow ①$$

If $n \to$ finite, $y(n) < \infty$. [∵ $y(n) = n+1$].

But $n \to \infty$, $y(\infty) = 0 + \cdots + 0 + 1 + 1 + \cdots + u(\infty)$ [using eqn ①]

$$= 0 + \cdots + 0 + \underbrace{1 + 1 + \cdots + 1}_{'\infty'\text{ times }'1'.}$$

$$y(\infty) = \infty + 1$$

$$y(\infty) = \infty.$$

∴ When $n \to \infty$, $y(\infty) = \infty$. ∴ The given system is unstable.

5. $y(n) = \sum_{k=-\infty}^{n+1} x(k).$

Soln:

$$y(n) = \sum_{k=-\infty}^{n+1} \underset{\downarrow u(k)}{x(k)}.$$

$$= u(-\infty) + \cdots + u(-1) + u(0) + u(1) + \cdots + u(n) + u(n+1)$$

$$y(n) = 0 + \cdots + 0 + 1 + 1 + \cdots + u(n) + u(n+1) \rightarrow ①$$

If 'n' is finite, $y(n) < \infty$. [$y(n) = n+2$]

But, $n \to \infty$, $y(\infty) = 0 + \cdots + 0 + 1 + 1 + \cdots + u(\infty) + u(\infty + 1)$

$$= 0 + \cdots + 0 + \underbrace{1 + 1 + \cdots + 1}_{'\infty'\text{ times }'1'.}$$ [∵ using eqn ①].

$$y(\infty) = \infty + 2$$

$$y(\infty) = \infty$$

∴ The given system isunstable.

6. $y(n) = n\,x(n)$.

Soln:

$$y(n) = n\,\underset{\substack{\downarrow \\ u(n)}}{x(n)}$$

If 'n' is finite, $y(n) < \infty$. But, $n \to \infty$,

$$y(\infty) = \infty \cdot u(\infty) = \infty \cdot 1 = \infty.$$

∴ The given system is unstable system.

7. $y(t) = t\,x(t)$.

Soln:

$$y(t) = t\,\underset{\substack{\downarrow \\ u(t)}}{x(t)}.$$

If 't' is finite, $y(t) < \infty$. But, $t \to \infty$, $y(\infty) = \infty . u(\infty)$

$$y(\infty) = \infty \cdot 1 = \infty$$

∴ The given system is unstable system.

8. $y(t) = \int_{-\infty}^{2t} x(\tau)\,d\tau$.

Soln:

$$y(t) = \int_{-\infty}^{2t} \underset{\substack{\downarrow \\ u(\tau)}}{x(\tau)}\,d\tau$$

$$y(t) = u(-\infty) + \dots + u(-1) + u(0) + u(1) + \dots + u(2t).$$

If 't' is finite, $y(t) < \infty$.

But, $t \to \infty$, $y(\infty) = u(-\infty) + \dots + u(-1) + \underbrace{u(0) + u(1) + \dots + u(2 \cdot \infty)}$

$$y(\infty) = 0 + \dots + 0 + \underbrace{1 + 1 + 1 + \dots + 1}_{\text{'}\infty\text{' times '1'}}$$

$$y(\infty) = \infty$$

∴ The given system is unstable system. $[2 \cdot \infty = \infty]$

9. $y(t) = x(t-2) + x(2-t)$.

Soln:

$$y(t) = \underset{\overset{\downarrow}{u(t-2)}}{x(t-2)} + \underset{\overset{\downarrow}{u(2-t)}}{x(2-t)}.$$

As long as $x(t-2)$ and $x(2-t)$ are bounded, the output $y(t)$ is also bounded.

$\therefore$ The given system is stable.

10. $y(n) = \underset{\overset{\downarrow}{u(n^2)}}{x(n^2)}$.

As long as $x(n^2)$ is bounded, the output $y(n)$ is also bounded. $\therefore$ The given system is stable.

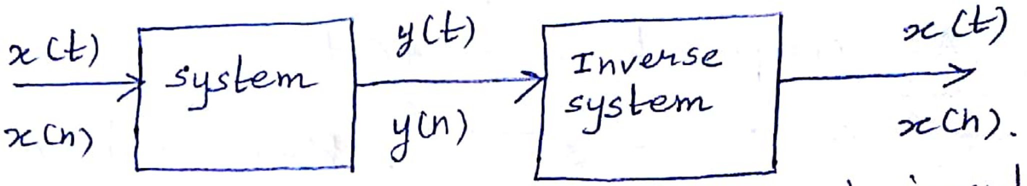
11. $y(n) = \underset{\overset{\downarrow}{u(-n)}}{x(-n)}$.

As long as $x(-n)$ is bounded, the output $y(n)$ is also bounded. $\therefore$ The given system is stable.

f) Inverse system :

The system is said to be inverse system (or) invertible, if the system produces unique output for every unique input.

Inverse system model:



U.Q. → Is the system output $y(t) = x^2(t)$ is invertible or not?

Soln:

$$y(t) = x^2(t).$$

Here, the system performs squaring operation on the given input $x(t)$. The inverse system performs square root operation on its input signal.

So, the inverse system output $= \sqrt{y(t)} = \sqrt{x^2(t)}$

$$= \pm x(t).$$

The inverse system output is not equal to the input of main system $x(t)$.

∴ The given system is not invertible.

U.Q. → A signal $x(t) = \cos 2\pi ft$ whose input and output is related by $y(t) = x^2(t)$. What are the frequency components in the output?

Soln:

$$x(t) = \cos 2\pi ft$$

$$y(t) = x^2(t)$$

$$y(t) = \cos^2 2\pi f t$$

$$y(t) = \frac{1+\cos 2(2\pi f t)}{2} \qquad \left[\because \cos^2\theta = \frac{1+\cos 2\theta}{2}\right]$$

$$y(t) = \frac{1}{2} + \frac{\cos 4\pi f t}{2}$$

DC component $= \frac{1}{2}$

AC component $= \frac{\cos 4\pi f t}{2}$

U.Q. → Given $g(n) = 2e^{-2n-3}$. Write out and simplify the functions i) $g(2-n)$ ii) $g(\frac{n}{10}+4)$.

Soln:

i) given: $g(n) = 2e^{-2n-3}$

$$g(2-n) = 2e^{-2(2-n)-3} = 2e^{-4+2n-3}$$

$$\boxed{g(2-n) = 2e^{2n-7}}$$

ii) given: $g(n) = 2e^{-2n-3}$

$$g(\tfrac{n}{10}+4) = 2e^{-2(\frac{n}{10}+4)-3} = 2e^{-\frac{n}{5}-8-3}$$

$$\boxed{g(\tfrac{n}{10}+4) = 2e^{-\frac{n}{5}-11}}$$

U.Q → Is diode a linear device? Give your reason.

No. We know that diode operates (or) conducts only when it is forward biased. It does not conduct when it is reverse biased. Therefore, diode is a non linear device.

2 mark related questions.

Evaluate the following integrals:

1) $\int_{-\infty}^{\infty} e^{-at^2} \delta(t-10)\,dt$

Soln: $$\int_{-\infty}^{\infty} e^{-at^2} \delta(t-10)\,dt = e^{-at^2}\Big|_{t=10} = e^{-100a}$$

$$\boxed{\text{Property: } \int_{-\infty}^{\infty} x(t)\,\delta(t-t_0)\,dt = x(t_0)} \quad \boxed{\delta(t-10) = \begin{cases} 1 & ; t=10 \\ 0 & ; t\neq 10 \end{cases}}$$

2) $\int_{0}^{\infty} t^2 \delta(t-3)\,dt$

Soln: $$\int_{0}^{\infty} t^2 \delta(t-3)\,dt = t^2\Big|_{t=3} = 3^2 = 9$$

3) $$\int_{0}^{5} \delta(t) \sin 2\pi t\,dt = \sin 2\pi t\Big|_{t=0} = \sin 0 = 0$$

4) $$\int_{-\infty}^{\infty} \delta(t+3)\, e^{-t}\,dt = e^{-t}\Big|_{t=-3} = e^{3}$$

5) $$\int_{-\infty}^{\infty} (t-3)^2 \delta(t-3)\,dt = (t-3)^2\Big|_{3} = 0$$

6) $$\int_{-\infty}^{\infty} [\delta(t)\cos t + \delta(t-1)\sin t]\,dt = \cos t\Big|_{t=0} + \sin t\Big|_{t=1}$$
$$= \cos 0 + \sin 1 = 1 + \sin 1$$

7) $\int_{-\infty}^{\infty} \delta(t)\, e^{-j\Omega t}\, dt = e^{-j\Omega t}\Big|_{t=0} = e^{0} = 1$

8) $\int_{-\infty}^{\infty} \delta(\tau) \cdot x(t-\tau)\, dt = x(t-\tau)\Big|_{\tau=0} = x(t)$

9) $\int_{-\infty}^{\infty} \delta(t-2) \sin \pi t\, dt = \sin \pi t\Big|_{t=2} = \sin 2\pi = 0$

10) $\int_{-\infty}^{\infty} \delta(t) \cos \pi t\, dt = \cos \pi t\Big|_{t=0} = \cos 0 = 1$

11) $\int_{-\infty}^{\infty} x(2-t)\, \delta(3-t)\, dt = x(2-t)\Big|_{t=3} = x(2-3) = x(-1)$

12) $\int_{-\infty}^{\infty} (t^3+4)\, \delta(1-t)\, dt = (t^3+4)\Big|_{t=1} = 1^3+4 = 5$

13) $\int_{-\infty}^{\infty} e^{-t}\, \delta(2t-2)\, dt = e^{-t}\Big|_{t=1} = e^{-1}$

14) $\int_{-1}^{1} (2t^2+3)\, \delta(t) \cdot dt = (2t^2+3)\Big|_{t=0} = 3.$ [Nov/Dec 2018]

15) $\int_{-\infty}^{\infty} e^{-2t}\, \delta(t+2)\, dt = e^{-2t}\Big|_{t=-2} = e^{4}.$

[Apr/May 2016]

Find the following summations:

$$\boxed{\delta(n) = \begin{cases} 1 ; & n=0 \\ 0 ; & n \neq 0 \end{cases}}$$

1) $$\sum_{n=-\infty}^{\infty} e^{2n} \delta(n-2) = e^{2n}\Big|_{n=2} = e^{4}$$

2) $$\sum_{n=-\infty}^{\infty} \delta(n-1) \sin 2n = \sin 2n\Big|_{n=1} = \sin 2$$

3) $$\sum_{n=-\infty}^{\infty} n^2 \delta(n+2) = n^2\Big|_{n=-2} = (-2)^2 = 4$$

4) $$\sum_{n=0}^{\infty} \delta(n-1) e^{n^2} = e^{n^2}\Big|_{n=1} = e^{1} = e$$

5) $$\sum_{n=0}^{5} \delta(n+1) 2^{n} = 2^{n}\Big|_{n=-1} = 2^{-1} = \frac{1}{2}$$

6) $$\sum_{n=2}^{\infty} \delta(n-1) \sin 2n = \sin 2n\Big|_{n=1} = \sin 1$$ [April/May 2017]

7) $$\sum_{n=0}^{\infty} x(n) \delta(n-2) = x(n)\Big|_{n=2} = x(2)$$

8) $$\sum_{n=-\infty}^{\infty} a^{n-2} \delta(n+3) = a^{n-2}\Big|_{n=-3} = a^{-3-2} = a^{-5} = \frac{1}{a^5}$$

Basic Signal Operations (Transformations in variable of a signal)

Priority to perform basic signal operations

1. Time Reversal or Time Folding
2. Time Shifting
3. Time Scaling
4. Addition, Multiplication

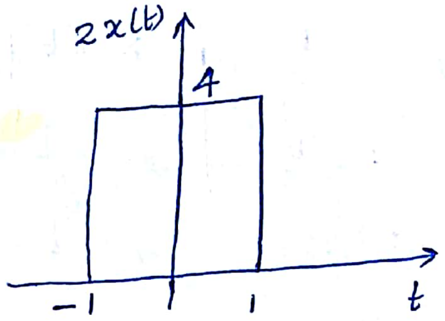
Signal Basic operations [Tranformations in Variable of a signal]

There are two variable parameters in general:

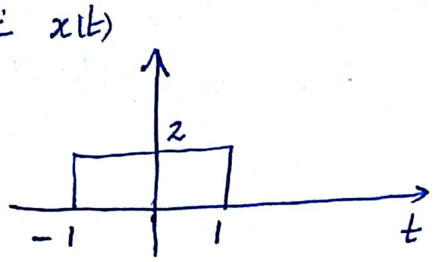
a) Amplitude b) Time

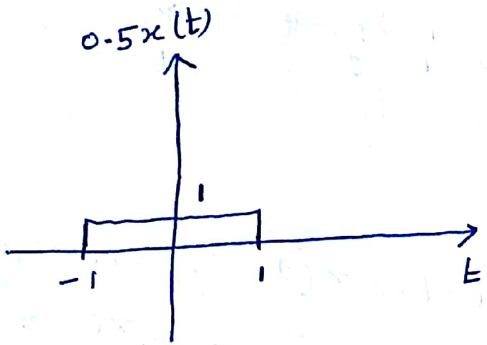
i) Amplitude Scaling:

$c\,x(t)$ is a amplitude scaled version of $x(t)$ whose amplitude is scaled by a factor c.



Ex:





ii) Time scaling:

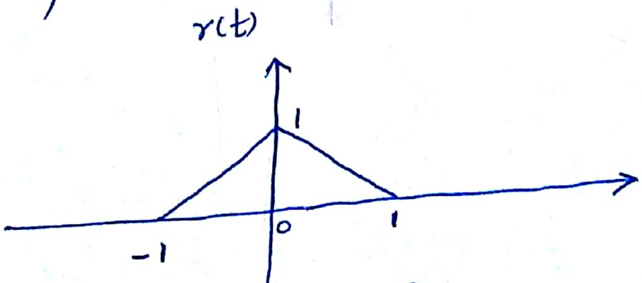
$x(at)$ is time scaled version of the signal $x(t)$.

where 'a' is always positive.

$a > 1$; compression of the signal

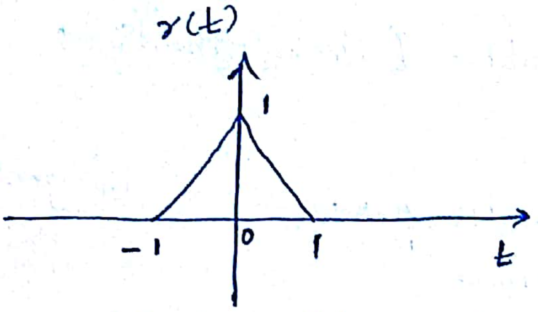
$0 < a < 1$; expansion of the signal:

Example:



To find $r(\frac{t}{2})$ and $r(2t)$:

Soln:



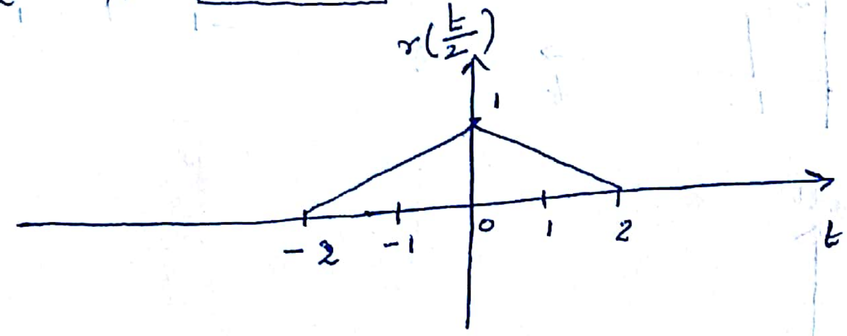
$r(-1) = 0, \quad r(0) = 1, \quad r(1) = 0.$

* To find $r\left(\frac{t}{2}\right)$:

$\frac{t}{2} = -1 \Rightarrow \boxed{t = -2}$; $r\left(\frac{t}{2}\right) = r\left(\frac{-2}{2}\right) = r(-1) = 0.$

$\frac{t}{2} = 0 \Rightarrow \boxed{t = 0}$; $r\left(\frac{t}{2}\right) = r\left(\frac{0}{2}\right) = r(0) = 1$

$\frac{t}{2} = 1 \Rightarrow \boxed{t = 2}$; $r\left(\frac{t}{2}\right) = r\left(\frac{2}{2}\right) = r(1) = 0.$

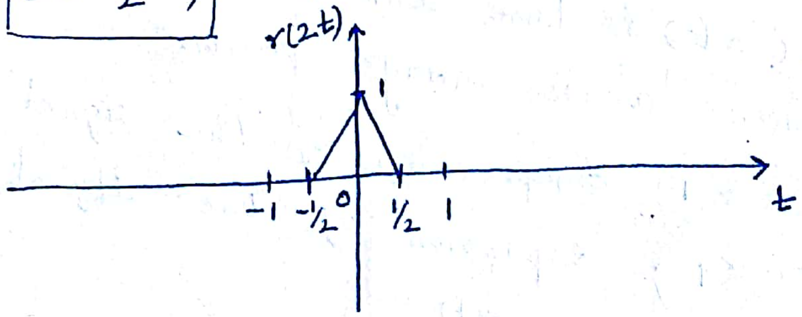


* To find $r(2t)$:

$2t = -1 \Rightarrow \boxed{t = -\frac{1}{2}}$; $r(2t) = r\left(2\times -\frac{1}{2}\right) = r(-1) = 0$

$2t = 0 \Rightarrow \boxed{t = 0}$; $r(2t) = r(2\times 0) = r(0) = 1.$

$2t = 1 \Rightarrow \boxed{t = \frac{1}{2}}$; $r(2t) = r\left(2\times\frac{1}{2}\right) = r(1) = 0$



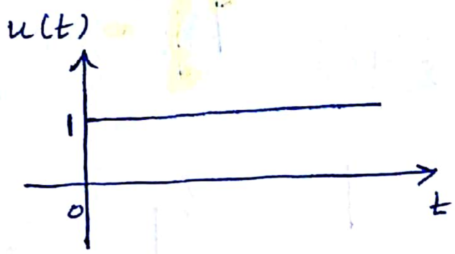
iii) Time shifting:

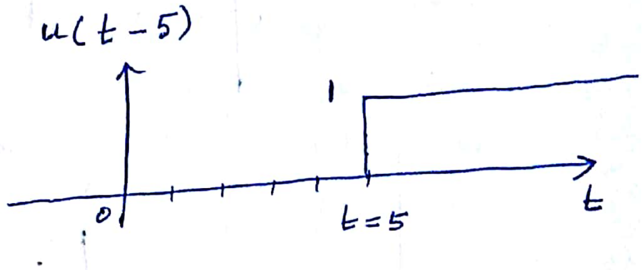
$x(t \pm t_0)$ is time shifted version of the signal $x(t)$.

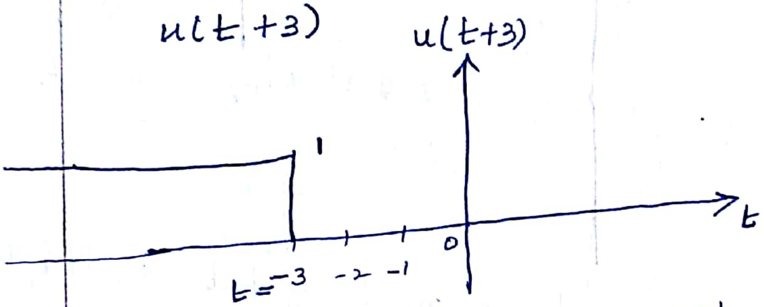
$x(t+t_0) \rightarrow$ negative shift [delayed version (or) left side shift]

$x(t-t_0) \rightarrow$ positive shift [advanced version (or) Right side shift]

Example:



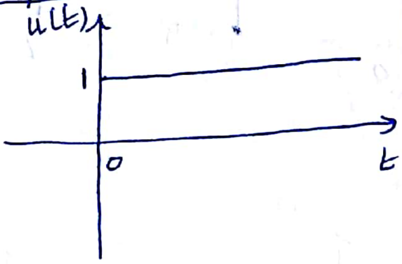


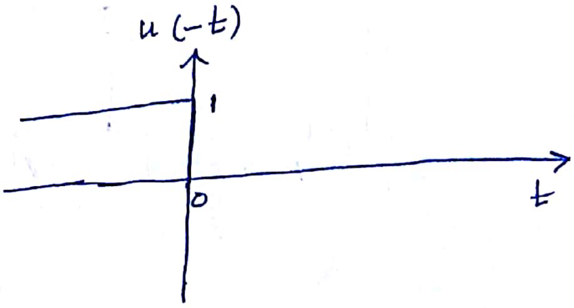


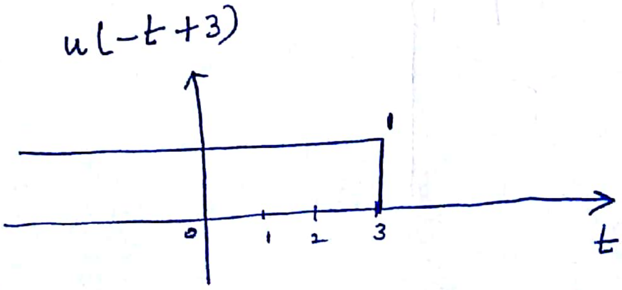
iv) Time reversal (or) Time folding:

$x(-t)$ is the time reversal (or) folded version of the signal $x(t)$.

Example :

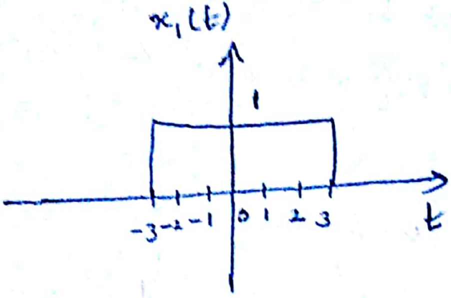


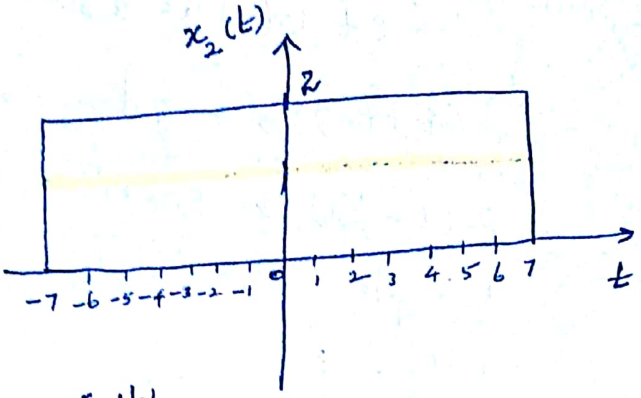


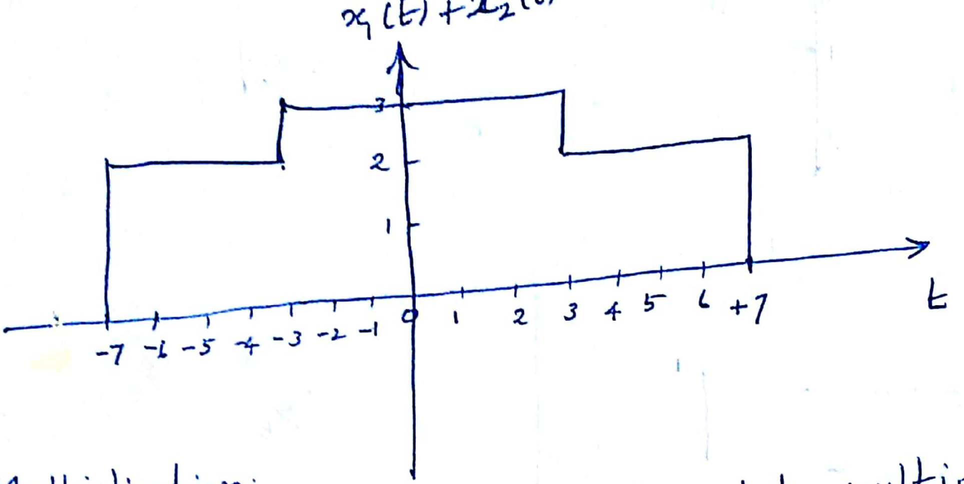


v) Addition:

Addition of two signals is nothing but addition of their corresponding amplitudes.

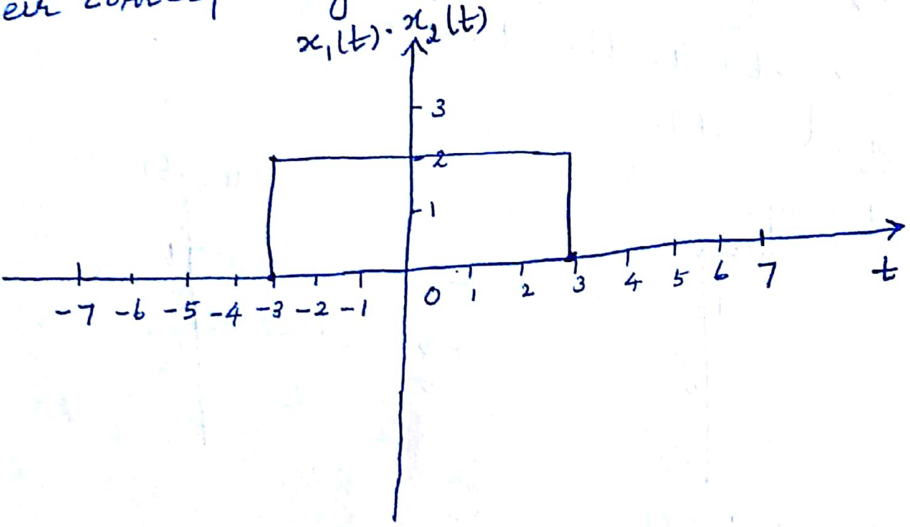






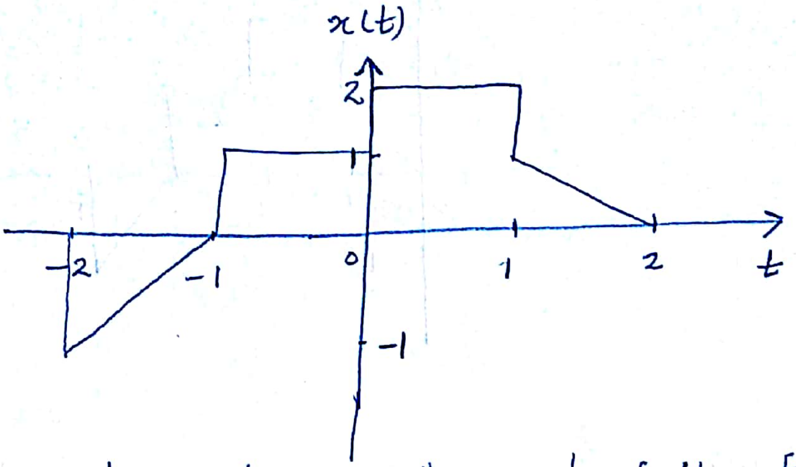
vi) Multiplication:

Multiplication of two signals is nothing but multiplication of their corresponding amplitudes.



Exercise Problems: Page No : 59 TB1
1.21

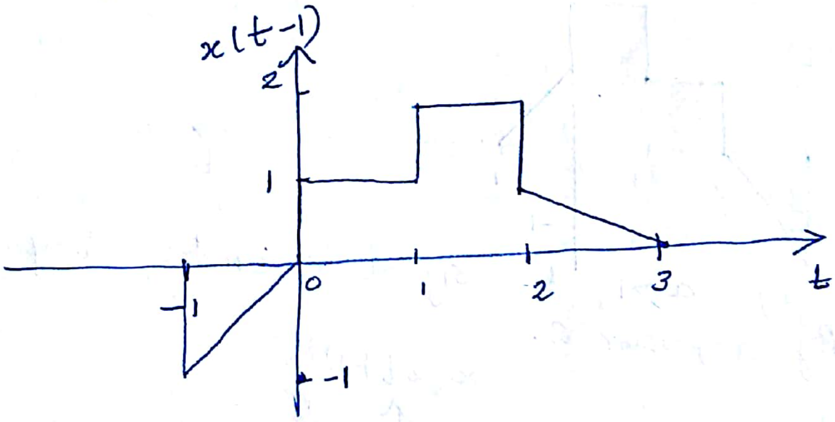
1) A continuous time signal $x(t)$ is given by



sketch and Label carefully each of the following signals.

a) $x(t-1)$:

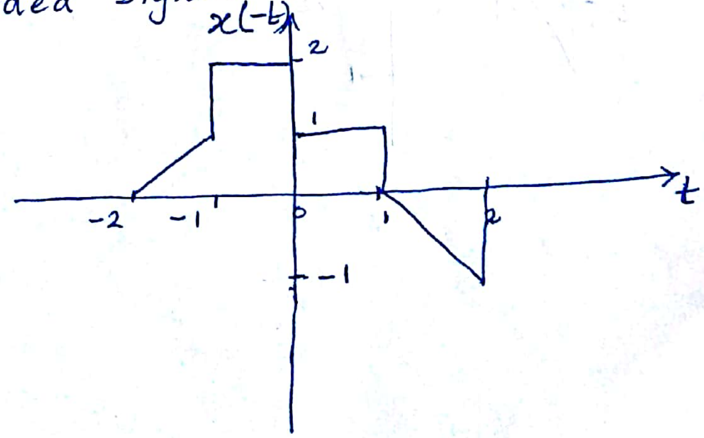
$x(t)$ is shifted to the right by 1 unit.



b) $x(2-t)$.

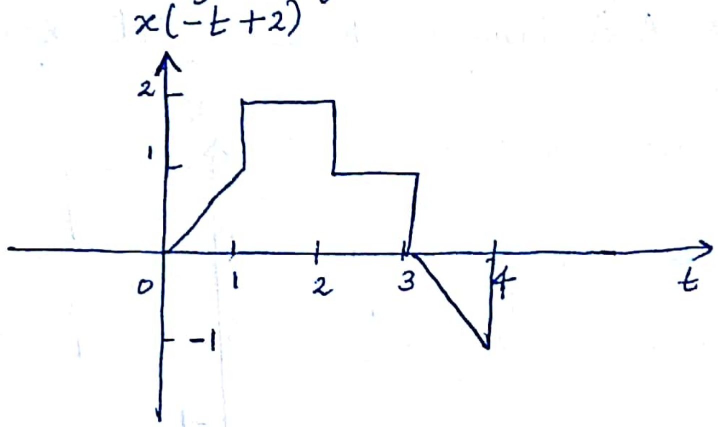
$x(2-t) = x(-t+2)$.

The folded signal of $x(t)$ is $x(-t)$.



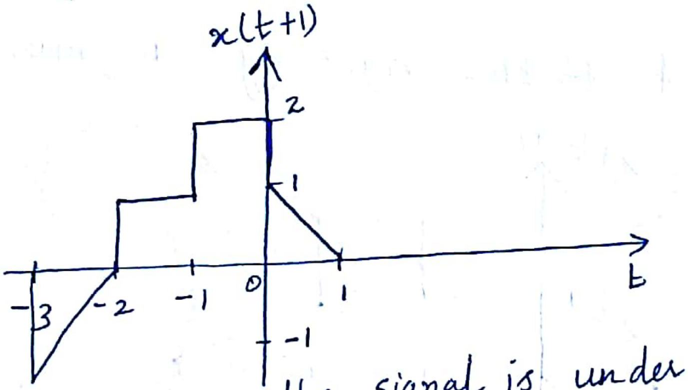
①

$x(-t)$ is shifted to right by 2 units.



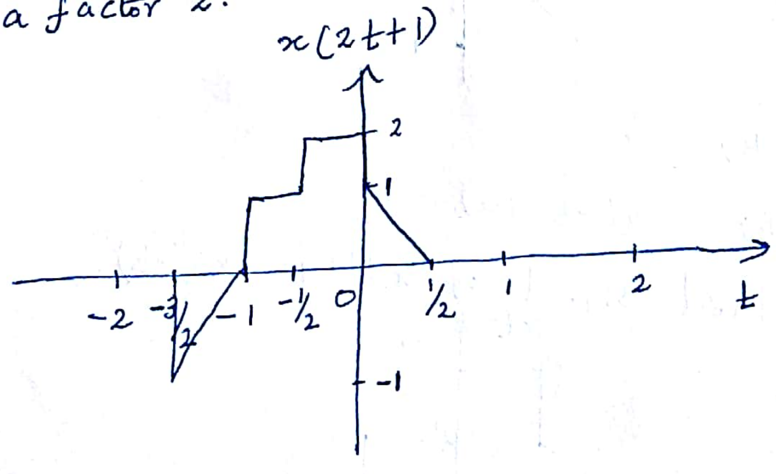
c) $x(2t+1)$.

To find $x(t+1)$: $x(t)$ is shifted to left by 1 unit



Here, $a=2$

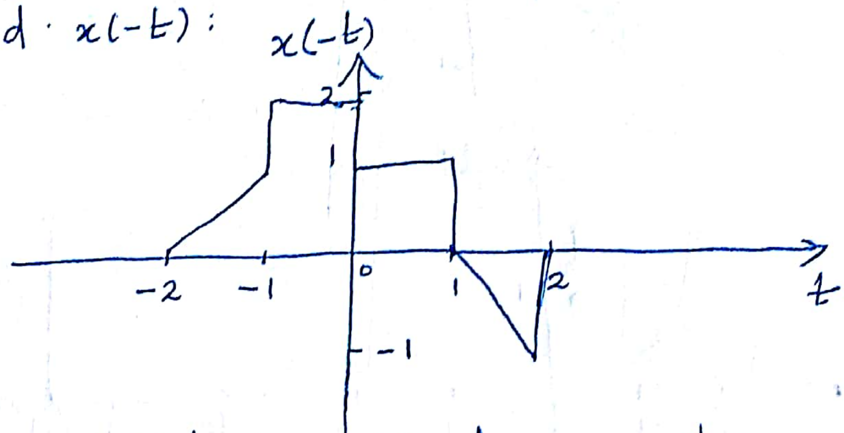
If $a>1$, the signal is under compression By a factor 2.



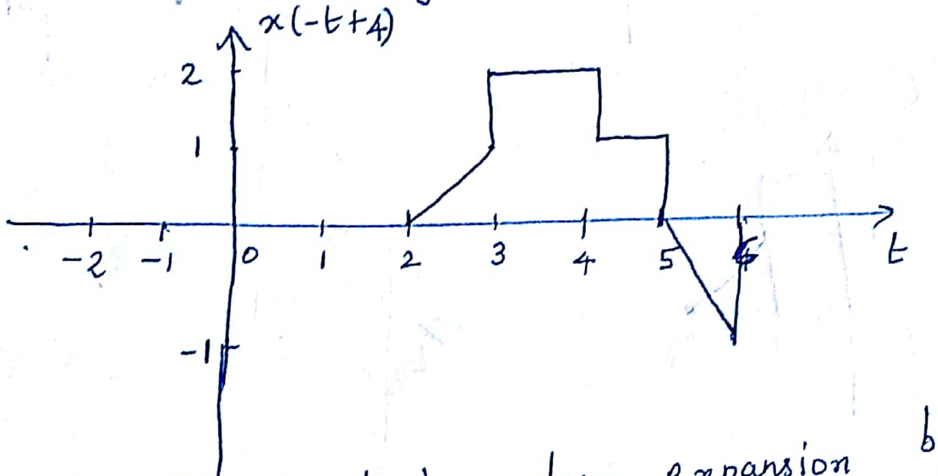
d) $x(4-\frac{t}{2})$:

$x(4-\frac{t}{2}) = x(-\frac{t}{2}+4)$

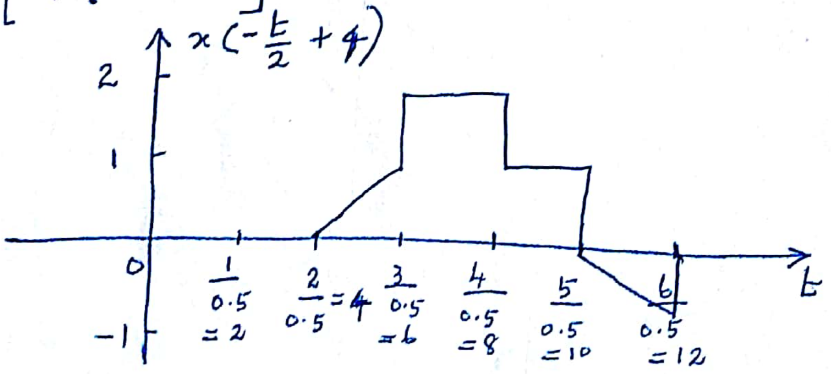
To find $x(-t)$:



$x(-t)$ is shifted to right by 4 units.



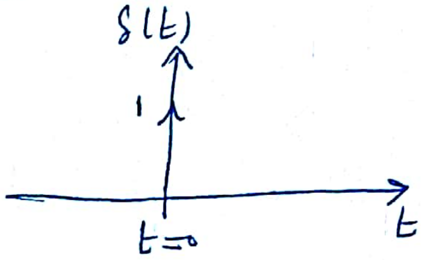
If $a<1$, the signal is under expansion by a factor of ~~2~~ 0.5. [$1 \div 0.5 = 2$]

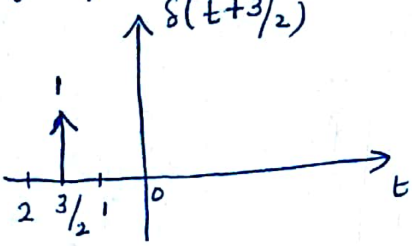


f) $x(t)\left[\delta(t+3/2) - \delta(t-3/2)\right]$:

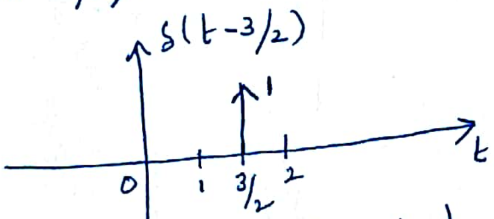
$\delta(t+3/2)$

$\delta(t)$ is shifted to the left by $3/2$ units.

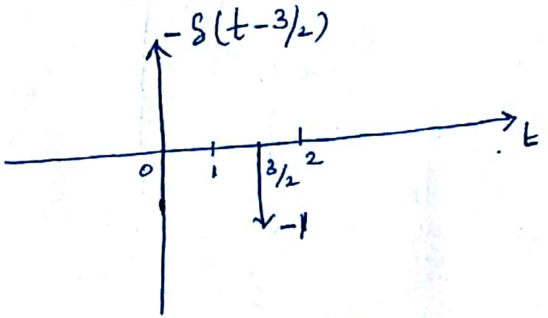




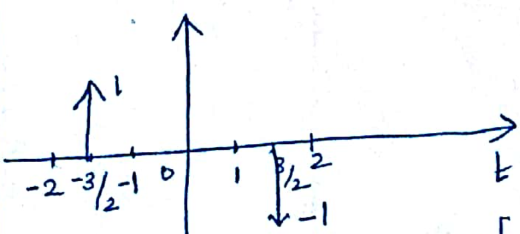
$\delta(t-3/2)$: $\delta(t)$ is shifted to the right by $3/2$ units.



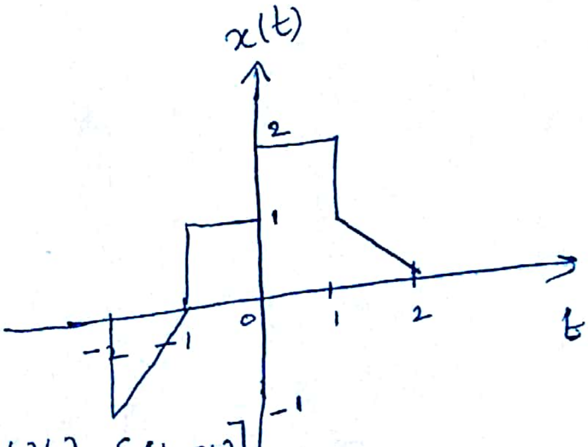
$-\delta(t-3/2)$ is inverted version of $\delta(t-3/2)$.



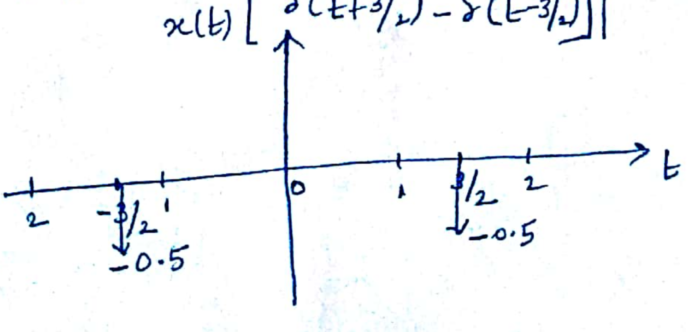
$\delta(t+3/2) - \delta(t-3/2)$



×



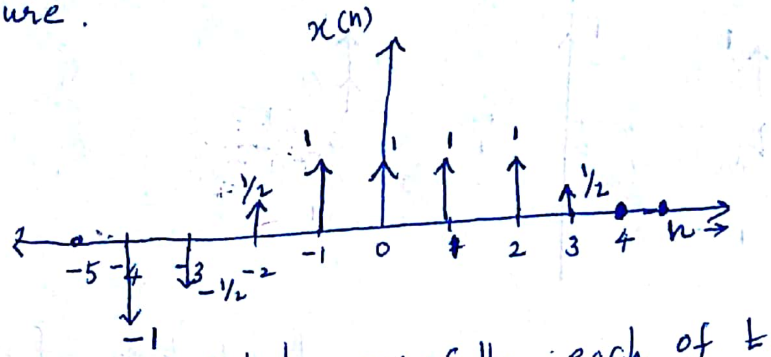
Answer:



Exercise Problems : Page No: 59 TB 1

1.22.

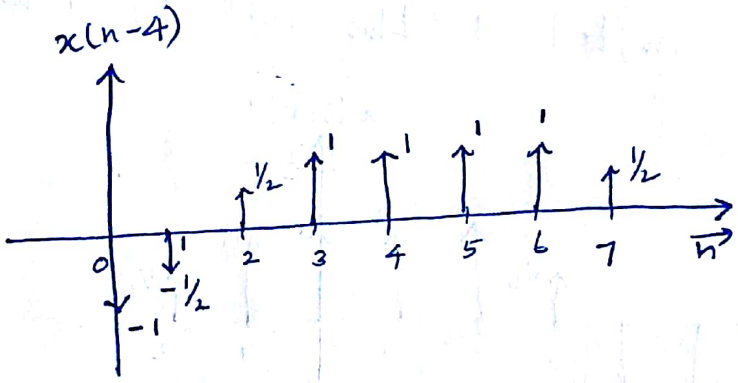
② A discrete- time signal is shown in the following figure.



Sketch and label carefully each of the following signals.

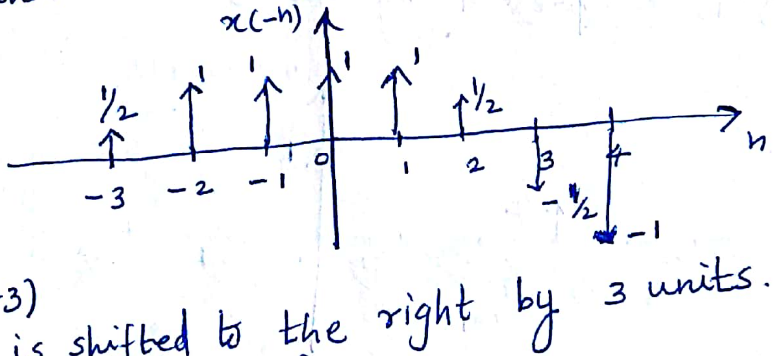
a) x(n-4)

x(n) is shifted to the right by 4 units.



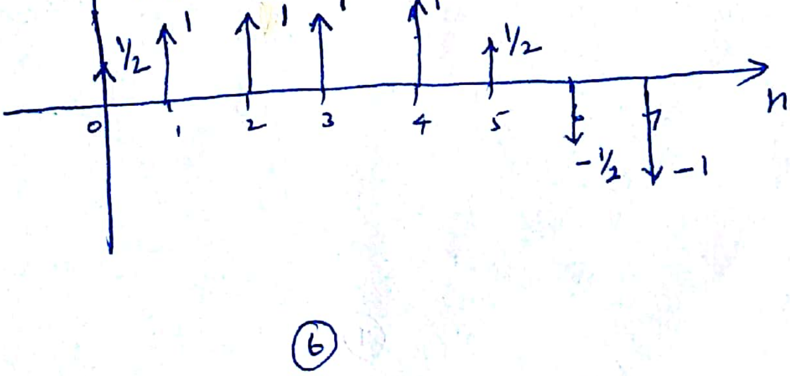
b) x(3-n)

The folded version of x(n) is x(-n).

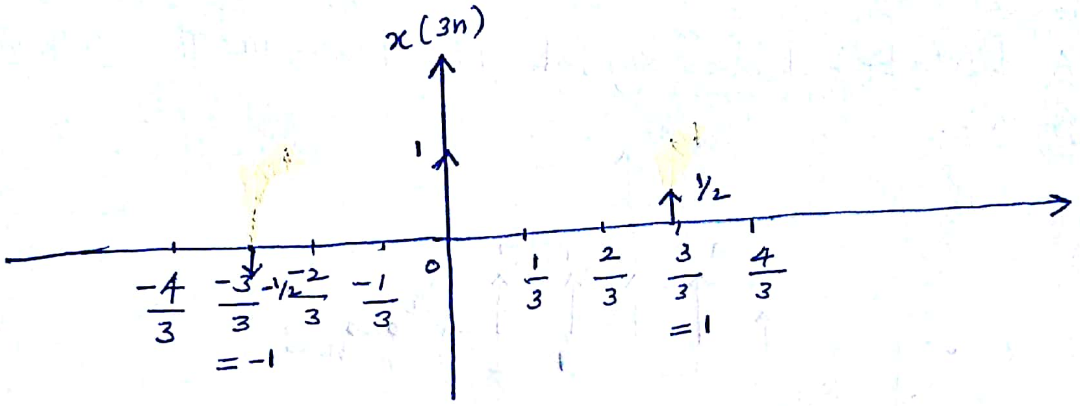


x(-n+3)

x(-n) is shifted to the right by 3 units.



c) $x(3n)$.

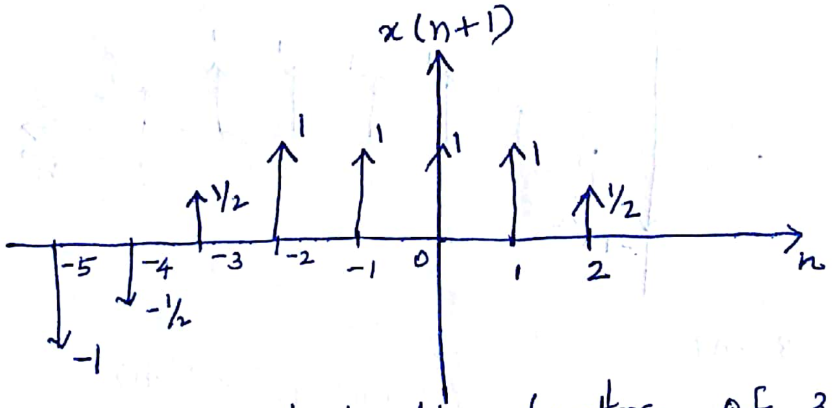


The signal $x(n)$ is compressed by the factor of 3.

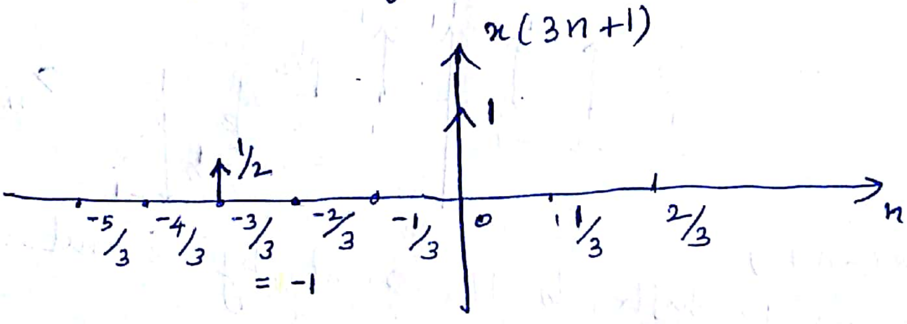
d) $x(3n+1)$

$x(n+1)$

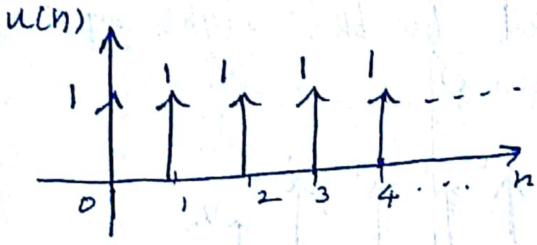
$x(n)$ is shifted to the left by 1 unit.

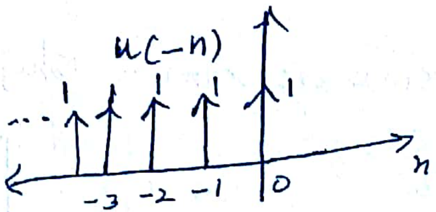


$x(n+1)$ is compressed by the factor of 3.



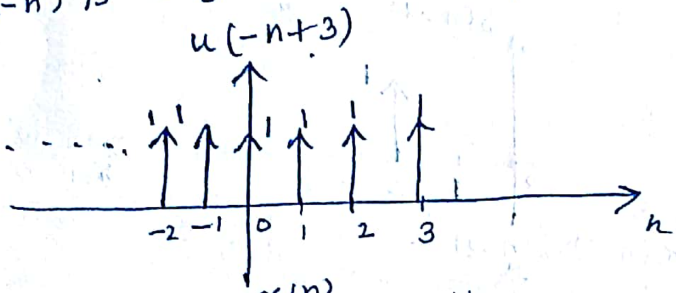
e) $x(n)\, u(3-n)$

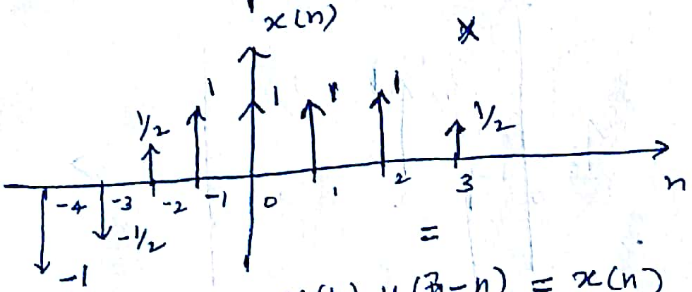




$u(-n+3)$

$u(-n)$ is shifted to the right by 3 units.

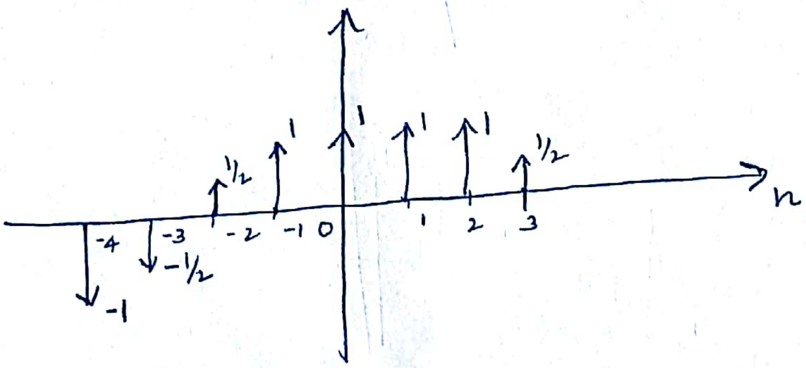




×

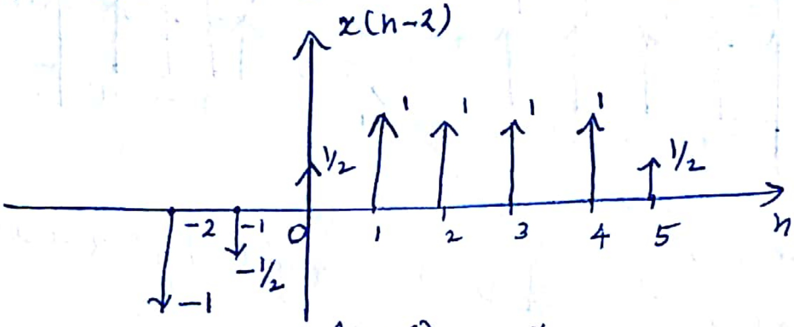
=

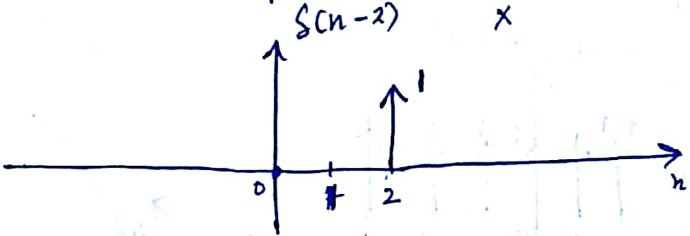
$x(n)\, u(3-n) = x(n)$.

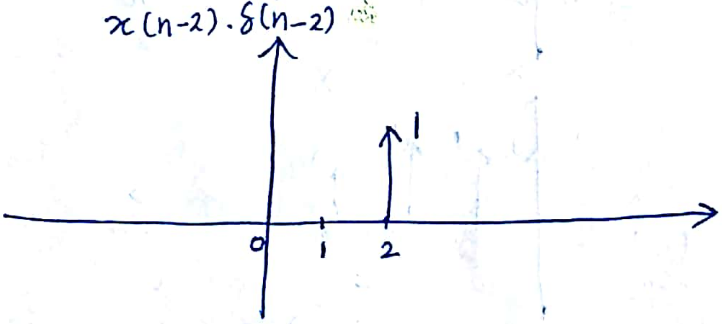


f) $x(n-2)\,\delta(n-2)$

$x(n-2)$: $x(n)$ is shifted to the right by 2 units.







9) $\frac{1}{2} x(n) + \frac{1}{2} (-1)^n x(n)$.

To find $\frac{1}{2} (-1)^n x(n)$:

$n = -4$; $\frac{1}{2} (-1)^{-4} x(-4) = \frac{x(-4)}{2} = \frac{-1}{2}$

$n = -3$; $\frac{1}{2} (-1)^{-3} x(-3) = -\frac{x(-3)}{2} = \frac{-(-1/2)}{2} = \frac{1}{4}$

$n = -2$; $\frac{1}{2} (-1)^{2} x(-2) = \frac{x(-2)}{2} = \frac{\frac{1}{2}}{2} = \frac{1}{4}$

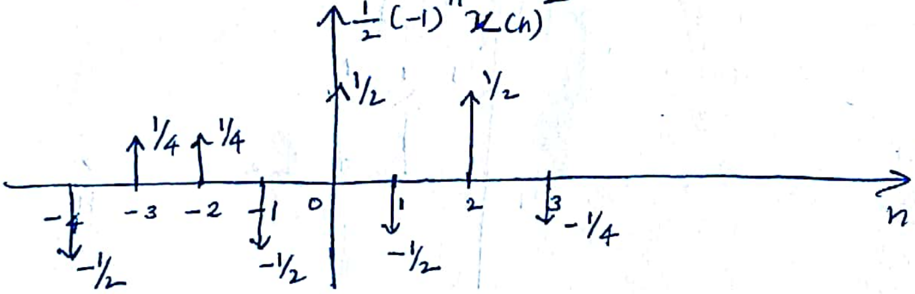
$n = -1$; $\frac{1}{2} (-1)^{-1} x(-1) = -\frac{x(-1)}{2} = -\frac{1}{2}$

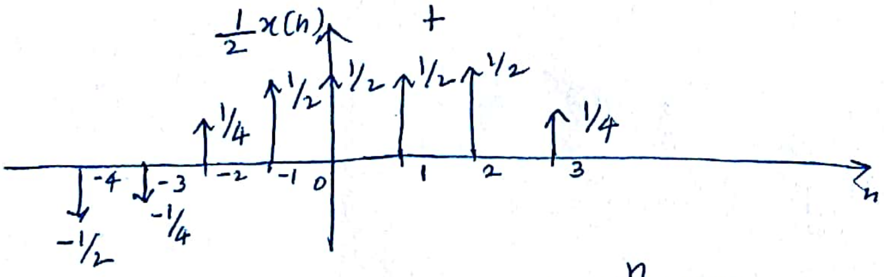
$n = 0$; $\frac{1}{2} (-1)^{0} x(0) = \frac{x(0)}{2} = \frac{1}{2}$

$n = 1$; $\frac{1}{2} (-1)^{1} x(1) = -\frac{x(1)}{2} = -\frac{1}{2}$

$n = 2$; $\frac{1}{2} (-1)^{2} x(2) = \frac{x(2)}{2} = \frac{1}{2}$

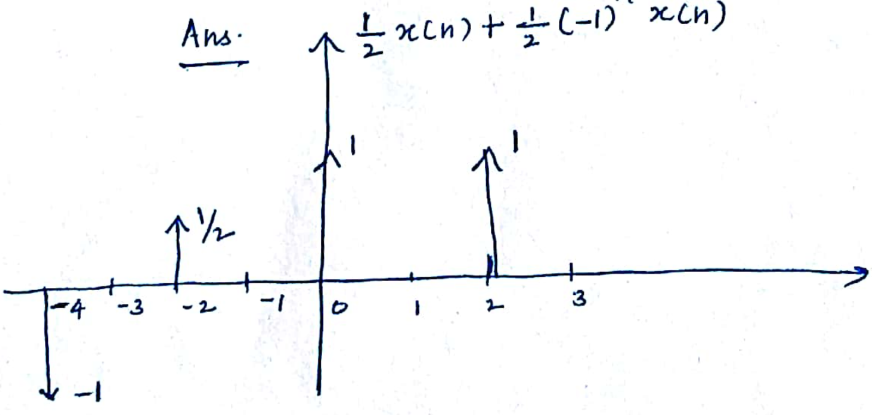
$n = 3$; $\frac{1}{2} (-1)^{3} x(3) = -\frac{x(3)}{2} = \frac{-1/2}{2} = -\frac{1}{4}$





+

Ans.



h) $x[(n-1)^2]$

$n=-4$; $x[(-4-1)^2] = x(25) = 0$

$n=-3$; $x[(-3-1)^2] = x(16) = 0$

$n=-2$; $x[(-2-1)^2] = x(9) = 0$

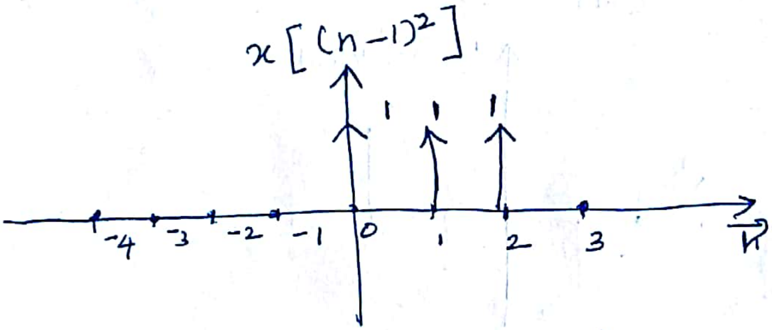
$n=-1$; $x[(-1-1)^2] = x(4) = 0$

$n=0$; $x[(0-1)^2] = x(1) = 1$

$n=1.$; $x[(1-1)^2] = x(0) = 1$

$n=2$; $x[(2-1)^2] = x(1) = 1$

$n=3$; $x[(3-1)^2] = x(4) = 0$



Unit II

Analysis of continuous Time signals

Fourier Series Analysis of continuous Time Periodic signals:

Need:

When the signal is passed through a system, the amplitude and phase characteristics of input signal frequency components are modified. Therefore, analysis of output signal is necessary.

Fourier series:

It is used to represent the periodic signal interms of its DC and AC components (Harmonics).

It is used to analyse periodic signal in time domain.

It is used Fourier Series:

a) Quadrature (or) Trignometric Fourier Series:

$$x(t) = \underset{\substack{\downarrow \\ (DC)\\ comp.}}{a_0} + \sum_{n=1}^{\infty} \underbrace{\left[a_n \cos n\omega_0 t + b_n \sin n\omega_0 t\right]}_{a.c.\ comp.} \longrightarrow ①$$

where, $a_0 = \frac{1}{T} \int_{t_0}^{t_0+T} x(t)\, dt$

$$a_n = \frac{2}{T} \int_{t_0}^{t_0+T} x(t) \cos n\omega_0 t \; dt$$

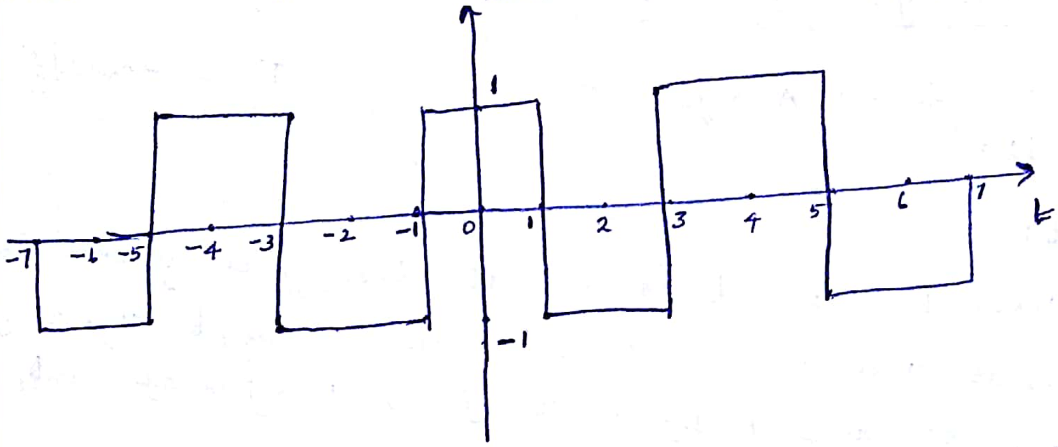
$$b_n = \frac{2}{T} \int_{t_0}^{t_0+T} x(t) \sin n\omega_0 t \; dt$$

$T \rightarrow$ Fundamental period.

$W_0 = \frac{2\pi}{T}$; $w_0 \rightarrow$ Fundamental Frequency.

Problems :

1. Find the Trignometic Fourier series of the following signal.



Soln:

From the graph,

Fundamental Period : T= 4 (from -1 to 3)

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{4} = \frac{\pi}{2}$$

* To find a_0:

Since , the given signal is symmetric with x-axis.

$a_0 = 0$.

* To find b_n:

Since, the given signal is even signal. i.e, $x(t) = x(-t)$.

$b_n = 0$.

* To find a_n:

$$a_n = \frac{2}{T}\int_{t_0}^{t_0+T} x(t) \cos n\omega_0 t \, dt$$

$$= \frac{2}{4}\int_{-1}^{3} x(t) \cos(n\frac{\pi}{2})t \, dt$$

$$= \frac{1}{2}\left[\int_{-1}^{1} (1) \cos(\frac{n\pi}{2})t \, dt + \int_{1}^{3} (-1) \cos(\frac{n\pi}{2})t \, dt\right.$$

$$= \frac{1}{2}\left[\left(\frac{\sin\frac{n\pi}{2}t}{\frac{n\pi}{2}}\right)_{-1}^{1} - \left(\frac{\sin\frac{n\pi}{2}t}{\frac{n\pi}{2}}\right)_{1}^{3}\right]$$

$$a_n = \frac{1}{2} \times \frac{2}{n\pi} \left[\left(\sin \frac{n\pi}{2} + \sin \frac{n\pi}{2} \right) - \left(\sin \frac{3n\pi}{2} - \sin \frac{n\pi}{2} \right) \right]$$

$$= \frac{1}{n\pi} \left[3 \sin \frac{n\pi}{2} - \sin 3 \frac{n\pi}{2} \right]$$

$$\boxed{\begin{array}{l} \sin 3 \frac{n\pi}{2} = - \sin \frac{n\pi}{2} \\ \sin 270 = \sin (180+90) \\ \quad = - \sin 90 \end{array}}$$

$$= \frac{1}{n\pi} \left[3 \sin \frac{n\pi}{2} + \sin \frac{n\pi}{2} \right]$$

$$\boxed{a_n = \frac{1}{n\pi} \times 4 \sin \frac{n\pi}{2}}$$

The Trignometric Fourier series of given signal $x(t)$ is,

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos n\omega_0 t + b_n \sin n\omega_0 t \right]$$

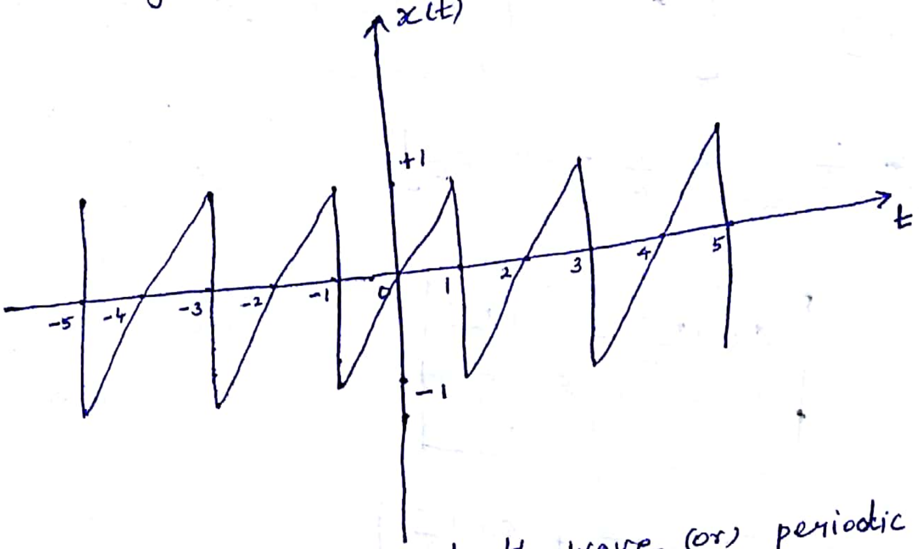
$[\because a_0 = 0, \ b_n = 0]$

$$x(t) = \sum_{n=1}^{\infty} a_n \cos n\omega_0 t .$$

$$\boxed{x(t) = \sum_{n=1}^{\infty} \left(\frac{4}{n\pi} \sin \frac{n\pi}{2} \right) \cos \left(\frac{n\pi}{2} \right) t .}$$

$\left[\because \omega_0 = \frac{\pi}{2} \right]$

2) Find the Trignometric Fourier series of the following signal.



Soln:

The given signal is saw tooth wave (or) periodic ramp signal.

$(x_1, y_1) = (-1,-1)$
$(x_2, y_2) = (1, 1)$

Slope: $= \frac{y_2-y_1}{x_2-x_1}$

$= \frac{1-(-1)}{1-(-1)} = \frac{2}{2}$

$= 1$

$x(t) = \frac{t}{slope}$; $-1 \leq t \leq 1$

$x(t) = t$; $-1 \leq t \leq 1$.

From the graph,

Fundamental period : $T = 2$ (from -1 to 1)

$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi$

* To find a_0 :

$a_0 = 0$, since the given signal is symmetric with 'x'- axis.

* To find a_n :

$a_n = 0$, since the given signal is odd signal.

$x(t) = -x(-t)$.

* To find b_n :

$$b_n = \frac{2}{T} \int_{t_0}^{t_0+T} x(t) \sin n\omega_0 t \, dt.$$

$$b_n = \frac{2}{2}\int_{-1}^{1} t \sin n\pi t \, dt.$$

$$b_n = \int_{-1}^{1} t \sin n\pi t \, dt.$$

Bernauli's theorem,

$$\int u\,dv = uv - u'v_1 + u''v_2 - u'''v_3 + \cdots$$

$u', u'', u''' \rightarrow$ successive differentiation

$v_1, v_2, v_3 \rightarrow$ successive integration

$$dv = \sin n\pi t \, dt$$

$$u = t$$

$$v = \int dv = \int \sin n\pi t \, dt = -\frac{\cos n\pi t}{n\pi}$$

$$u' = 1$$

$$v_1 = \int -\frac{\cos n\pi t}{n\pi} \, dt$$

$$u'' = 0$$

$$v_1 = -\frac{\sin n\pi t}{(n\pi)(n\pi)} = -\frac{\sin n\pi t}{n^2\pi^2}$$

$$b_n = \left[\frac{-t\cos n\pi t}{n\pi}\right]_{-1}^{1} - \left[1 \cdot \left(\frac{-\sin n\pi t}{n^2\pi^2}\right)\right]_{-1}^{1}$$

$$b_n = \frac{-1}{n\pi}\left[t\cos n\pi t\right]_{-1}^{1} + \frac{1}{n^2\pi^2}\left[\sin n\pi t\right]_{-1}^{1}$$

$$b_n = -\frac{1}{n\pi}\left[\cos n\pi - (-1 \cdot \cos n\pi(-1)\right] + \frac{1}{n^2\pi^2}\left[\sin n\pi - \sin n\pi(-1)\right]$$

$$b_n = \frac{-1}{n\pi}\left[\cos n\pi + \cos n\pi\right] + \frac{1}{n^2\pi^2}\left[\sin n\pi + \sin n\pi\right]$$

$$\sin n\pi = 0$$

$$\cos n\pi = \begin{cases} -1 ; 1, 3, 5, \ldots \\ +1 ; 2, 4, 6, \ldots \end{cases}$$

$$= (-1)^n$$

$$b_n = -\frac{2\cos n\pi}{n\pi}$$

(or)

$$b_n = -2\frac{(-1)^n}{n\pi}$$

Fourier series of the given signal $x(t)$ is,

$$x(t) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos n\omega_0 t + b_n \sin n\omega_0 t\right]$$

$$x(t) = \sum_{n=1}^{\infty} b_n \sin n\omega_0 t.$$

$$\boxed{x(t) = \sum_{n=1}^{\infty}\left(-\frac{2\cos n\pi}{n\pi}\right)\sin n\pi t}$$

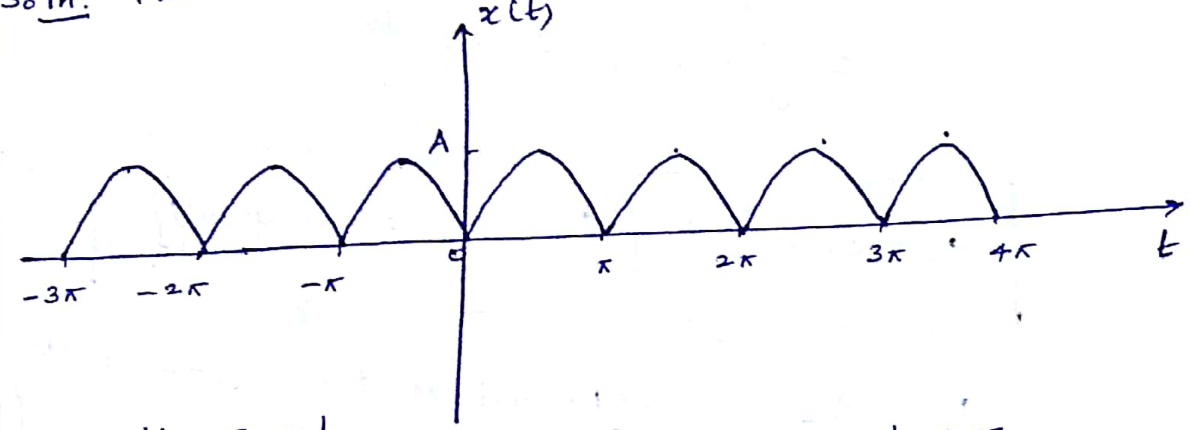
$\because \omega_0 = \pi$

(or)

$$\boxed{x(t) = \sum_{n=1}^{\infty}\left(\frac{-2(-1)^n}{n\pi}\right)\sin n\pi t.}$$

3) Find the quadrature Fourier series for the full wave rectifier output signal.

Soln: Full wave rectifier output signal : $x(t) = A \sin t$.



From the graph,

$$x(t) = A \sin t \quad ; \; 0 \leq t \leq \pi$$

Fundamental Period : $T = \pi$ (from 0 to π)

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{\pi} = 2$$

* To find a_0 :

$$a_0 = \frac{1}{T} \int_{t_0}^{t_0+T} x(t)\, dt$$

$$a_0 = \frac{1}{\pi} \int_0^{\pi} A \sin t \, dt$$

$$a_0 = \frac{A}{\pi} \int_0^{\pi} \sin t \, dt$$

$$a_0 = \frac{A}{\pi} \Big[-\cos t \Big]_0^{\pi}$$

$$a_0 = -\frac{A}{\pi} \left[\cos \pi - \cos 0 \right]$$

$$a_0 = -\frac{A}{\pi} \left[-1 - 1 \right]$$

$$\boxed{a_0 = \frac{2A}{\pi}}$$

* To find a_n :

$$a_n = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\cos n\omega_0 t \, dt$$

$$a_n = \frac{2}{\pi}\int_{0}^{\pi} A \sin t \, \cos 2nt \, dt$$

$$\boxed{\sin A \cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]}$$

$$a_n = \frac{2A}{\pi}\times\frac{1}{2}\left[\int_0^{\pi} \sin(1+2n)t \, dt + \int_0^{\pi} \sin(1-2n)t \, dt\right]$$

$$a_n = \frac{A}{\pi}\left[\left(-\frac{\cos(1+2n)t}{(1+2n)}\right)_0^{\pi} + \left(-\frac{\cos(1-2n)t}{(1-2n)}\right)_0^{\pi}\right]$$

$$a_n = -\frac{A}{\pi}\left[\left(\frac{\cos(1+2n)\pi - \cos 0}{(1+2n)}\right) + \left(\frac{\cos(1-2n)\pi - \cos 0}{(1-2n)}\right)\right]$$

$$\boxed{\begin{array}{l} \cos(1+2n)\pi = -1 \;;\; n = 1, 2, 3, \cdots \\ \cos(1-2n)\pi = -1 \;;\; n = 1, 2, 3, \cdots \\ \hline (1+2n)(1-2n) = 1^2 - (2n)^2 = 1 - 4n^2 \end{array}}$$

$$a_n = \frac{-A}{\pi}\left[\frac{(-1-1)}{(1+2n)} + \frac{(-1-1)}{(1-2n)}\right]$$

$$a_n = \frac{-A}{\pi}\left[\frac{(-2)}{(1+2n)} + \frac{(-2)}{(1-2n)}\right] = \frac{2A}{\pi}\left[\frac{1}{(1+2n)} + \frac{1}{(1-2n)}\right]$$

$$a_n = \frac{2A}{\pi}\left[\frac{1-\cancel{2n}+1+\cancel{2n}}{(1+2n)(1-2n)}\right]$$

$$\boxed{a_n = \frac{4A}{\pi(1-4n^2)}}$$

* To find b_n :

$b_n = 0$, since the given signal is even signal

i.e, $x(t) = x(-t)$.

The fourier series of Fullwave Rectifier output signal $x(t)$ is given by,

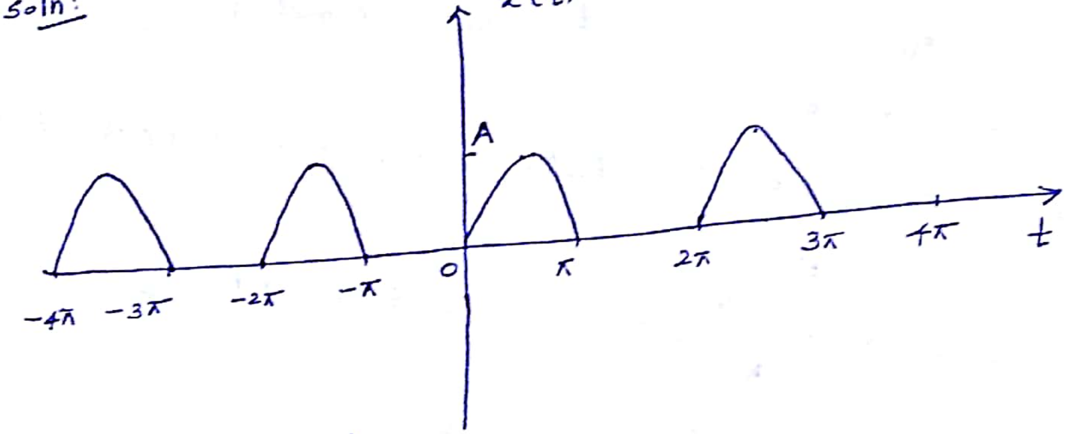
$$x(t) = a_0 + \sum_{n=1}^{\infty} [a_n \cos n\omega_0 t + b_n \sin n\omega_0 t]$$

$$x(t) = \frac{2A}{\pi} + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t$$

$$\boxed{x(t) = \frac{2A}{\pi} + \sum_{n=1}^{\infty} \left(\frac{4A}{\pi(1-4n^2)}\right) \cos 2nt.}$$ $\because \omega_0 = 2$

4) Find the trignometric Fourier Series for half wave rectifier output signal.

<u>soln:</u>



From the graph,

$$x(t) = A\sin t \quad ; \quad 0 \le t \le \pi$$

Fundamental period : $T = 2\pi$ (from 0 to 2π)

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$$

*To find a_0 :

$$a_0 = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)\,dt$$

$$= \frac{1}{2\pi}\int_0^{\pi} A\sin t\,dt$$

$$= \frac{A}{2\pi}\left[-\cos t\right]_0^{\pi}$$

$$= -\frac{A}{2\pi}\left[\cos\pi - \cos 0\right]$$

$$= \frac{-A}{2\pi}\left[-1-1\right]$$

$$= \frac{2A}{2\pi}$$

$$\boxed{a_0 = \frac{A}{\pi}}$$

* To Find a_n :

$$a_n = \frac{2}{T}\int_{t_0}^{t_0+T} x(t) \cos n\omega_0 t \, dt$$

$$= \frac{2}{2\pi}\int_{0}^{\pi} A \sin t \cos nt \, dt$$

$$\boxed{\sin A \cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]}$$

$$= \frac{A}{\pi} \times \frac{1}{2}\left[\int_0^{\pi} \sin(1+n)t \, dt + \int_0^{\pi} \sin(1-n)t \, dt\right]$$

$$= \frac{A}{2\pi}\left[\left(\frac{-\cos(1+n)t}{(1+n)}\right)_0^{\pi} + \left(\frac{-\cos(1-n)t}{(1-n)}\right)_0^{\pi}\right]$$

$$= \frac{-A}{2\pi}\left[\left(\frac{\cos(1+n)\pi - \cos 0}{(1+n)}\right) + \left(\frac{\cos(1-n)\pi - \cos 0}{(1-n)}\right)\right]$$

$\cos(1+n)\pi = 1$; $n = 1, 3, 5, \ldots$

$\cos(1+n)\pi = -1$; $n = 2, 4, 6, \ldots$

$\cos(1-n)\pi = 1$; $n = 1, 3, 5, \ldots$

$\cos(1-n)\pi = -1$; $n = 2, 4, 6, \ldots$

When $n = 1, 3, 5, \ldots$; the above equation becomes zero.

$$= -\frac{A}{2\pi}\left[\frac{(-1-1)}{(1+n)} + \frac{(-1-1)}{(1-n)}\right] ; \; n = 2, 4, 6, \ldots$$

$$= \frac{-A}{2\pi}\left[\left(\frac{-2}{1+n}\right) + \left(\frac{-2}{1-n}\right)\right]$$

$$= \frac{\cancel{2}A}{\cancel{2}\pi}\left[\frac{1}{1+n} + \frac{1}{1-n}\right]$$

$$= \frac{A}{\pi}\left[\frac{1-n+1+n}{(1+n)(1-n)}\right]$$

$$\boxed{a_n = \frac{2A}{\pi(1-n^2)}} \quad ; \; n = 2, 4, 6, \ldots$$

* To find b_n :

$$b_n = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\, \sin n\omega_0 t \; dt$$

$$= \frac{2}{2\pi}\int_0^{\pi} A \sin t \; \sin nt \; dt = \frac{A}{\pi}\int_0^{\pi} \sin t \; \sin nt \; dt \longrightarrow ①$$

$$\boxed{\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]}$$

$$= \frac{A}{\pi}\times\frac{1}{2}\left[\int_0^{\pi} \cos(1-n)t \; dt - \int_0^{\pi} \cos(1+n)t \; dt\right]$$

$$= \frac{A}{2\pi}\left[\left(\frac{\sin(1-n)t}{(1-n)}\right)_0^{\pi} - \left(\frac{\sin(1+n)t}{(1+n)}\right)_0^{\pi}\right]$$

$$= \frac{A}{2\pi}\left[\left(\frac{\sin(1-n)\pi - \sin 0}{(1-n)}\right) - \left(\frac{\sin(1+n)\pi - \sin 0}{(1+n)}\right)\right]$$

$\sin 0 = 0$

$\sin n\pi = 0 \quad ; \; n = 1, 2, 3, \ldots$

When $n = 1$; $b_1 = \frac{0}{0} (\neq 0)$

When $n = 2, 3, 4, \ldots \; b_n = 0$

①⇒

When $n=1$, $b_1 = \frac{A}{\pi}\int_0^{\pi} \sin t \sin t \, dt$

$$= \frac{A}{\pi}\int_0^{\pi} \sin^2 t \, dt = \frac{A}{\pi}\int_0^{\pi}\left[\frac{1-\cos 2t}{2}\right] dt$$

$$= \frac{A}{2\pi}\left[\int_0^{\pi} dt - \int_0^{\pi} \cos 2t \, dt\right]$$

$$= \frac{A}{2\pi}\left[(t)_0^{\pi}\right] - \left[\frac{\sin 2t}{2}\right]_0^{\pi}$$

$$= \frac{A}{2\pi}\left[(\pi - 0) - \left(\frac{\sin 2\pi - \sin 0}{2}\right)\right]$$

$$= \frac{A}{2\pi}\left[\pi - \left(\frac{0-0}{2}\right)\right]$$

$$= \frac{A}{2\pi} \times \pi$$

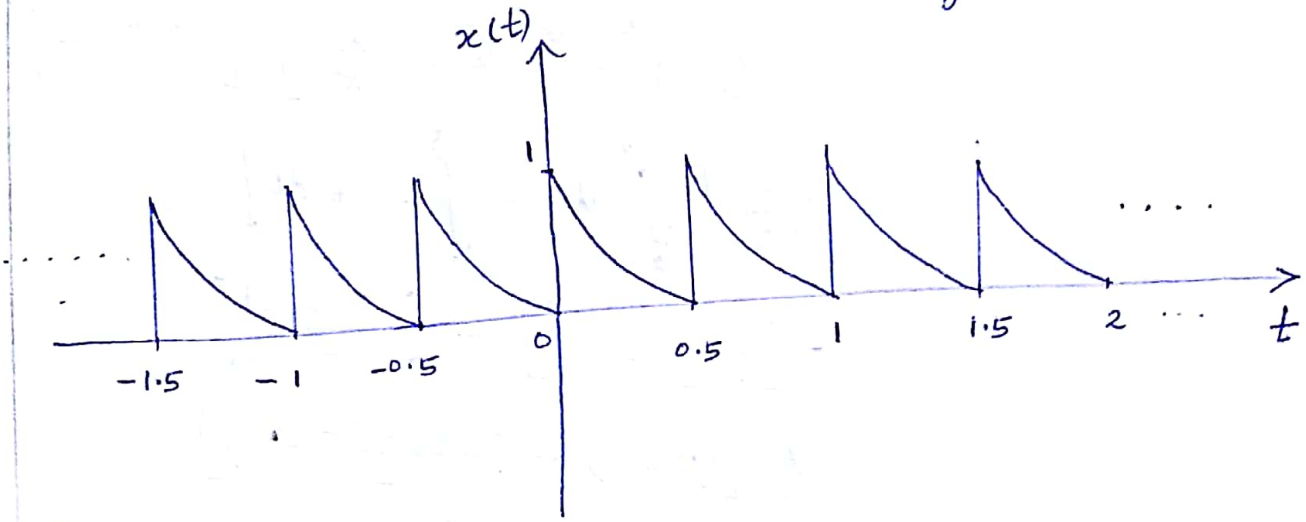
$$\boxed{b_1 = \frac{A}{2}}$$

Fourier series of Half Wave Rectifier output signal:

$$x(t) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos n\omega_0 t + b_n \sin n\omega_0 t\right]$$

$$\boxed{x(t) = \frac{A}{\pi} + \sum_{n=2,4..}^{\infty}\left(\frac{2A}{\pi(1-n^2)}\right)\cos nt + \frac{A}{2}\sin t} \quad \because \omega_0 = 1$$

5) Find the Fourier series of the following signal $x(t)$.



Soln:

The given signal is exponentially decaying signal

$x(t) = e^{-t} \quad ; \quad 0 \leq t \leq 0.5$

Fundamental period : $T = 0.5$ (0 to 0.5)

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{0.5} = 4\pi$$

Trignometric Fourier series

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos n\omega_0 t + b_n \sin n\omega_0 t\right] \longrightarrow ①$$

To find a_0 :

$$a_0 = \frac{1}{T} \int_{t_0}^{t_0+T} x(t)\, dt$$

$$= \frac{1}{0.5} \int_0^{0.5} e^{-t}\, dt = 2 \int_0^{0.5} e^{-t}\, dt = 2\left[\frac{e^{-t}}{-1}\right]_0^{0.5}$$

$$= -2\left[e^{-0.5} - e^{0}\right] = -2\left[0.6065 - 1\right] = 0.79$$

$$\boxed{a_0 = 0.79}$$

To Find a_n:

$$a_n = \frac{2}{T}\int_{t_0}^{t_0+T} x(t) \cdot \cos n\omega_0 t \, dt$$

$$= \frac{2}{0.5}\int_{0}^{0.5} e^{-t} \cos n.4\pi t \, dt$$

$$\boxed{\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2}\left[a \cos bx + b \sin bx\right]}$$

Here, $a = -1$ & $b = 4\pi n$, $x = t$

$$a_n = 4\left[\frac{e^{-t}}{1+16\pi^2 n^2}\left[-\cos 4\pi n t + 4\pi n \sin 4\pi n t\right]\right]_0^{0.5}$$

$$= \frac{4}{1+16\pi^2 n^2}\left\{e^{-0.5}\left(-\cos 2\pi n + 4\pi n \sin 2\pi n\right) - e^{0}\left(-\cos 0 + 4\pi n \sin 0\right)\right\}$$

$$= \frac{4}{1+16\pi^2 n^2}\left\{0.6065\left(-1+0\right) - 1\left(-1+0\right)\right\}$$

$$\boxed{\begin{aligned} \because \cos 2\pi n = 1 \ ; \ n = 1,2,3,\ldots \\ \sin 2\pi n = 0 \ ; \ n = 1,2,3,\ldots \end{aligned}}$$

$$a_n = \frac{4}{1+16\pi^2 n^2}\left\{-0.6065 + 1\right\}$$

$$\boxed{a_n = \frac{1.574}{1+16\pi^2 n^2}}$$

To find b_n :

$$b_n = \frac{2}{T} \int_{t_0}^{t_0+T} x(t) \cdot \sin n\omega_0 t \ dt$$

$$= \frac{2}{0.5} \int_0^{0.5} e^{-t} \sin n\, 4\pi t \ dt$$

$$b_n = 4 \int_0^{0.5} e^{-t} \sin 4\pi n t \ dt$$

$$\boxed{\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}\left[a \sin bx - b \cos bx\right]}$$

Here, $a=-1$, $b = 4\pi n$, $x=t$.

Now,

$$b_n = 4 \times \left[\frac{e^{-t}}{1+16\pi^2 n^2}\left[-\sin 4\pi n t - 4\pi n \cos 4\pi n t\right]\right]_0^{0.5}$$

$$= \frac{4}{1+16\pi^2 n^2}\left[e^{-0.5} (-\sin 2\pi n - 4\pi n \cos 2\pi n) - e^{0}(-\sin 0 - 4\pi n \cos 0)\right]$$

$$= \frac{4}{1+16\pi^2 n^2}\left[0.6065(0 - 4\pi n) - 1 \cdot (0 - 4\pi n)\right]$$

$$= \frac{4}{1+16\pi^2 n^2}\left[-2.426\pi n + 4\pi n\right] = \frac{4\left[1.574\pi n\right]}{1+16\pi^2 n^2}$$

$$\boxed{b_n = \frac{6.296\pi n}{1+16\pi^2 n^2}}$$

sub. a_0, a_n and b_n in eqn. ①, also sub. $\omega_0 = 4\pi$ in eqn. ①.

① ⇒

$$x(t) = 0.79 + \sum_{n=1}^{\infty} \left[\frac{1.574}{1+16\pi^2 n^2} \cdot \cos 4\pi n t + \frac{6.296\,\pi n}{1+16\pi^2 n^2} \cdot \sin 4\pi n t \right]$$

b) Cosine (or) Polar Fourier series

$$x(t) = D_0 + \sum_{n=1}^{\infty} D_n \cos[\omega_0 nt + \phi_n]$$

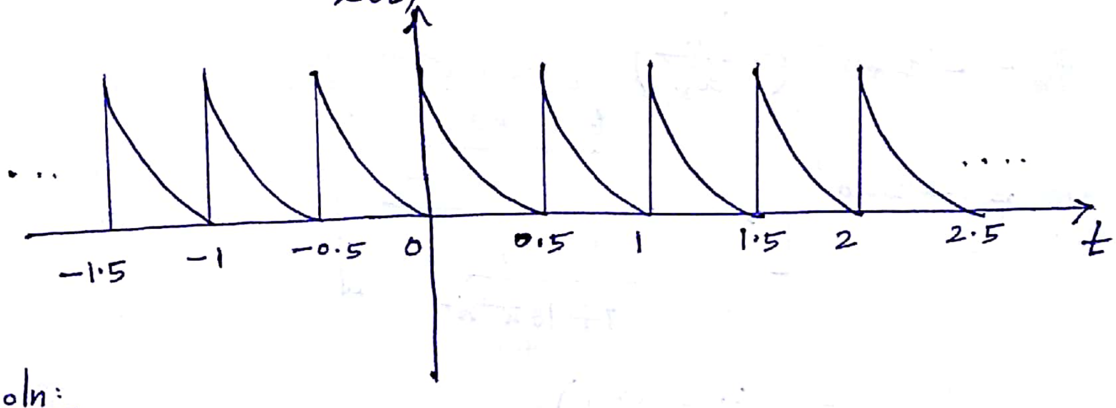
where, $D_0 = a_0$

$$D_n = \sqrt{a_n^2 + b_n^2}$$

$$\phi_n = -\tan^{-1}\left(\frac{b_n}{a_n}\right)$$

Problems:

1. Find the cosine Fourier series of the given signal, Also plot the magnitude and phase spectrum.



Soln:

From the graph,

$$x(t) = e^{-t} \quad ; \quad 0 \leq t \leq 0.5$$

Fundamental Period : T = 0.5 (0 to 0.5)

Fundamental Frequency : $\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{0.5} = 4\pi$

Refer Previous Problem :

$$D_0 = a_0 = 0.79$$

$$D_n = \sqrt{a_n^2 + b_n^2} = \sqrt{\left(\frac{1.574}{1+16\pi^2 n^2}\right)^2 + \left(\frac{6.296\pi n}{1+16\pi^2 n^2}\right)^2}$$

$$= \sqrt{\frac{2.477 + 39.64\pi^2 n^2}{(1+16\pi^2 n^2)^2}} = \sqrt{\frac{2.477(1+16\pi^2 n^2)}{(1+16\pi^2 n^2)^2}}$$

$$= \sqrt{\frac{2.477}{1+16\pi^2 n^2}}$$

$$\boxed{D_n = \frac{1.574}{\sqrt{1+16\pi^2 n^2}}}$$

$$\phi_n = -\tan^{-1}\left(\frac{b_n}{a_n}\right)$$

$$= -\tan^{-1}\left[\frac{\frac{6.296\pi n}{1+16\pi^2 n^2}}{\frac{1.574}{1+16\pi^2 n^2}}\right]$$

$$= -\tan^{-1}\left(\frac{6.296\pi n}{1.574}\right)$$

$$\boxed{\phi_n = -\tan^{-1}(4\pi n)}$$

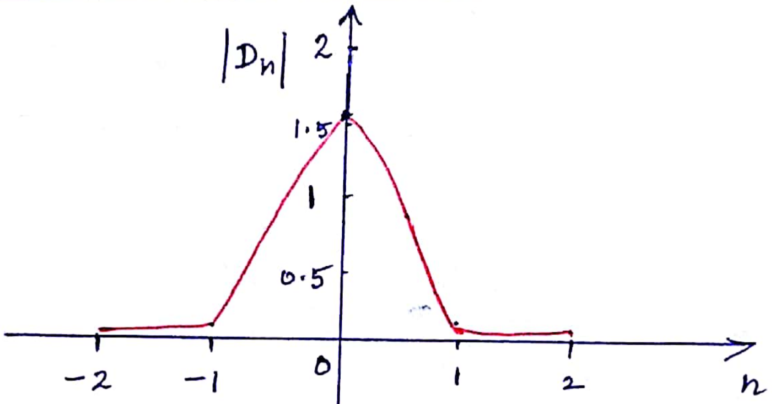
Polar Fourier series:

$$x(t) = D_0 + \sum_{n=1}^{\infty} D_n \cos[w_0 n t + \phi_n]$$

$$\boxed{x(t) = 0.79 + \sum_{n=1}^{\infty} \frac{1.574}{\sqrt{1+16\pi^2 n^2}} \cdot \cos\left[4\pi n t - \tan^{-1}(4\pi n)\right]}$$

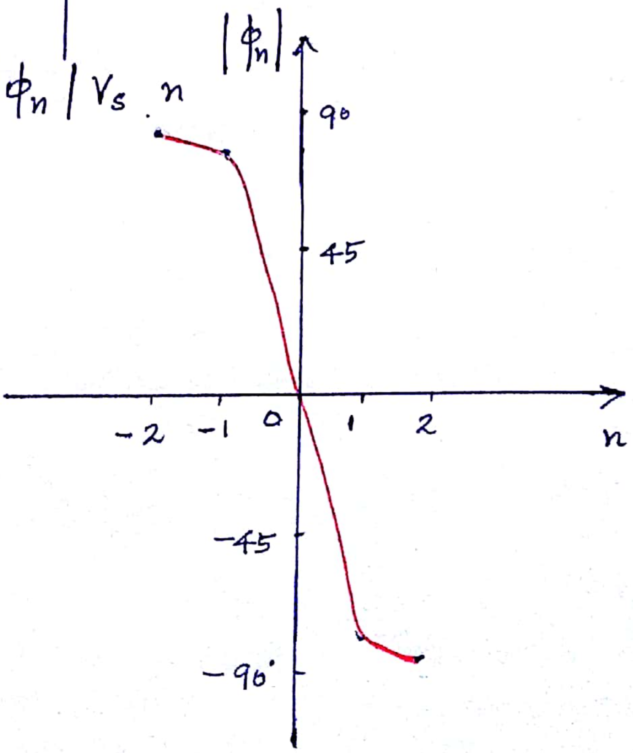
→Magnitude spectrum : $|D_n|$ Vs n

n	$\lvert D_n \rvert$
−2	0.063
−1	0.125
0	1.574
1	0.125
2	0.063



→Phase spectrum $|\phi_n|$ Vs n

n	ϕ_n
−2	87.72
−1	85.45
0	0
1	−85.45
2	−87.72



* Complex Exponential Fourier series.

$$x(t) = \sum_{n=1}^{\infty} C_n \, e^{j\omega_0 n t}$$

where,

$$c_n = \frac{1}{T} \int_{t_0}^{t_0+T} x(t) \, e^{-j\omega_0 n t} \, dt$$

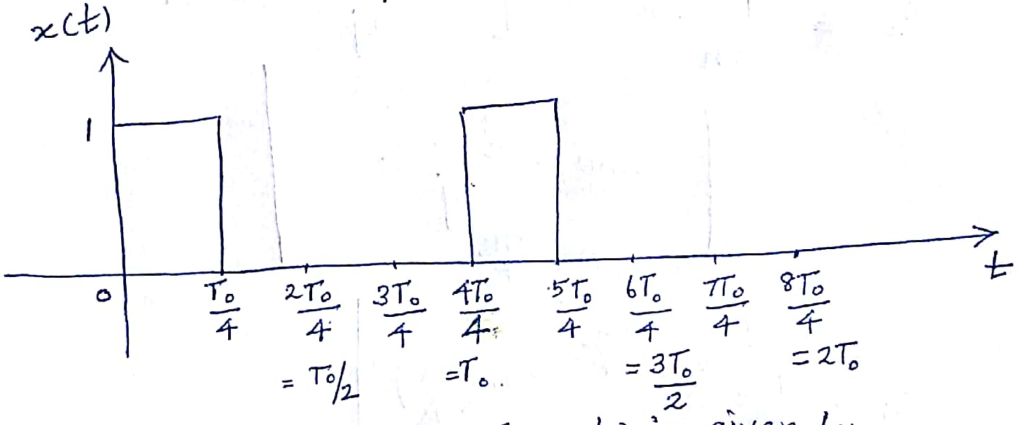
Problems:

1) The signal described below is repeated with 'T_0'. Find out the Fourier series co-efficient of the resultant signal and its Magnitude spectrum.

$$x(t) = \begin{cases} 1 & ; \quad 0 \le t \le \frac{T_0}{4} \\ 0 & ; \quad \frac{T_0}{4} \le t \le T_0 . \end{cases}$$

Soln:

$$x(t) = \begin{cases} 1 & ; \quad 0 \le t \le \frac{T_0}{4} \\ 0 & ; \quad \frac{T_0}{4} \le t \le T_0 \end{cases}$$



Complex Exponential series of $x(t)$ is given by,

$$x(t) = \sum C_n \, e^{+j\omega_0 n t} \longrightarrow ①$$

From the graph,

Fundamental period : $T = T_0$

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{T_0}$$

→ To find C_n :

$$C_n = \frac{1}{T} \int_{t_0}^{t_0+T} x(t) \cdot e^{-j\omega_0 n t} \, dt$$

$$c_n = \frac{1}{T_0} \int_0^{T_0/4} 1 \cdot e^{-j \frac{2\pi}{T_0} n t} \, dt.$$

$$= \frac{1}{T_0} \left[\frac{e^{-j \frac{2\pi}{T_0} n t}}{-j \frac{2\pi}{T_0} n} \right]_0^{T_0/4}$$

$$= \frac{1}{T_0} \times \frac{T_0}{-j 2\pi n} \left[e^{-j \frac{2\pi}{T_0} n \times \frac{T_0}{4}} - e^{0} \right]$$

$$= \frac{1}{-j 2\pi n} \left[e^{-j \frac{\pi n}{2}} - 1 \right] = \frac{1}{j 2\pi n} \left[1 - e^{-j \frac{\pi n}{2}} \right]$$

$$= \frac{1}{j 2\pi n} \left[e^{j \frac{\pi n}{4}} \cdot e^{-j \frac{\pi n}{4}} - e^{-j \frac{\pi n}{4}} e^{-j \frac{\pi n}{4}} \right]$$

$$c_n = \frac{1}{\pi n} e^{-j \frac{\pi n}{4}} \left[\frac{e^{j \frac{\pi n}{4}} - e^{-j \frac{\pi n}{4}}}{2j} \right]$$

$$\boxed{\begin{array}{l} \sin\theta = \dfrac{e^{j\theta} - e^{-j\theta}}{2j} \\ \text{Here, } \theta = \dfrac{\pi n}{4} \end{array}}$$

$$\boxed{c_n = \frac{1}{\pi n} e^{-j \frac{\pi n}{4}} \sin\left(\frac{n\pi}{4}\right)}$$

Now,

$$① \Rightarrow x(t) = \sum_{n=1}^{\infty} \left[\frac{1}{\pi n} \cdot e^{-j \frac{\pi n}{4}} \cdot \sin\left(\frac{n\pi}{4}\right) \right] \cdot e^{+j \frac{2\pi}{T_0} n t}.$$

$\hookrightarrow$ complex Exponential Fourier Series.

→ Magnitude spectrum

$|c_n|$ Vs n

$$|c_n| = \left| \frac{e^{-j\frac{n\pi}{4}} \cdot \sin \frac{n\pi}{4}}{\pi n} \right|$$

$$\boxed{|c_n| = \left| \frac{\sin \frac{n\pi}{4}}{\pi n} \right|} \quad \because |e^{-j\theta}| = 1, \therefore |e^{-j\frac{n\pi}{4}}| = 1$$

n	$\lvert c_n \rvert$
−2	0.16
−1	0.23
0	0.25
1	0.23
2	0.16

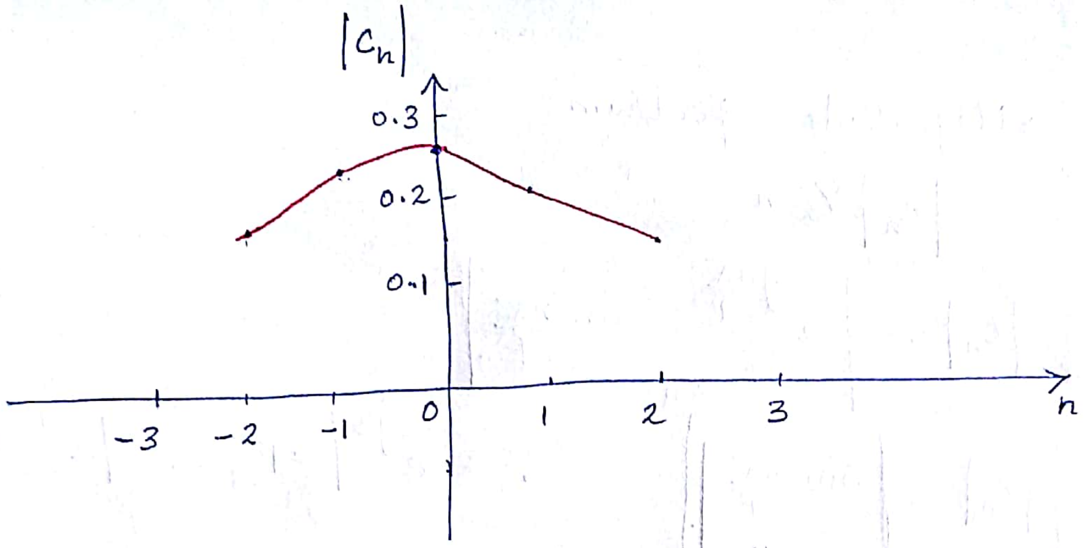
If $n = 0$, $|c_n| = \left|\frac{\sin 0}{0}\right| = \frac{0}{0}$. [indet.]

using L'Hopital rue, to find $|c_0|$

$$|c_0| = \left. \frac{\frac{d}{dn} \sin \frac{n\pi}{4}}{\frac{d}{dn} \pi n.} \right|_{n=0} = \left. \frac{\cos\left(\frac{n\pi}{4}\right) \times \frac{\pi}{4}}{\pi} \right|_{n=0}$$

$$|c_0| = \frac{\cos 0 \times \frac{\pi}{4}}{\pi} = \frac{1 \times \frac{\pi}{4}}{\pi} = \frac{\pi}{4\pi} = \frac{1}{4}$$

$$\boxed{|c_0| = 0.25}$$


|c_n|
0.3
0.2
0.1
-3
-2
-1
0
1
2
3
n

* Properties of Fourier Series:

$$x(t) \xrightarrow{\text{F.S.Coeff}} C_n$$

i) Time shifting property.

$$x(t-t_0) \xrightarrow[\text{F.S.C}]{} e^{-j\omega_0 n t_0} \cdot C_n$$

ii) Time Reversal property

$$x(-t) \xrightarrow[\text{F.S.C}]{} C_{-n}$$

iii) Conjugate property.

$$x^*(t) \xrightarrow[\text{F.S.C}]{} C^*_{-n}$$

iv) Differentiation property

$$\frac{d}{dt} x(t) \xrightarrow[\text{F.S.C}]{} j\omega_0 n \, C_n$$

V.Q v) Parseval's theorem:

The average power of any periodic signal is equal to sum of square of fourier series co-efficients.

$$P = \frac{1}{T} \int_{t_0}^{t_0+T} |x(t)|^2 dt = \sum_{n=1}^{\infty} |C_n|^2$$

Note: $|C_n|^2 = C_n \cdot C_n^*$

Proof:

$$P = \frac{1}{T} \int_{t_0}^{t_0+T} |x(t)|^2 dt = \frac{1}{T} \int_{t_0}^{t_0+T} x(t) \cdot x^*(t) \, dt$$

$$= \frac{1}{T} \int_{t_0}^{t_0+T} x(t) \left[\sum_{n=1}^{\infty} C_n \, e^{jn\omega_0 t} \right]^* dt$$

$$= \frac{1}{T} \int_{t_0}^{t_0+T} x(t) \sum_{n=1}^{\infty} c_n^* \, e^{-jnw_0 t} \, . dt$$

$$= \sum_{n=1}^{\infty} c_n^* \cdot \frac{1}{T} \int_{t_0}^{t_0+T} x(t) \, e^{-jnw_0 t} \, dt$$

$$P = \sum_{n=1}^{\infty} c_n^* \; c_n = \sum_{n=1}^{\infty} |c_n|^2.$$

Hence. proved.

U.Q. → Dirichilet conditions.

Fourier series of $x(t)$ can be convergent if it satisfies the following conditions:

i) The signal $x(t)$ should be absolutely integrable one.

$$\int_{-\infty}^{\infty} |x(t)| \, dt < \infty .$$

ii) The signal $x(t)$ should have finite number of minimas and maximas within the fundamental period 'T'.

iii) The signal $x(t)$ should be a single valued function within the fundamental period 'T'.

Previously Asked Anna University Questions:

1) Let $x(t)$ be a periodic signal with fundamental period $T_0 = 2$.

i) Determine the value of a_0.

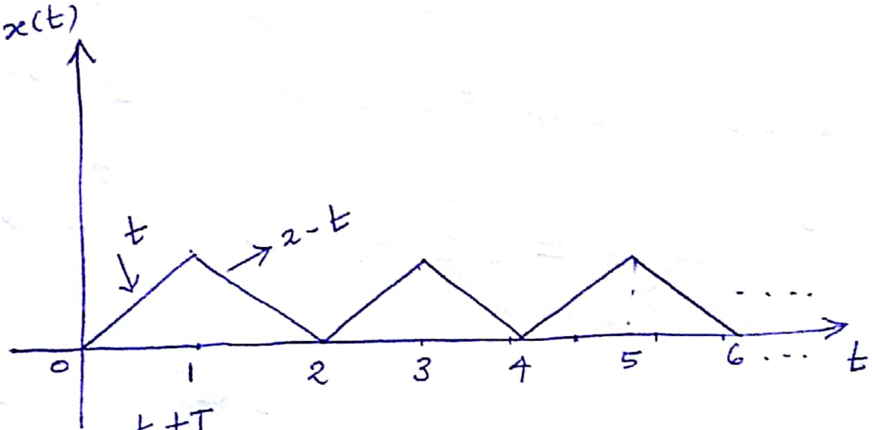
ii) Determine the fourier representation of $\frac{d}{dt}x(t)$.

iii) Use the result of ii) and the differentiation property of continuous Time Fourier series to determine the fourier series co-efficient of $x(t)$.

$$x(t) = \begin{cases} t & ; \quad 0 \le t \le 1 \\ 2-t & ; \quad 1 \le t \le 2 . \end{cases}$$

Soln:

Given fundamental period : $T = T_0 = 2$



$$a_0 = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)\, dt$$

$$= \frac{1}{2}\left[\int_0^1 t\, dt + \int_1^2 (2-t)\, dt\right]$$

$$= \frac{1}{2}\left[\left(\frac{t^2}{2}\right)_0^1 + \left(2t - \frac{t^2}{2}\right)_1^2\right]$$

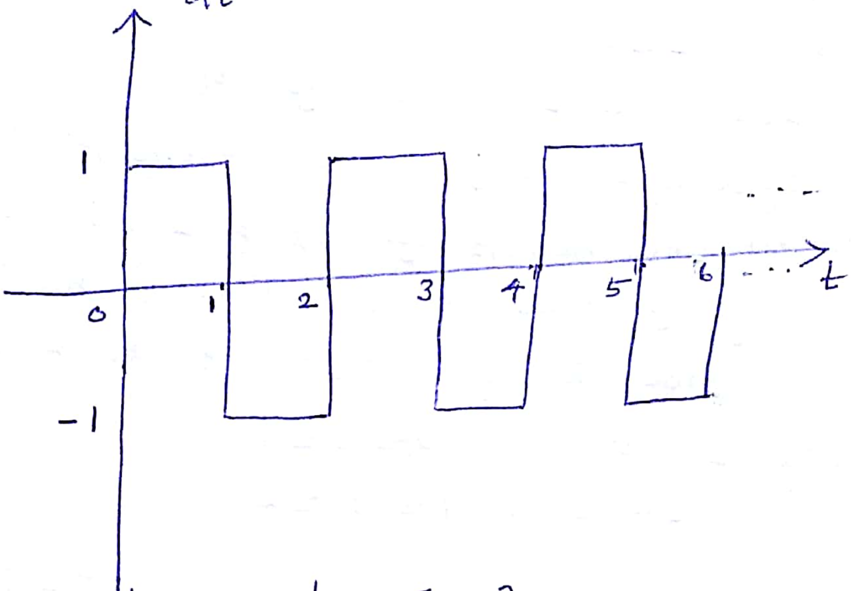
$$= \frac{1}{2}\left[\left(\frac{1}{2} - 0\right) + \left(2.2 - \frac{2^2}{2} - \left(2.1 - \frac{1^2}{2}\right)\right)\right]$$

$$= \frac{1}{2}\left[\frac{1}{2} + 4 - \frac{4}{2} - 2 + \frac{1}{2}\right] = \frac{1}{2}\left[\frac{1+8-4-4+1}{2}\right] = \frac{1}{4} \times 2$$

$$\boxed{a_0 = \frac{1}{2}}$$

ii) To find Fourier series of $\frac{d}{dt}x(t)$.

$$x'(t) = \frac{d}{dt}x(t).$$



Fundamental period : $T = 2$

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi$$

Complex Exponential Fourier Series of $x'(t)$ is given by:

$$x'(t) = \sum_{n=1}^{\infty} c_n' \, e^{j\omega_0 n t} \longrightarrow ①$$

where,
$$c_n' = \frac{1}{T}\int_{t_0}^{t_0+T} x'(t) \, e^{-j\omega_0 n t} \, dt$$

$$= \frac{1}{2}\left[\int_0^1 1 \cdot e^{-j\pi n t} \, dt + \int_1^2 -1 \cdot e^{-j\pi n t} \, dt\right]$$

$$= \frac{1}{2}\left\{\left(\frac{e^{-j\pi n t}}{-j\pi n}\right)_0^1 - \left(\frac{e^{-j\pi n t}}{-j\pi n}\right)_1^2\right\}$$

$$= \frac{1}{-2j\pi n}\left\{\left(e^{-j\pi n} - e^{0}\right) - \left(e^{-j2\pi n} - e^{-j\pi n}\right)\right\}$$

$$= \frac{1}{-j2\pi n}\left\{e^{-j\pi n} - 1 - e^{-j2\pi n} + e^{-j\pi n}\right\}$$

$$c_n' = \frac{-1}{-j2\pi n}\left[1 - 2e^{-j\pi n} + e^{-j2\pi n}\right]$$

$$c_n' = \frac{1}{j2\pi n}\left[1 - e^{-j\pi n}\right]^2$$

$$\boxed{c_n' = \frac{1}{j2\pi n}\left[1 - (-1)^n\right]^2}$$

$$\left[\because e^{-j\pi n} = \cos \pi n - j\,\cancel{\sin \pi n}^{\,0} = (-1)^n \; ; n = 1, 2, 3 \ldots\right]$$

Now, ① ⇒ $$x'(t) = \sum_{n=1}^{\infty} c_n' \, e^{j\omega_0 n t}$$

$$\boxed{x'(t) = \sum_{n=1}^{\infty} \frac{1}{j2\pi n} \cdot \left[1 - (-1)^n\right]^2 \cdot e^{j\pi n t}}$$

iii) Differentiation Property of Fourier Series.

$$\frac{dx(t)}{dt} \xrightarrow{F.S.C} c_n' = j\omega_0 n \, c_n$$

$$c_n = \frac{c_n'}{j\omega_0 n} = \frac{c_n'}{j\pi n} \quad [\because \omega_0 = \pi]$$

$$c_n = \frac{1}{j\pi n} \times \frac{1}{j2\pi n}\left[1 - (-1)^n\right]^2$$

$$\boxed{c_n = \frac{-1}{2\pi^2 n^2}\left[1 - (-1)^n\right]^2}$$

Frequently asked GATE Questions:

1. Find the Fourier Co-efficient of $x(t) = \cos\pi t$ ..

Soln:

$$x(t) = \cos\pi t$$

$$x(t) = \frac{e^{j\pi t} + e^{-j\pi t}}{2}$$

$$x(t) = \frac{1}{2} e^{-j(-1)\pi t} + \frac{1}{2} e^{-j(1)\pi t}$$

Fourier series Co-efficients:

$$C_{-1} = \frac{1}{2}$$

$$C_1 = \frac{1}{2}.$$

2. Find the fourier series co-efficient of $x(n) = \sin\omega_0 n$.

Soln:

$$x(n) = \sin\omega_0 n$$

$$x(n) = \frac{e^{j\omega_0 n} - e^{-j\omega_0 n}}{2j}$$

$$x(n) = \frac{1}{2j} e^{-j(-1)\omega_0 n} - \frac{1}{2j} e^{-j(1)\omega_0 n}$$

Fourier series Co-efficients:

$$C_{-1} = \frac{1}{2j} = \frac{-j}{2}$$

$$C_1 = -\frac{1}{2j} = \frac{j}{2}$$

$$\left[\frac{1}{j} = \frac{1}{j} \times \frac{j}{j} = \frac{j}{j^2} = \frac{j}{-1} = -j\right]$$

Note:

F.S. Coeff of $\sin\pi t$.

$$C_{-1} = \frac{-j}{2}$$

$$C_1 = \frac{j}{2}$$

3. Find the Fourier Series Co-efficient of the signal

$x(t) = 2 + \cos\left(\frac{2\pi}{3}\right)t + 4\sin\left(\frac{5\pi}{3}\right)t.$

Soln:

$x(t) = 2 + \cos\left(\frac{2\pi}{3}\right)t + 4\sin\left(\frac{5\pi}{3}\right)t.$

$$x(t) = 2 + \frac{e^{j\frac{2\pi}{3}t} + e^{-j\frac{2\pi}{3}t}}{2} + \overset{2}{\cancel{4}}\left(\frac{e^{j\frac{5\pi}{3}t} - e^{-j\frac{5\pi}{3}t}}{\cancel{2}j}\right)$$

$$x(t) = 2 + \frac{1}{2}e^{-j(-2)\frac{\pi}{3}t} + \frac{1}{2}e^{-j(2)\frac{\pi}{3}t} + \frac{2}{j}e^{-j(-5)\frac{\pi}{3}t} - \frac{2}{j}e^{-j(5)\frac{\pi}{3}t}$$

Fourier Series Co-efficients:

$C_0 = 2$

$C_{-1} = 0$ & $C_{+1} = 0$

$C_{-2} = \frac{1}{2}$ & $C_2 = \frac{1}{2}$

$C_{-3} = 0$ & $C_3 = 0$

$C_{-4} = 0$ & $C_4 = 0$

$C_{-5} = \frac{2}{j} = -j2$ & $C_5 = -\frac{2}{j} = j2.$

Fourier Transform

→ **Need :**

i) To analyse both the continuous Time periodic and aperiodic signals.

ii) To analyse stable systems

iii) It can convert the signals in time domain to frequency domain.

→ **Limitations**

i) It cannot be used to analyse unstable system.

ii) It can analyse the signal which satisfies the following condition.

$$\int_{-\infty}^{\infty} |x(t)|\, dt < \infty \quad [\because \text{absolutely integrable}]$$

→ **Properties of Continuous Time Fourier Transform [CTFT]**

* **Analysis Equation**

$$F.T[x(t)] = X(j\Omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\Omega t}\, dt$$

* **synthesis Equation**

$$\text{Inverse } F.T[X(j\Omega)] = x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega)\, e^{+j\Omega t}\, d\Omega$$

i) Linearity :

$$x_1(t) \xrightarrow{F.T} X_1(j\Omega)$$

$$x_2(t) \xrightarrow{F.T} X_2(j\Omega)$$

$$F.T\left[a x_1(t) + b x_2(t)\right] = a X_1(j\Omega) + b X_2(j\Omega).$$

Proof: $F\cdot T[x(t)] = \int_{-\infty}^{\infty} x(t)\, e^{-j\Omega t}\, dt$

$$F\cdot T[a x_1(t) + b x_2(t)] = \int_{-\infty}^{\infty} [a x_1(t) + b x_2(t)]\, e^{-j\Omega t}\, dt$$

$$= a \int_{-\infty}^{\infty} x_1(t)\, e^{-j\Omega t}\, dt + b \int x_2(t)\, e^{-j\Omega t}\, dt$$

$$F.T[a x_1(t) + b x_2(t)] = a\, X_1(j\Omega) + b\, X_2(j\Omega)$$

Hence proved.

ii) **Time shifting:**

$$x(t) \xrightarrow{F.T} X(j\Omega)$$

$$F.T[x(t-t_0)] = e^{-j\Omega t_0}\, X(j\Omega).$$

Proof:

$$F\cdot T[x(t)] = \int_{-\infty}^{\infty} x(t)\, e^{-j\Omega t}\, dt$$

$$F\cdot T[x(t-t_0)] = \int_{-\infty}^{\infty} x(t-t_0)\, e^{-j\Omega t}\, dt$$

$$\boxed{\begin{aligned} \text{Let,}\quad m &= t - t_0 \\ t &= m + t_0 \\ dt &= dm \end{aligned}}$$

$$= \int_{-\infty}^{\infty} x(m)\, e^{-j\Omega(m+t_0)}\, dm$$

$$= e^{-j\Omega t_0} \int_{-\infty}^{\infty} x(m).\, e^{-j\Omega m}\, dm$$

$$F\cdot T[x(t-t_0)] = e^{-j\Omega t_0}\, X(j\Omega)$$

Hence proved.

iii) Frequency shifting:

$$x(t) \xrightarrow{F.T} X(j\Omega)$$

$$F.T[e^{j\Omega_0 t} x(t)] = X(j(\Omega - \Omega_0))$$

Proof:

$$F.T[x(t)] = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\Omega t} dt$$

$$F.T[e^{j\Omega_0 t} x(t)] = \int_{-\infty}^{\infty} e^{j\Omega_0 t} x(t) \cdot e^{-j\Omega t} dt$$

$$= \int_{-\infty}^{\infty} x(t) \; e^{-j(\Omega - \Omega_0)t} dt$$

$$F.T[e^{j\Omega_0 t} x(t)] = X[j(\Omega - \Omega_0)]$$

Hence proved.

iv) Time scaling:

$$x(t) \xrightarrow{F.T} X(j\Omega)$$

$$F.T[x(at)] = \frac{1}{a} X\left(\frac{j\Omega}{a}\right)$$

Proof:

$$F.T[x(t)] = \int_{-\infty}^{\infty} x(t) \, e^{-j\Omega t} dt$$

$$F.T[x(at)] = \int_{-\infty}^{\infty} x(at) \, e^{-j\Omega t} dt$$

Let, $m = at$

$t = \frac{m}{a}$

$dt = \frac{1}{a} . dm$

$$F.T[x(at)] = \int_{-\infty}^{\infty} x(m) . e^{-j\Omega(\frac{m}{a})} . \frac{1}{a} dm.$$

$$= \frac{1}{a}\int_{-\infty}^{\infty} x(m)\, e^{-j\left(\frac{\Omega}{a}\right)m} dm$$

$$F.T[x(at)] = \frac{1}{a} X\left(\frac{j\Omega}{a}\right)$$

Hence proved.

V) <u>Differentiation in time domain</u>

$$x(t) \xrightarrow{F.T} X(j\Omega)$$

$$F.T\left[\frac{dx(t)}{dt}\right] = j\Omega\, X(j\Omega)$$

<u>Proof:</u>

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega) e^{j\Omega t} d\Omega$$

$$\frac{dx(t)}{dt} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega)\, \frac{d}{dt}(e^{j\Omega t})\, d\Omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega)\, e^{j\Omega t} . (j\Omega) . d\Omega$$

$$= j\Omega . \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega) e^{j\Omega t} d\Omega$$

$$\frac{dx(t)}{dt} = j\Omega . x(t)$$

$$F.T\left[\frac{dx(t)}{dt}\right] = F.T\left[j\Omega\, x(t)\right] = j\Omega\, F.T[x(t)]$$

$$F.T\left[\frac{d}{dt}x(t)\right] = j\Omega\, X(j\Omega)$$

Hence proved

vi) Differentiation in Frequency domain

$$x(t) \xrightarrow{F.T} X(j\Omega)$$

$$F.T[t\,x(t)] = j\frac{d}{d\Omega}X(j\Omega)$$

Proof:

$$F.T[x(t)] = X(j\Omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\Omega t}\, dt$$

$$\frac{dX(j\Omega)}{d\Omega} = \int_{-\infty}^{\infty} x(t).\ \frac{d}{d\Omega}(e^{-j\Omega t})\, dt$$

$$= \int_{-\infty}^{\infty} x(t).\ e^{-j\Omega t}(-jt)\, dt$$

$$\frac{dX(j\Omega)}{d\Omega} = -j\int_{-\infty}^{\infty} t\,x(t)\, e^{-j\Omega t}\, dt$$

$$\frac{1}{-j}\frac{dX(j\Omega)}{d\Omega} = \int_{-\infty}^{\infty} t\,x(t)\, e^{-j\Omega t}\, dt$$

$$j\frac{dX(j\Omega)}{d\Omega} = F.T[t\,x(t)]$$

Hence proved.

vii) Integration in time domain

$$x(t) \xrightarrow{F.T} X(j\Omega)$$

$$F.T\left[\int_{-\infty}^{t} x(t)\, dt\right] = \frac{1}{j\Omega} X(j\Omega).$$

Soln: synthesis eqn.

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega)\, e^{j\Omega t}\, d\Omega$$

$$\int_{-\infty}^{t} x(t)\, dt = \int_{-\infty}^{t} \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega)\, e^{j\Omega t}\, d\Omega\right] dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega)\left[\int_{-\infty}^{\infty} e^{j\Omega t}\, dt\right] d\Omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega)\left(\frac{e^{j\Omega t}}{j\Omega}\right) d\Omega$$

$$= \frac{1}{j\Omega}\cdot\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega)\, e^{j\Omega t}\, d\Omega$$

$$\int_{-\infty}^{t} x(t)\, dt = \frac{1}{j\Omega}\, x(t)$$

$$F.T\left[\int_{-\infty}^{t} x(t)\, dt\right] = F.T\left[\frac{1}{j\Omega}\, x(t)\right] = \frac{1}{j\Omega}\, F.T[x(t)]$$

$$\boxed{F.T\left[\int_{-\infty}^{t} x(t)\, dt\right] = \frac{1}{j\Omega}\, X(j\Omega)}$$

Hence proved.

viii) Convolution Property

$$x_1(t) \xrightarrow{F.T} X_1(j\Omega)$$

$$x_2(t) \xrightarrow{FT} X_2(j\Omega)$$

$$F.T[x_1(t) * x_2(t)] = X_1(j\Omega) \cdot X_2(j\Omega)$$

Proof:

w.k.T.

convolution integral formula

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(m)\, x_2(t-m)\, dm \rightarrow ①$$

$$F.T[x(t)] = \int_{-\infty}^{\infty} x(t)\, e^{-j\Omega t}\, dt$$

$$F.T[x_1(t) * x_2(t)] = \int_{-\infty}^{\infty} [x_1(t) * x_2(t)]\, e^{-j\Omega t}\, dt$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x_1(m)\, x_2(t-m)\, dm \right] e^{-j\Omega t}\, dt$$

Multiblying by $e^{-j\Omega m}$ & $e^{+j\Omega m}$.

$$= \int_{-\infty}^{\infty} x_1(m)\, e^{-j\Omega m}\, dm \int_{-\infty}^{\infty} x_2(t-m) \cdot e^{j\Omega m}\, e^{-j\Omega t}\, dt$$

$$= X_1(j\Omega) \cdot \int_{-\infty}^{\infty} x_2(t-m)\, e^{-j\Omega(t-m)}\, dt$$

$$\boxed{\begin{aligned} \text{Let,}\quad k &= t-m \\ dk &= dt \end{aligned}}$$

$$= X_1(j\Omega) \int_{-\infty}^{\infty} x_2(k)\, e^{-j\Omega k}\, dk$$

$$= X_1(j\Omega) \cdot X_2(j\Omega)$$

Hence proved.

ix) Parseval's theorem for continuous Time Aperiodic signals.

$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\Omega)|^2 \, d\Omega$$

Energy of a signal in time domain = Energy of a signal in frequency domain.

Proof:

$$|x(t)|^2 = x(t) \cdot x^*(t)$$

$$|X(j\Omega)|^2 = X(j\Omega) \, X^*(j\Omega)$$

synthesis eqn.

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\Omega) \, e^{j\Omega t} \, d\Omega$$

$$x^*(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(j\Omega) \, e^{-j\Omega t} \, d\Omega$$

$$L.H.S = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} x(t) \cdot x^*(t) \, dt$$

$$= \int_{-\infty}^{\infty} x(t) \cdot \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(j\Omega) \, e^{-j\Omega t} \, d\Omega \right] dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(j\Omega) \, d\Omega \cdot \underbrace{\int_{-\infty}^{\infty} x(t) \, e^{-j\Omega t} \, dt}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(j\Omega) \cdot X(j\Omega) \, d\Omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\Omega)|^2 \, d\Omega = R.H.S.$$

Problems based on Fourier Transform

1. Find the Fourier Transform of $x(t) = e^{-at} u(t)$.

Soln:

$$F.T[x(t)] = X(j\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt$$

$$F.T[e^{-at} u(t)] = \int_{-\infty}^{\infty} e^{-at} \cdot u(t) \cdot e^{-j\Omega t} dt$$

$$= \int_{0}^{\infty} e^{-(j\Omega + a)t} dt \quad \left[\because u(t) = \begin{matrix} 1 & ; t \geq 0 \\ 0 & ; t < 0 \end{matrix} \right]$$

$$= \left[\frac{e^{-(j\Omega + a)t}}{-(j\Omega + a)} \right]_{0}^{\infty}$$

$$= \frac{1}{-(j\Omega + a)} \left[e^{-\infty} - e^{0} \right] = \frac{1}{-(j\Omega + a)} [0 - 1]$$

$$\boxed{F.T[e^{-at} u(t)] = \frac{1}{j\Omega + a}}$$

2. $x(t) = e^{at} u(t)$.

Soln:

$$F.T[x(t)] = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt$$

$$F.T[e^{at} u(t)] = \int_{-\infty}^{\infty} e^{at} u(t) e^{-j\Omega t} dt$$

$$= \int_{0}^{\infty} e^{-(j\Omega - a)t} dt \quad \left[\because u(t) = \begin{matrix} 1 & ; t \geq 0 \\ 0 & ; t < 0 \end{matrix} \right]$$

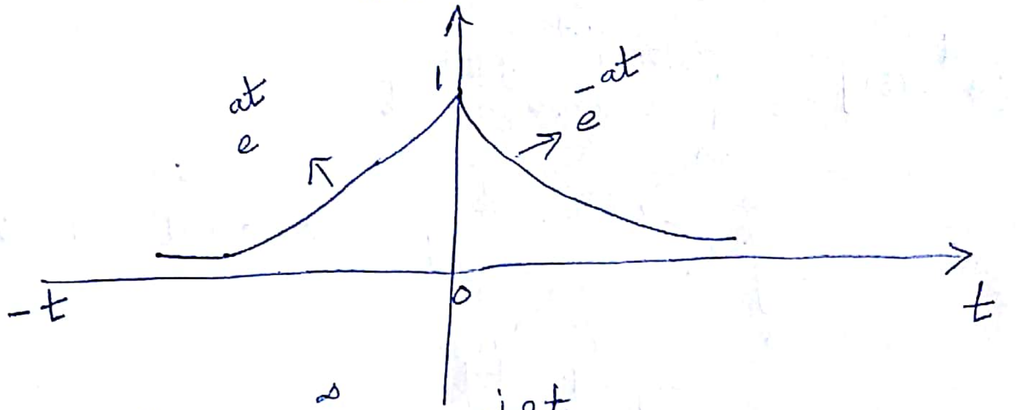
$$= \left[\frac{e^{-(j\Omega - a)t}}{-(j\Omega - a)} \right]_{0}^{\infty} = \frac{1}{-(j\Omega - a)} \left[e^{-\infty} - e^{0} \right] = \frac{1}{-(j\Omega - a)} [0 - 1]$$

$$\boxed{F.T[e^{at} u(t)] = \frac{1}{j\Omega - a}}$$

3) Find the Fourier Transform of $x(t) = e^{-a|t|}$.

Soln:

$x(t) = e^{-a|t|}$.



$$F.T[x(t)] = \int_{-\infty}^{\infty} x(t)\, e^{-j\Omega t}\, dt$$

$$F.T[e^{-a|t|}] = \int_{-\infty}^{\infty} e^{-a|t|}\, e^{-j\Omega t}\, dt$$

$$= \int_{-\infty}^{0} e^{-a(-t)}\, e^{-j\Omega t}\, dt + \int_{0}^{\infty} e^{-a(t)}\, e^{-j\Omega t}\, dt$$

$$= \int_{-\infty}^{0} e^{(-j\Omega+a)t}\, dt + \int_{0}^{\infty} e^{-(j\Omega+a)t}\, dt$$

$$= \int_{0}^{\infty} e^{-(-j\Omega+a)t}\, dt + \int_{0}^{\infty} e^{-(j\Omega+a)t}\, dt$$

$$= \left[\frac{e^{-(-j\Omega+a)t}}{-(-j\Omega+a)}\right]_{0}^{\infty} + \left[\frac{e^{-(j\Omega+a)t}}{-(j\Omega+a)}\right]_{0}^{\infty}$$

$$= \frac{1}{-(-j\Omega+a)}\left[e^{-\infty} - e^{0}\right] + \frac{1}{-(j\Omega+a)}\left[e^{-\infty} - e^{0}\right]$$

$$= \frac{1}{-(-j\Omega+a)}[0-1] + \frac{1}{-(j\Omega+a)}[0-1]$$

$$= \frac{1}{(-j\Omega + a)} + \frac{1}{(j\Omega + a)} = \frac{1}{a - j\Omega} + \frac{1}{a + j\Omega}$$

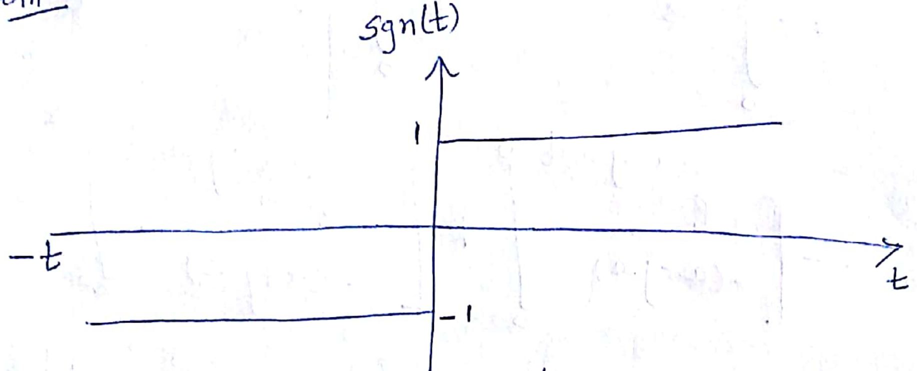
$$= \frac{a + j\Omega + a - j\Omega}{(a - j\Omega)(a + j\Omega)} = \frac{2a}{a^2 - (j\Omega)^2} = \frac{2a}{a^2 - j^2\Omega^2}$$

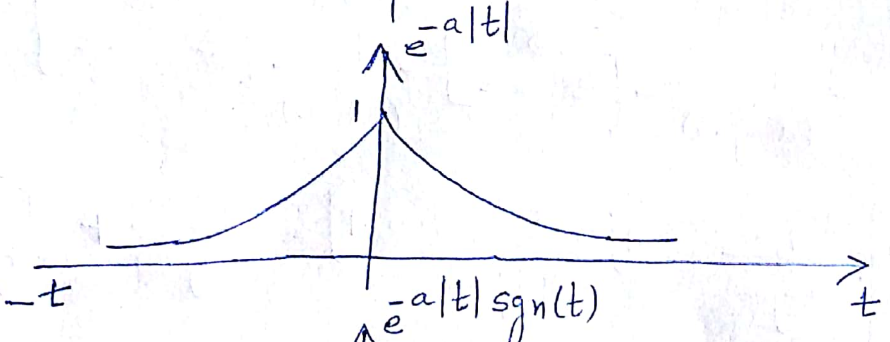
$$= \frac{2a}{a^2 - (-1 \cdot \Omega^2)} = \frac{2a}{a^2 + \Omega^2}$$

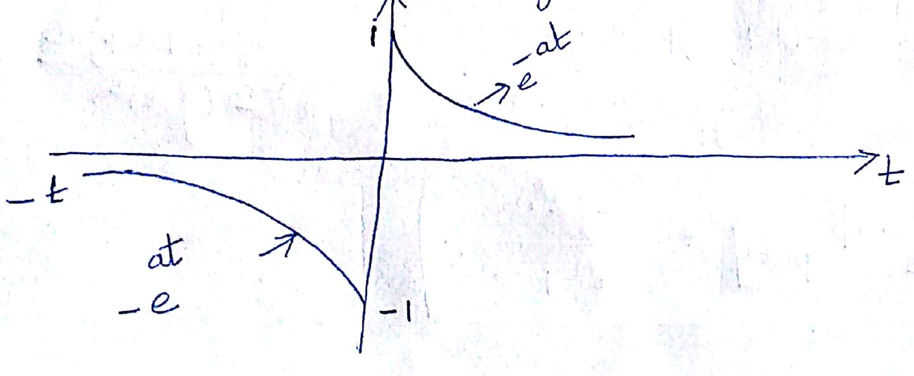
$$\boxed{F.T\left[e^{-a|t|}\right] = \frac{2a}{a^2 + \Omega^2}}$$

7) Find the fourier transform of $e^{-a|t|}$ $sgn(t)$

Soln:







$$F.T[x(t)] = \int_{-\infty}^{\infty} x(t)\, e^{-j\Omega t}\, dt$$

$$F.T[e^{-a|t|} sgn(t)] = \int_{-\infty}^{\infty} e^{-a|t|}\, sgn(t)\, e^{-j\Omega t}\, dt$$

$$= \int_{-\infty}^{0} - e^{at}\, e^{-j\Omega t}\, dt + \int_{0}^{\infty} e^{-at}\, e^{-j\Omega t}\, dt.$$

$$= -\int_{-\infty}^{0} e^{(a-j\Omega)t} + \int_{0}^{\infty} e^{-(a+j\Omega)t}\, dt$$

$$= -\int_{0}^{\infty} e^{-(a-j\Omega)t} + \int_{0}^{\infty} e^{-(a+j\Omega)t}\, dt$$

$$= -\left[\frac{e^{-(a-j\Omega)t}}{-(a-j\Omega)}\right]_{0}^{\infty} + \left[\frac{e^{-(a+j\Omega)t}}{-(a+j\Omega)}\right]_{0}^{\infty}$$

$$= \frac{1}{(a-j\Omega)}\left[e^{-\infty} - e^{0}\right] - \frac{1}{(a+j\Omega)}\left[e^{-\infty} - e^{0}\right]$$

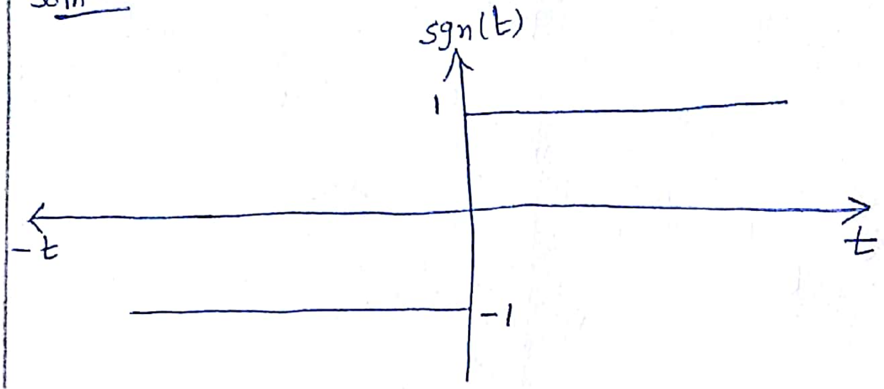
$$= \frac{1}{(a-j\Omega)}[0-1] - \frac{1}{(a+j\Omega)}[0-1] = \frac{-1}{(a-j\Omega)} + \frac{1}{(a+j\Omega)}$$

$$= \frac{-(a+j\Omega)+(a-j\Omega)}{(a-j\Omega)(a+j\Omega)} = \frac{-a-j\Omega+a-j\Omega}{a^2-(j\Omega)^2} = \frac{-j2\Omega}{a^2-(-1.\Omega^2)}$$

$$\boxed{F.T[e^{-a|t|} sgn(t)] = \frac{-j2\Omega}{a^2+\Omega^2}}$$

5) Find the fourier transform of sgn(t).

Soln:



The signal sgn(t) does not satisfy the absolute integrable property. So, the fourier transform of sgn(t) is obtained by the following way.

$$F.T\left[sgn(t)\right] = \underset{a\to 0}{Lt} \; F.T\left[e^{-a|t|}\, sgn(t)\right]$$

$$= \underset{a\to 0}{Lt}\left[\frac{-2j\Omega}{a^2+\Omega^2}\right] = \frac{-2j\Omega}{\Omega^2}$$

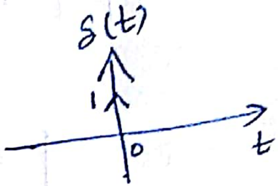
$$F.T\left[sgn(t)\right] = \frac{-2j}{\Omega} = \frac{-2j}{\Omega}\times\frac{j}{j}$$

$$\boxed{F.T\left[sgn(t)\right] = \frac{2}{j\Omega}}$$

Group II

Problems based on Fourier Transform.

1) Fourier Transform of $\delta(t)$.



Soln:

w.k.T. $\delta(t)=\begin{cases}1 ; & t=0\\ 0 ; & t\neq 0.\end{cases}$

$$F.T[x(t)] = X(j\Omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\Omega t}\, dt.$$

$$FT[\delta(t)] = \delta(j\Omega) = \int_{-\infty}^{\infty} \delta(t) . e^{-j\Omega t}\, dt$$

$$= \delta(t)\, e^{-j\Omega t}\Big|_{t=0}$$

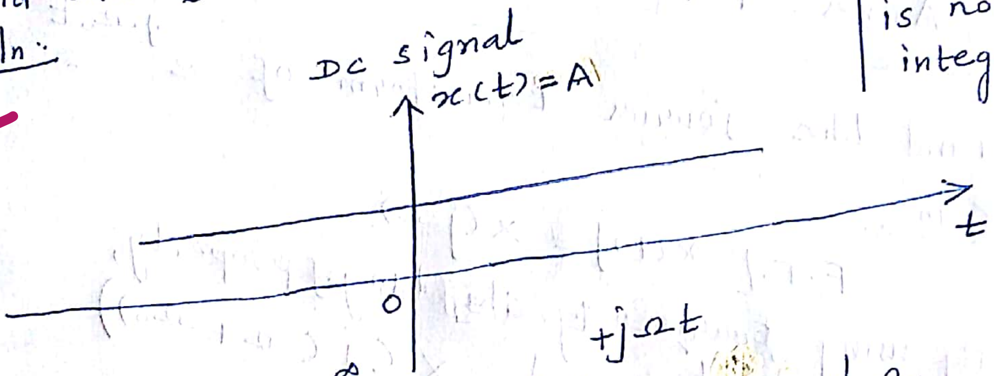
$$= \delta(0)\, e^{0} = 1.1 = 1$$

$$\boxed{F.T[\delta(t)] = \delta(j\Omega) = 1}$$

2) Find the fourier transform of DC signal.

Note: DC signal is not an absolutely integrable signal.

Soln:



$$F.T^{-1}[X(j\Omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega)\, e^{+j\Omega t}\, d\Omega$$

$$FT^{-1}[\delta(j\Omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(j\Omega)\, e^{+j\Omega t}\, d\Omega$$

$$\boxed{\delta(j\Omega)=\delta(\Omega)=\begin{cases}1 ; & \Omega=0\\ 0 ; & \Omega\neq 0\end{cases}}$$

$FT^{-1}[\delta(j\Omega)] = \frac{1}{2\pi}[1]$

$\boxed{2\pi\,\delta(\Omega) = FT[1]}$ $\quad \because \delta(j\Omega) = \delta(\Omega)$

So, $\boxed{FT[A] = A \cdot 2\pi\,\delta(\Omega)}$ $\quad$ [Always, $\delta(\Omega) = \begin{cases} 1 ; & \Omega = 0 \\ 0 ; & \Omega \neq 0 \end{cases}$]

---

3) Find the fourier transform of $e^{j\Omega_0 t}$.

Soln:

$F\cdot T[x(t)] = X(j\Omega)$.

By using frequency shifting prop,

$F\cdot T[e^{j\Omega_0 t}\, x(t)] = X(j(\Omega - \Omega_0))$

$\therefore F\cdot T[e^{j\Omega_0 t} \cdot 1] = 2\pi\,\delta(\Omega - \Omega_0)$

$\boxed{\text{Answer : } F\cdot T[e^{j\Omega_0 t}] = 2\pi\,\delta(\Omega - \Omega_0)}$

4) Find the fourier transform of $e^{-j\Omega_0 t}$.

Soln:

$F\cdot T\cdot[x(t)] = X(j\Omega)$.

By using frequency shifting property,

$F\cdot T[e^{-j\Omega_0 t}\, x(t)] = X(j(\Omega + \Omega_0))$

$\therefore FT[e^{-j\Omega_0 t} \cdot 1] = 2\pi\,\delta(\Omega + \Omega_0)$

$\boxed{\text{Answer: } F\cdot T[e^{-j\Omega_0 t}] = 2\pi\,\delta(\Omega + \Omega_0)}$

5) Find the fourier transform of $\cos \Omega_o t$.

Soln:

$$F.T[\cos \Omega_o t] = F.T\left[\frac{e^{+j\Omega_o t} + e^{-j\Omega_o t}}{2}\right]$$

$$= \frac{1}{2}\left\{FT[e^{j\Omega_o t}] + FT[e^{-j\Omega_o t}]\right\}$$

$$= \frac{1}{2}\left\{2\pi\,\delta(\Omega - \Omega_o) + 2\pi\,\delta(\Omega + \Omega_o)\right\}$$

$$\boxed{F.T[\cos \Omega_o t] = \pi\,\delta(\Omega - \Omega_o) + \pi\,\delta(\Omega + \Omega_o)}$$

6) Find the fourier transform of $\sin \Omega_o t$.

Soln:

$$F.T[\sin \Omega_o t] = F.T\left[\frac{e^{+j\Omega_o t} - e^{-j\Omega_o t}}{2j}\right]$$

$$= \frac{1}{2j}\left\{F.T(e^{+j\Omega_o t}) - F.T(e^{-j\Omega_o t})\right\}$$

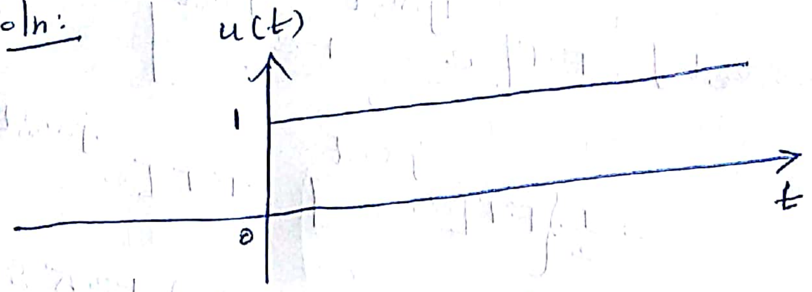
$$= \frac{1}{2j}\left\{2\pi\,\delta(\Omega - \Omega_o) - 2\pi\,\delta(\Omega + \Omega_o)\right\}$$

$$= \frac{\pi}{j}\left\{\delta(\Omega - \Omega_o) - \delta(\Omega + \Omega_o)\right\}$$

$$\boxed{\begin{aligned} F.T[\sin \Omega_o t] &= -j\pi\left\{\delta(\Omega - \Omega_o) - \delta(\Omega + \Omega_o)\right\} \\ &\text{(or)} \\ FT[\sin \Omega_o t] &= j\pi\left\{\delta(\Omega + \Omega_o) - \delta(\Omega - \Omega_o)\right\} \end{aligned}}$$

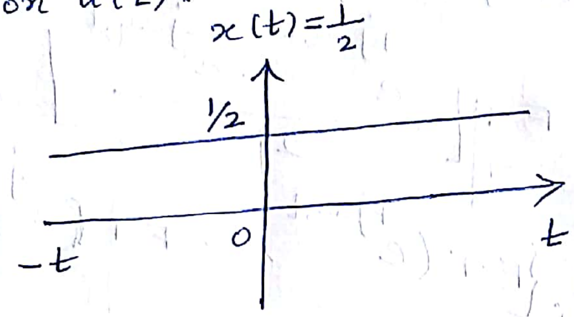
7). Find the fourier transform of $u(t)$.

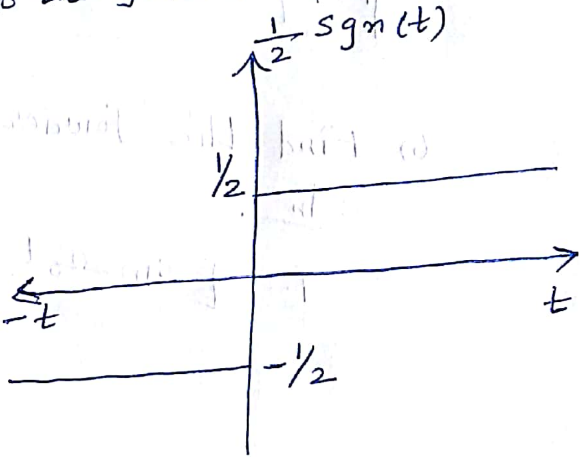
Soln:

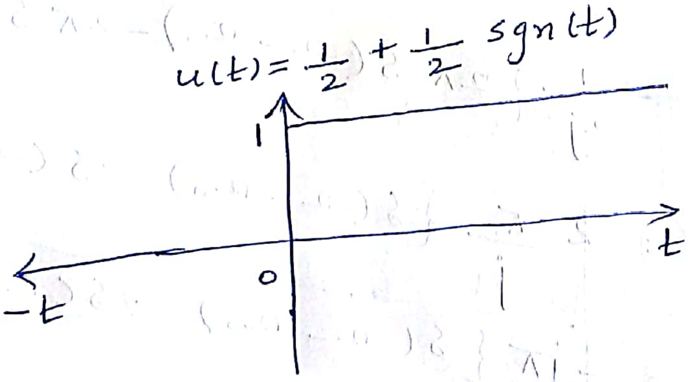


$$u(t) = \begin{cases} 1 & ; t \geq 0 \\ 0 & ; t < 0. \end{cases}$$

$u(t)$ is not an absolutely integrable signal. So, we cannot apply directly to the fourier transform on $u(t)$.







$$\therefore F[u(t)] = F.T\left[\frac{1}{2} + \frac{1}{2} \operatorname{sgn}(t)\right]$$

$$= \frac{1}{2} F.T[1] + \frac{1}{2} FT[\operatorname{sgn}(t)]$$

$$= \frac{1}{2} \times 2\pi\delta(\Omega) + \frac{1}{2} \times \frac{2}{j\Omega}$$

$$\boxed{FT[u(t)] = \pi\delta(\Omega) + \frac{1}{j\Omega}}$$

8) Find the Fourier Transform of $te^{-at}u(t)$.

Soln:

$$F.T[e^{-at}u(t)] = \frac{1}{j\Omega+a}$$

W.k.T., Differentiation in frequency domain property.

$$F.T[t\,x(t)] = j\frac{d}{d\Omega}X(j\Omega).$$

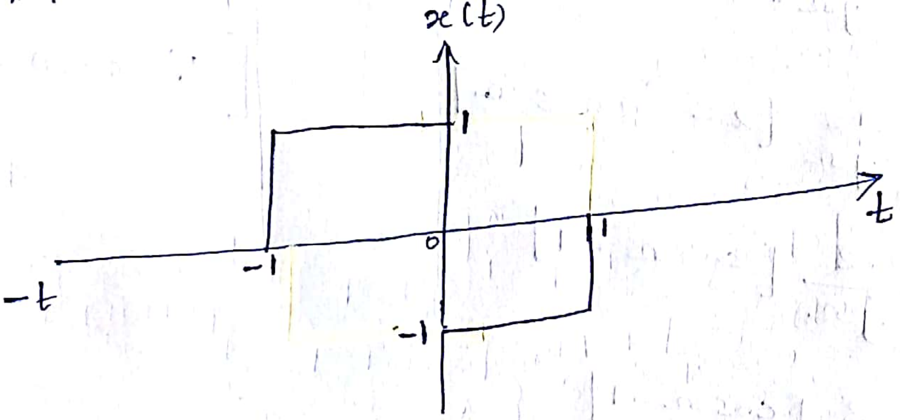
$$F.T[t\,e^{-at}u(t)] = j\frac{d}{d\Omega}\left[\frac{1}{j\Omega+a}\right] = j\frac{d}{d\Omega}\left[(j\Omega+a)^{-1}\right]$$

$$= j\times -1\times(j\Omega+a)^{-2}\times(j)$$

$$= (j\Omega+a)^{-2} \qquad [\because j^{2}=-1]$$

$$F.T[t\,e^{-at}u(t)] = \frac{1}{(j\Omega+a)^{2}}$$

9) Find the fourier transform of the following signal.



Soln:

$$F.T[x(t)] = \int_{-\infty}^{\infty} x(t)\,e^{-j\Omega t}\,dt$$

$$= \int_{-1}^{0} 1 \cdot e^{-j\Omega t}\, dt + \int_{0}^{1} -1 \cdot e^{-j\Omega t}\, dt$$

$$= \int_{-1}^{0} e^{-j\Omega t}\, dt - \int_{0}^{1} e^{-j\Omega t}\, dt$$

$$= \left[\frac{e^{-j\Omega t}}{-j\Omega} \right]_{-1}^{0} - \left[\frac{e^{-j\Omega t}}{-j\Omega} \right]_{0}^{1}$$

$$= \frac{1}{-j\Omega} \left[e^{0} - e^{+j\Omega} \right] + \frac{1}{j\Omega} \left[e^{-j\Omega} - e^{0} \right]$$

$$= \frac{1}{-j\Omega} \left[1 - e^{+j\Omega} \right] + \frac{1}{j\Omega} \left[e^{-j\Omega} - 1 \right]$$

$$= \frac{1}{-j\Omega} \left[1 - e^{+j\Omega} - e^{-j\Omega} + 1 \right]$$

$$= \frac{1}{-j\Omega} \left[2 - \left(e^{j\Omega} + e^{-j\Omega} \right) \right]$$

$$= \frac{1}{-j\Omega} \left[2 - 2\cos\Omega \right] \qquad \left[\because \cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2} \right]$$

$$= \frac{-2}{-j\Omega} \left[\cos\Omega - 1 \right]$$

$$= \frac{2}{j\Omega} \left[\cos\Omega - 1 \right]$$

Group III

→ Problems based on Fourier Transform

1) Find the Fourier Transform of $\cos\Omega_0 t\ u(t)$.

Soln:

$$F.T\left[\cos\Omega_0 t\ u(t)\right] = F.T\left[\left(\frac{e^{j\Omega_0 t}+e^{-j\Omega_0 t}}{2}\right)u(t)\right]$$

$$=\frac{1}{2}\left\{FT\left(e^{j\Omega_0 t}u(t)\right)+FT\left(e^{-j\Omega_0 t}u(t)\right)\right\}$$

$F.T[u(t)] = \frac{1}{j\Omega} + \pi\delta(\Omega)$

&

Frequency shifting property of F.T.

$FT[e^{j\Omega_0 t}x(t)] = X(j(\Omega-\Omega_0))$

$$=\frac{1}{2}\left\{\frac{1}{j(\Omega-\Omega_0)}+\pi\delta(\Omega-\Omega_0)+\frac{1}{j(\Omega+\Omega_0)}+\pi\delta(\Omega+\Omega_0)\right\}$$

$$=\frac{1}{2}\left\{\frac{1+j(\Omega-\Omega_0)\pi\delta(\Omega-\Omega_0)}{j(\Omega-\Omega_0)}+\frac{1+j(\Omega+\Omega_0)\pi\delta(\Omega+\Omega_0)}{j(\Omega+\Omega_0)}\right\}$$

$$=\frac{1}{j2}\left\{\frac{\Omega+\Omega_0+j(\Omega^2-\Omega_0^2).\pi\delta(\Omega-\Omega_0)+\Omega-\Omega_0+j(\Omega^2-\Omega_0^2)\pi\delta(\Omega+\Omega_0)}{(\Omega-\Omega_0)(\Omega+\Omega_0)}\right\}$$

$$=\frac{-j}{2(\Omega^2-\Omega_0^2)}\left\{2\Omega+j\pi(\Omega^2-\Omega_0^2)\left[\delta(\Omega-\Omega_0)+\delta(\Omega+\Omega_0)\right]\right\}$$

2) Find the Fourier Transform of $\sin\Omega_0 t\, u(t)$.

Soln:

$$FT[\sin\Omega_0 t\; u(t)] = FT\left[\left(\frac{e^{j\Omega_0 t} - e^{-j\Omega_0 t}}{2j}\right) u(t)\right]$$

$$= \frac{1}{2j}\left\{ FT\left[e^{j\Omega_0 t} u(t)\right] - FT\left[e^{-j\Omega_0 t} u(t)\right]\right\}$$

$F\cdot T[u(t)] = \frac{1}{j\Omega} + \pi\delta(\Omega)$

&

Frequency shifting property of F.T

$F\cdot T[e^{j\Omega_0 t} x(t)] = X(j(\Omega - \Omega_0))$

$$= \frac{1}{2j}\left\{\frac{1}{j(\Omega-\Omega_0)} + \pi\delta(\Omega-\Omega_0) - \left(\frac{1}{j(\Omega+\Omega_0)} + \pi\delta(\Omega+\Omega_0)\right)\right\}$$

$$= \frac{1}{2j}\left\{\frac{1+j(\Omega-\Omega_0)\pi\delta(\Omega-\Omega_0)}{j(\Omega-\Omega_0)} - \left(\frac{1+j(\Omega+\Omega_0)\pi\delta(\Omega+\Omega_0)}{j(\Omega+\Omega_0)}\right)\right\}$$

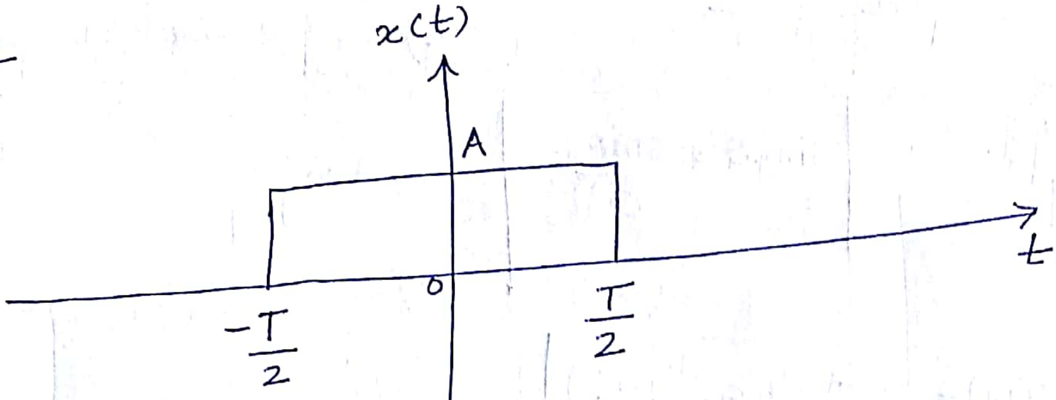
$$= \frac{1}{2j\times j}\left\{\frac{\Omega+\Omega_0+j(\Omega^2-\Omega_0^2)\pi\delta(\Omega-\Omega_0) - (\Omega-\Omega_0+j(\Omega^2-\Omega_0^2)\pi\delta(\Omega+\Omega_0)}{(\Omega-\Omega_0)(\Omega+\Omega_0)}\right\}$$

$$= -\frac{1}{2}\left\{\frac{\Omega+\Omega_0+j(\Omega^2-\Omega_0^2)\pi\delta(\Omega-\Omega_0) - \Omega+\Omega_0-j(\Omega^2-\Omega_0^2)\pi\delta(\Omega+\Omega_0)}{\Omega^2-\Omega_0^2}\right\}$$

$$= \frac{-1}{2(\Omega^2-\Omega_0^2)}\left\{2\Omega_0 + j\pi(\Omega^2-\Omega_0^2)\left[\delta(\Omega-\Omega_0) - \delta(\Omega+\Omega_0)\right]\right\}$$

3) Find the Fourier Transform of Gate function (or) Gate signal. Also draw the magnitude spectrum.

Soln:



$$F.T[x(t)] = \int_{-\infty}^{\infty} x(t)\, e^{-j\Omega t}\, dt$$

From the graph,

$$= \int_{-T/2}^{T/2} A\, e^{-j\Omega t}\, dt = A\int_{-T/2}^{T/2} e^{-j\Omega t}\, dt$$

$$= A\left[\frac{e^{-j\Omega t}}{-j\Omega}\right]_{-T/2}^{T/2} = -\frac{A}{j\Omega}\left[e^{-j\frac{\Omega T}{2}} - e^{+j\frac{\Omega T}{2}}\right]$$

$$= \frac{A}{j\Omega}\left[e^{j\frac{\Omega T}{2}} - e^{-j\frac{\Omega T}{2}}\right]$$

$$\boxed{\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}}$$

$$= \frac{A}{j\Omega} \times 2j\,\sin\left(\frac{\Omega T}{2}\right)$$

$$= \frac{2A}{\Omega}\sin\left(\frac{\Omega T}{2}\right)$$

(or)

$$= \frac{2A}{\Omega} \times \frac{\Omega T}{2} \times \frac{\sin\left(\frac{\Omega T}{2}\right)}{\left(\frac{\Omega T}{2}\right)}$$

$$\boxed{\operatorname{sinc}\theta = \frac{\sin\theta}{\theta}}$$

$$\boxed{X(j\Omega) = AT \operatorname{sinc}\left(\frac{\Omega T}{2}\right)}$$

↳ F.T of Gate signal [Rectangular pulse].

* Magnitude Spectrum.: $|X(j\Omega)|$ Vs Ω

Assume: $A = 1$ & $T = 1$

So, $|X(j\Omega)| = \left|\operatorname{sinc}\left(\frac{\Omega}{2}\right)\right| = \left|\frac{\sin(\Omega/2)}{\Omega/2}\right|$

Ω	$\lvert X(j\Omega)\rvert$
-4π	0
-3π	0.21
-2π	0
$-\pi$	0.64
0	1
π	0.64
2π	0
3π	0.21
4π	0

When $\Omega = 0$, $|X(0)| = \left|\frac{\sin 0}{0}\right| = \frac{0}{0}$ [not defined]

Using L'Hopital rule,

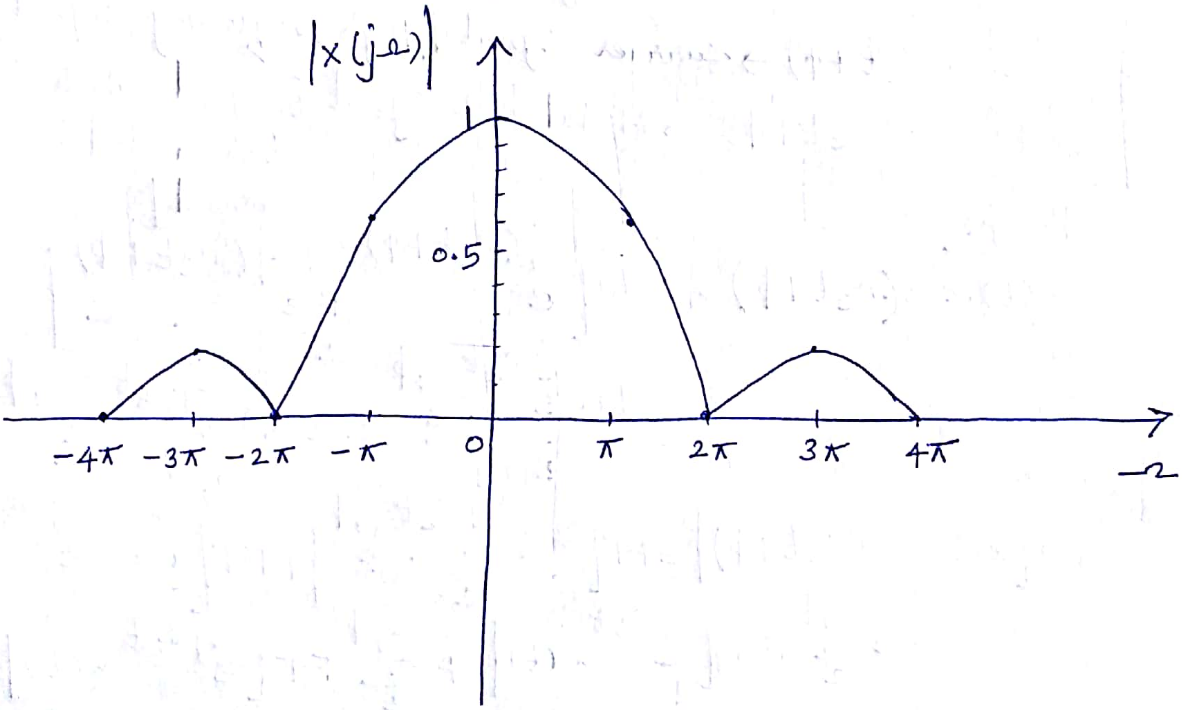
$$|X(0)| = \underset{\Omega \to 0}{Lt} \frac{\frac{d}{d\Omega} \sin(\Omega/2)}{\frac{d}{d\Omega}\left(\frac{\Omega}{2}\right)}$$

$$= \underset{\Omega \to 0}{Lt} \frac{\cos\left(\frac{\Omega}{2}\right) \times \frac{1}{2}}{1/2}$$

$$= \underset{\Omega \to 0}{Lt} \cos(\Omega/2)$$

$$= \cos(0/2) = \cos 0$$

$$\boxed{|X(0)| = 1}$$



* Previously asked questions in Anna University Questions :

1) Find the Fourier Transform of the signal $x(3t)$.

Soln :

$$x(t) = x(3t).$$

$$F.T[x(t)] = X(j\Omega).$$

Time scaling property of F.T

$$F.T[x(at)] = \frac{1}{a} X\left(\frac{j\Omega}{a}\right).$$

So,

$$\boxed{F.T[x(3t)] = \frac{1}{3} X\left(\frac{j\Omega}{3}\right)}$$

2). State and Prove modulation theorem using Fourier Transform.

Modulation Theorem:

$$F.T[x(t)\cdot\cos(\Omega_c t+\phi)] = \frac{e^{j\phi}}{2} X(\Omega-\Omega_c) + \frac{e^{-j\phi}}{2} X(\Omega+\Omega_c)$$

where, $x(t) \rightarrow$ Modulating signal [information signal]

$\cos(\Omega_c t+\phi) \rightarrow$ carrier signal [High frequency signal].

$x(t)\cdot\cos(\Omega_c t+\phi) \rightarrow$ Modulated signal [signal which is to be transmitted through the channel].

Proof:

$$x(t)\cdot\cos(\Omega_c t+\phi) = x(t)\left[\frac{e^{j(\Omega_c t+\phi)} + e^{-j(\Omega_c t+\phi)}}{2}\right]$$

$$= \frac{x(t)\cdot e^{j\Omega_c t}\cdot e^{j\phi}}{2} + \frac{x(t)\, e^{-j\Omega_c t}\cdot e^{-j\phi}}{2}$$

Now,

$$F.T[x(t)\cos(\Omega_c t+\phi)] = FT\left[\frac{x(t)\, e^{j\Omega_c t}\cdot e^{j\phi}}{2}\right] + FT\left[\frac{x(t)\cdot e^{-j\Omega_c t}\cdot e^{-j\phi}}{2}\right]$$

$$= \frac{e^{j\phi}}{2} F.T[e^{j\Omega_c t}\, x(t)] + \frac{e^{-j\phi}}{2} F.T[e^{-j\Omega_c t}\, x(t)]$$

$$= \frac{e^{j\phi}}{2} X(\Omega-\Omega_c) + \frac{e^{-j\phi}}{2} X(\Omega+\Omega_c)$$

[using frequency shifting property of fourier transform]

Hence, Modulation Theorem is proved.

→Laplace Transform [S - Transform]

× Need

i) To analyse unstable systems.

ii) To analyse complex exponential signals.

iii) To analyse signals which are absolutely not integrable.

× Definition:

Analysis equation

$$LT[x(t)] = X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt \rightarrow$$ Bilateral (or) Double sided Laplace Transform

$$LT[x(t)] = X(s) = \int_{0}^{\infty} x(t) e^{-st} dt \rightarrow$$ Unilateral (or) single sided Laplace Transform

Synthesis equation

$$\text{Inverse } L.T[X(s)] = LT^{-1}[X(s)] = x(t) = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} X(s) e^{st} ds.$$

*** Properties of Laplace Transform**

i) Linearity

$$x(t) \xrightarrow{LT} X(s)$$

$$L.T[a x_1(t) + b x_2(t)] = a X_1(s) + b X_2(s)$$

Proof:

$$L.T[x(t)] = X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$LT[a x_1(t) + b x_2(t)] = \int_{-\infty}^{\infty} [a x_1(t) + b x_2(t)] e^{-st} dt$$

$$= a \int_{-\infty}^{\infty} x_1(t) e^{-st} dt + b \int_{-\infty}^{\infty} x_2(t) e^{-st} dt$$

$$= a X_1(s) + b X_2(s).$$

Hence proved.

ii) Time shifing property:

$$x(t) \xrightarrow{LT} X(s)$$

$$LT[x(t-t_0)] = e^{-st_0} X(s).$$

Proof:

$$L.T[x(t)] = X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

$$LT[x(t-t_0)] = \int_{-\infty}^{\infty} x(t-t_0)\, e^{-st}\, dt$$

$$\boxed{\begin{aligned} \text{Let } m &= t-t_0 \\ t &= m+t_0 \\ dt &= dm \end{aligned}}$$

$$= \int_{-\infty}^{\infty} x(m).\ e^{-s(m+t_0)}\, dm$$

$$= e^{-st_0} \int_{-\infty}^{\infty} x(m)\, e^{-sm}\, dm$$

$$= e^{-st_0}\, X(s).$$

Hence proved.

iii) Frequency shifting property:

$$x(t) \xrightarrow{LT} X(s)$$

$$LT[e^{-at} x(t)] = X(s+a)$$

Proof:

$$LT[x(t)] = X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

$$LT[e^{-at}x(t)] = \int_{-\infty}^{\infty} e^{-at} x(t) e^{-st} dt$$

$$= \int_{-\infty}^{\infty} x(t) \cdot e^{-(s+a)t} dt$$

$$= X(s+a).$$

Hence proved.

Note:

$$LT[e^{at} x(t)] = X(s-a).$$

iv) Time scaling Property:

$$x(t) \xrightarrow{LT} X(s).$$

$$LT[x(at)] = \frac{1}{a} X\left(\frac{s}{a}\right)$$

Proof:

$$L \cdot T[x(t)] = X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$LT[x(at)] = \int_{-\infty}^{\infty} x(at) e^{-st} dt$$

$$\boxed{\begin{aligned} &\text{Let} \\ &m = at \\ &t = \frac{m}{a} \\ &dt = \frac{dm}{a} \end{aligned}}$$

$$= \int_{-\infty}^{\infty} x(m) \cdot e^{-s\frac{m}{a}} \cdot \frac{dm}{a}$$

$$= \frac{1}{a} \int_{-\infty}^{\infty} x(m) \cdot e^{-\left(\frac{s}{a}\right)m} dm$$

$$= \frac{1}{a} X\left(\frac{s}{a}\right).$$

Hence proved.

v) Differentiation in Time Domain:

$$x(t) \xrightarrow{LT} X(s).$$

$$L.T\left[\frac{dx(t)}{dt}\right] = sX(s).$$

Proof:

Synthesis equation

$$x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)\, e^{st}\, ds$$

$$\frac{dx(t)}{dt} = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)\, \frac{d}{dt}[e^{st}]\, ds$$

$$= \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s).\, e^{st}.\, s.\, ds$$

$$= s.\, \underbrace{\frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)\, e^{st}\, ds}$$

$$\frac{dx(t)}{dt} = s\,x(t)$$

$$L.T\left[\frac{dx(t)}{dt}\right] = LT[s\ x(t)$$

$$= s\ LT[x(t)]$$

$$= sX(s)$$

Hence proved.

vi) Differentiation in frequency domain:

$$LT[x(t)] = X(s)$$

$$LT[-t\,x(t)] = \frac{dX(s)}{ds}.$$

Proof:

$$L.T[x(t)] = X(s) = \int_{-\infty}^{\infty} x(t)\,e^{-st}\,dt$$

$$\frac{dX(s)}{ds} = \int_{-\infty}^{\infty} x(t)\,\frac{d}{ds}[e^{-st}]\,dt$$

$$= \int_{-\infty}^{\infty} x(t).\,e^{-st}.\,(-t).\,dt$$

$$= \int_{-\infty}^{\infty} [-t\,x(t)]\,e^{-st}\,dt$$

$$= L.T[-t\,x(t)].$$

Hence proved.

vii) Integration in Time Domain:

$$LT[x(t)] = X(s)$$

$$LT\left[\int_{-\infty}^{t} x(t)\,dt\right] = \frac{X(s)}{s}.$$

Proof:

Synthesis equation

$$x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)\,e^{st}\,ds$$

$$\int_{-\infty}^{t} x(t)\,dt = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s).\left[\int_{-\infty}^{t} e^{st}\,dt\right] ds$$

$$= \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s).\ \frac{e^{st}}{s}.\ ds$$

$$= \frac{1}{s} \times \underbrace{\frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)\, e^{st}\ ds}$$

$$\int_{-\infty}^{t} x(t)\, dt = \frac{1}{s}.\ x(t)$$

$$L.T\left[\int_{-\infty}^{t} x(t)\, dt\right] = L.T\left[\frac{1}{s}.\ x(t)\right]$$

$$= \frac{1}{s}.\ X(s) = \frac{X(s)}{s}.$$

Hence proved.

Viii) Convolution Property:

$$LT[x(t)] = X(s)$$

$$LT[x_1(t) * x_2(t)] = X_1(s).\ X_2(s).$$

Proof:

$$LT[x(t)] = X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

$$LT[x_1(t) * x_2(t)] = \int_{-\infty}^{\infty} [x_1(t) * x_2(t)]\, e^{-st}\, dt$$

Convolution integral formula

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(m) .\ x_2(t-m)\, dm$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x_1(m) \cdot x_2(t-m)dm \right] e^{-st} dt$$

Multiply the above expression by e^{-sm} and e^{sm}

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x_1(m)\, x_2(t-m)dm \right] e^{-st} dt \cdot e^{-sm} \cdot e^{sm}$$

Rearranging the terms.

$$= \int_{-\infty}^{\infty} \underbrace{\left[\int_{-\infty}^{\infty} x_1(m)\, e^{-sm} dm \right]} x_2(t-m)\, e^{-s(t-m)} \cdot dt$$

$$= X_1(s) \cdot \int_{-\infty}^{\infty} x_2(t-m)\, e^{-s(t-m)} dt$$

$$\boxed{\begin{array}{l} \text{Let} \\ \quad t-m = k \\ \quad t = k+m \\ \quad dt = dk \end{array}}$$

$$= X_1(s) \cdot \underbrace{\int_{-\infty}^{\infty} x_2(k)\, e^{-sk} dk}$$

$$= X_1(s) \cdot X_2(s).$$

Hence proved.

ix) Initial value theorem:

$$x(0) = \underset{s \to \infty}{Lt}\ sX(s)$$

x) Final value theorem

$$x(\infty) = \underset{s \to 0}{Lt}\ sX(s).$$

→ Relation between Laplace Transform and Fourier Transform:

Soln:

$$LT[x(t)] = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

Put, $s = \sigma + j\Omega$

$$= \int_{-\infty}^{\infty} x(t)\, e^{-(\sigma + j\Omega)t}\, dt$$

$$= e^{-\sigma t} \underbrace{\int_{-\infty}^{\infty} x(t) \cdot e^{-j\Omega t}\, dt}$$

$$LT[x(t)] = e^{-\sigma t}\; F.T[x(t)]$$

$$\boxed{X(s) = e^{-\sigma t}\; X(j\Omega)}$$

→ Region of convergence [ROC]

The range of values of 'σ' for which L.T[x(t)] converges is called Region of convergence [ROC].

Problems: Group I

1. Find the laplace transform of $e^{at} u(t)$ and plot the ROC.

Soln:

$$L.T[x(t)] = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$LT[e^{at} u(t)] = \int_{-\infty}^{\infty} e^{at} u(t). e^{-st} dt$$

$$u(t) = \begin{cases} 1 ; & t \geq 0 \\ 0 ; & t < 0 \end{cases}$$

$$= \int_{0}^{\infty} e^{-(s-a)t} dt$$

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)} \right]_{0}^{\infty}$$

$$= -\frac{1}{(s-a)} \left[e^{-(s-a).\infty} - e^{-(s-a).0} \right]$$

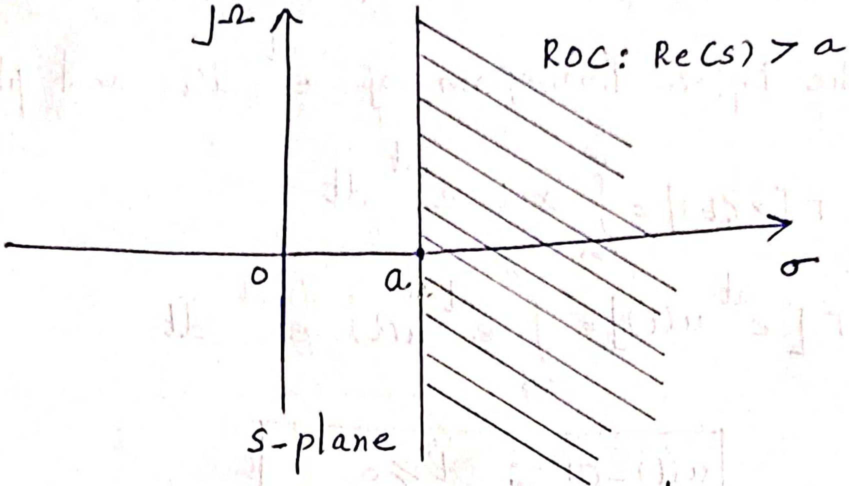
To converge e^{-x}, x should be +ve.

$\therefore$ $Re(s-a) > 0$

$Re(s) > a$

$$= -\frac{1}{(s-a)} \left[e^{-\infty} - e^{0} \right] = \frac{-1}{s-a} [0-1]$$

$$\boxed{LT[e^{at} u(t)] = \frac{1}{s-a} ; \ Roc : Re(s) > a}$$



2) Find the laplace transform of e^{-at} u(t) and plot ROC.

Soln:

$$LT[x(t)] = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

$$LT[e^{-at} u(t)] = \int_{-\infty}^{\infty} e^{-at}\, u(t) \cdot e^{-st}\, dt$$

$$\boxed{u(t) = \begin{cases} 1 \, ; & t \geq 0 \\ 0 \, ; & t < 0. \end{cases}}$$

$$= \int_{0}^{\infty} e^{-at}\, e^{-st}\, dt$$

$$= \int_{0}^{\infty} e^{-(s+a)t}\, dt$$

$$= \left[\frac{e^{-(s+a)t}}{-(s+a)} \right]_0^{\infty} = -\frac{1}{(s+a)} \left[e^{-(s+a)\infty} - e^{-(s+a)\cdot 0} \right]$$

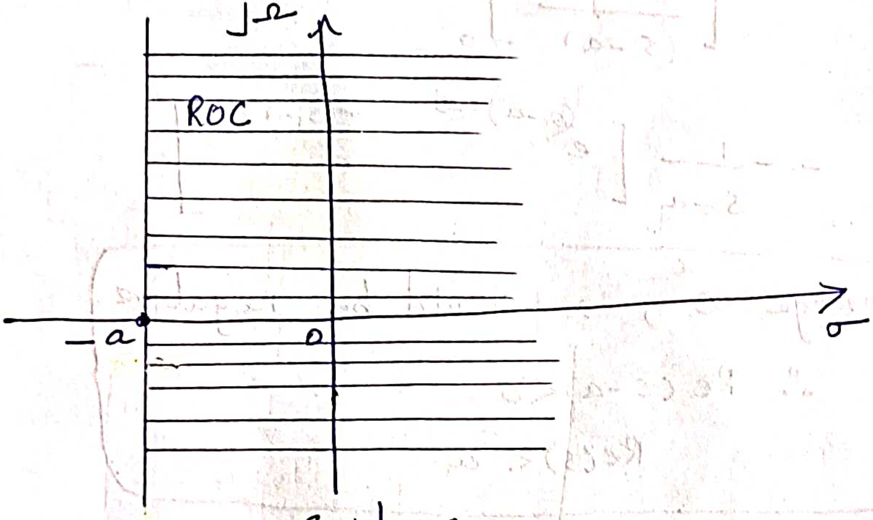
To converge e^{-x}, x should be positive.

$$\therefore \ Re(s+a) > 0$$

$$Re(s) > -a$$

$$= -\frac{1}{(s+a)}\left[e^{-\infty} - e^{0}\right] = \frac{-1}{(s+a)}[0-1].$$

$$L.T[e^{-at}u(t)] = \frac{1}{s+a} \quad ; \quad Roc: Re(s) > -a$$



s-plane

3. Laplace Transform of $x(t) = -e^{at}u(-t)$.

Soln:

$$LT[x(t)] = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

$$LT[-e^{at}u(-t)] = \int_{-\infty}^{\infty} -e^{at}\, u(-t)\, e^{-st}\, dt$$

$$u(t) = \begin{cases} 1 \ ; & t \geq 0 \\ 0 \ ; & t < 0 \end{cases}$$

$$u(-t) = \begin{cases} 1 \ ; & t \leq 0 \\ 0 \ ; & t > 0 \end{cases}$$

$$= -\int_{-\infty}^{0} e^{at} e^{-st} dt$$

$$= -\int_{0}^{\infty} e^{-at} e^{st} dt$$

$$= -\int_{0}^{\infty} e^{(s-a)t} dt$$

$$= -\left[\frac{e^{(s-a)t}}{(s-a)}\right]_0^{\infty}$$

$$= -\frac{1}{s-a}\left[e^{(s-a)\cdot\infty} - e^{(s-a)\cdot 0}\right].$$

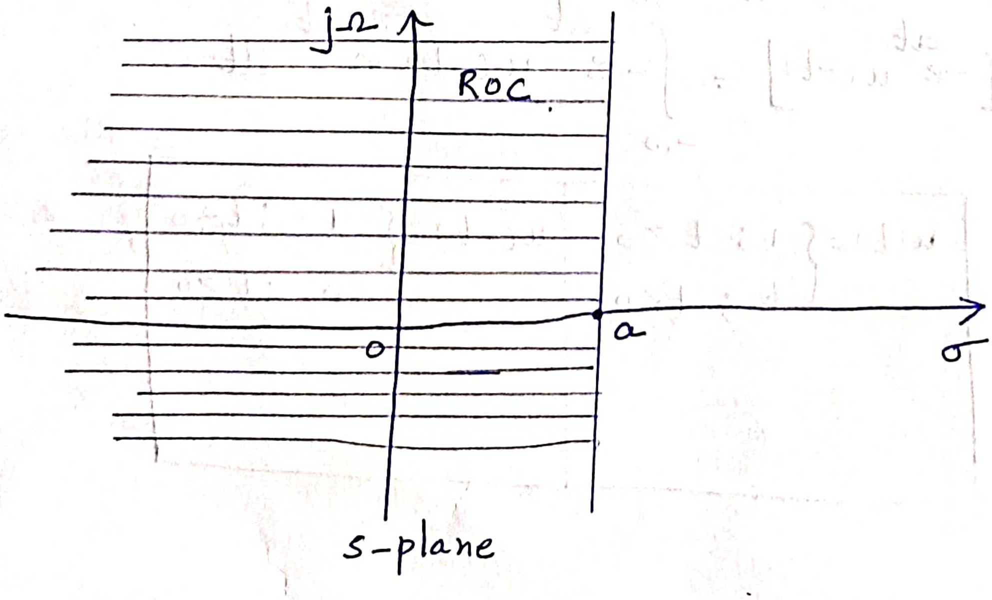
To converge e^{x}, x should be negative.

$\therefore$ $Re(s-a) < 0$

$Re(s) < a$

$$= -\frac{1}{s-a}\left[e^{-\infty} - e^{0}\right] = \frac{-1}{s-a}[0-1]$$

$$L.T\left[e^{at} u(-t)\right] = \frac{1}{s-a} \quad ; \quad ROC : Re(s) < a$$



4) Find the laplace transform of $x(t) = -e^{-at}u(-t)$ and plot Roc.

Soln:

$$LT[x(t)] = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt$$

$$LT[-e^{-at}u(-t)] = \int_{-\infty}^{\infty} -e^{-at}u(-t)\,e^{-st}\,dt.$$

$u(t) = \begin{cases} 1 ; & t \geq 0 \\ 0 ; & t < 0 \end{cases}$	$u(-t) = \begin{cases} 1 ; & t \leq 0 \\ 0 ; & t > 0 \end{cases}$

$$= -\int_{-\infty}^{0} e^{-at}e^{-st}\,dt$$

$$= -\int_{0}^{\infty} e^{at}e^{st}\,dt = -\int_{0}^{\infty} e^{(s+a)t}\,dt$$

$$= -\left[\frac{e^{(s+a)t}}{(s+a)}\right]_0^{\infty} = -\frac{1}{(s+a)}\left[e^{(s+a)\infty} - e^{(s+a)\cdot 0}\right]$$

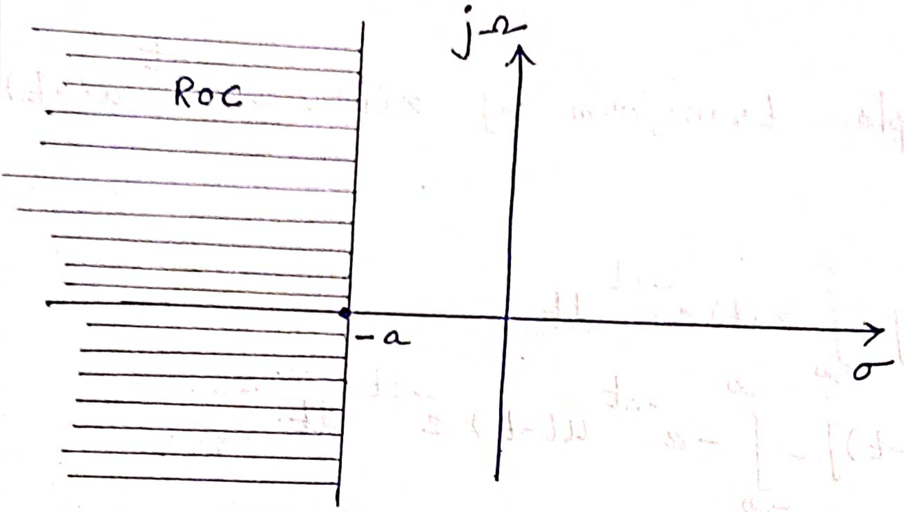
To converge e^{x}, x should be negative

$\therefore$ $Re(s+a) < 0$

$Re(s) < -a$

$$= -\frac{1}{(s+a)}\left[e^{-\infty} - e^{0}\right] = -\frac{1}{(s+a)}[0-1].$$

$$\boxed{L.T[-e^{-at}u(-t)] = \frac{1}{s+a} \; ; \quad Roc : Re(s) < -a}$$



S-plane

5. Laplace Transform of $x(t) = u(t)$. and plot Roc.

Soln:

$$L.T[x(t)] = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

$$LT[u(t)] = \int_{-\infty}^{\infty} u(t)\, e^{-st}\, dt$$

$$\boxed{u(t) = \begin{cases} 1 ; & t \geq 0 \\ 0 ; & t < 0 \end{cases}}$$

$$= \int_{0}^{\infty} e^{-st}\, dt$$

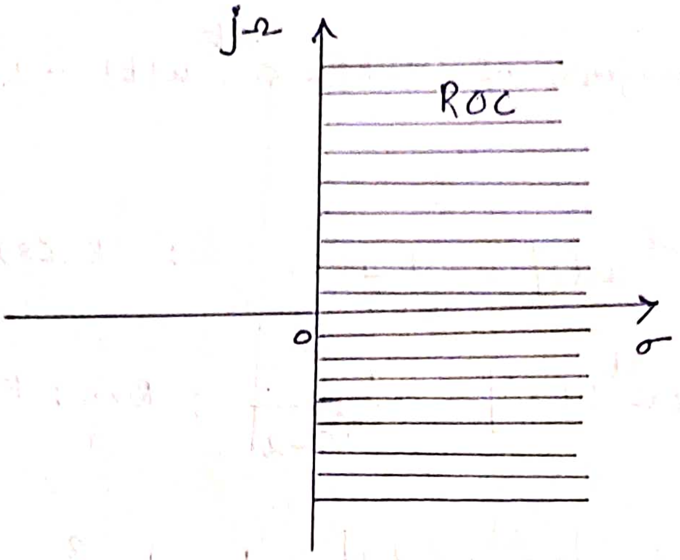
$$= \left[\frac{e^{-st}}{-s}\right]_0^{\infty} = -\frac{1}{s}\left[e^{-s.\infty} - e^{-s.0}\right]$$

To converge e^{-x}, x should be +ve

$\therefore$ Re(S) > 0.

$$= -\frac{1}{s}\left[e^{-\infty} - e^{0}\right] = -\frac{1}{s}[0-1]$$

$$\boxed{L.T[u(t)] = \frac{1}{s} \;; \quad \text{Roc : } Re(s) > 0}$$



s-plane

6. Find the laplace transform of $\delta(t)$ and plot ROC.

Soln:

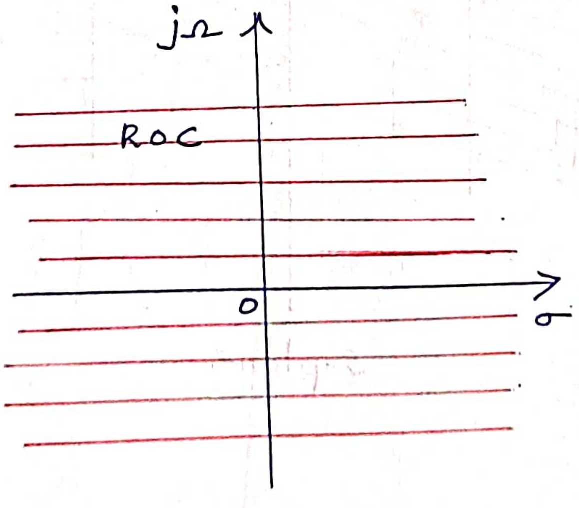
$$L.T[x(t)] = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

$$LT[\delta(t)] = \int_{-\infty}^{\infty} \delta(t)\, e^{-st}\, dt$$

$$\boxed{\delta(t) = \begin{cases} 1 & ; t = 0 \\ 0 ; & t \neq 0 \end{cases}}$$

$$= \delta(t)\, e^{-st} \Big|_{t=0} = \delta(0) .\ e^{0} = 1 \cdot 1 = 1 .$$

$L.T[\delta(t)] = 1$; ROC: Entire s-plane.



s-plane

7. Laplace Transform of $x(t) = e^{-2t} u(t) - 2e^{2t} u(-t)$ and plot RoC.

Soln:

$$L.T[e^{-2t} u(t)] = \frac{1}{s+2} \quad ; \text{ Roc: } \quad Re(s) > -2$$

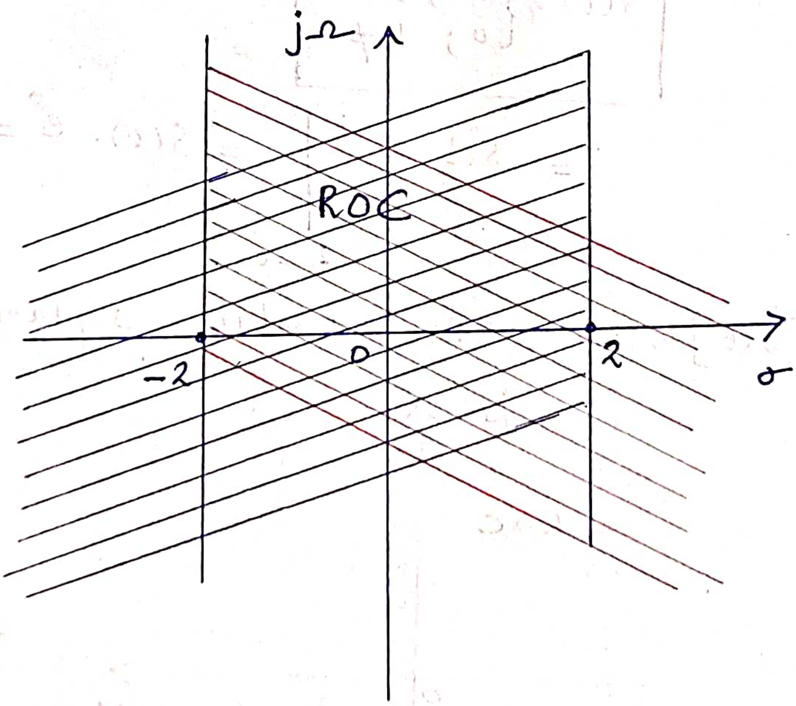
$$LT\left[-2e^{2t} u(-t)\right] = 2 \times \frac{1}{S-2} \quad ; \text{ Roc: } Re(s) < 2$$

$$LT\left[e^{-2t} u(t) - 2e^{2t} u(-t)\right] = \frac{1}{s+2} + \frac{2}{s-2} \quad ; \text{ Roc:} -2 < Re(s) < 2$$

$$= \frac{s-2+2(s+2)}{(s+2)(s-2)} \quad ; \text{ Roc:} -2 < Re(s) < 2$$

$$= \frac{s-2+2s+4}{s^2-2^2} \quad ; \text{ Roc: } -2 < Re(s) < 2$$

$$\boxed{LT[e^{-2t} u(t) - 2e^{2t} u(-t)] = \frac{3s+2}{s^2-4} \quad ; \text{ Roc : } -2 < Re(s) < 2}$$



s-plane

8. Find the laplace transform of $x(t) = 3e^{-2t}u(t) - 2e^{3t}u(-t)$ and plot ROC.

Soln:

$LT[3e^{-2t}u(t)] = 3 \times \frac{1}{s+2}$; ROC : $Re(s) > -2$

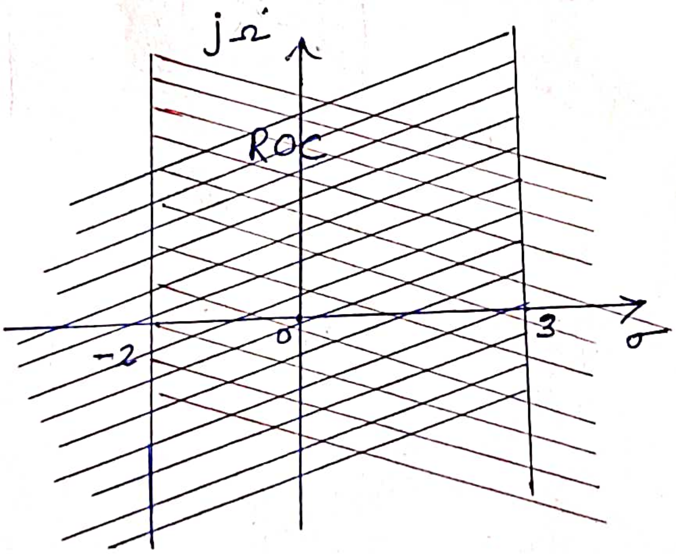
$LT[-2e^{3t}u(-t)] = 2 \times \frac{1}{s-3}$; ROC : $Re(s) < 3$

$LT[3e^{-2t}u(t) - 2e^{3t}u(-t)] = \frac{3}{s+2} + \frac{2}{s-3}$; ROC: $-2 < Re(s) < 3$

$= \frac{3(s-3)+2(s+2)}{(s+2)(s-3)}$; ROC : $-2 < Re(s) < 3$

$= \frac{3s-9+2s+4}{s^2-3s+2s-6}$; ROC : $-2 < Re(s) < 3$

$$\boxed{LT[3e^{-2t}u(t) - 2e^{3t}u(-t)] = \frac{5s-5}{s^2-s-6} \; ; \; ROC : -2 < Re(s) < 3}$$



s-plane

9. Find the laplace transform of $5e^{-4t}u(t)+3e^{-t}u(-t)$ and plot ROC.

soln:

$$LT\left[5e^{-4t}u(t)\right] = 5 \times \frac{1}{s+4} \quad ; \text{ Roc : } \quad Re(s) > -4$$

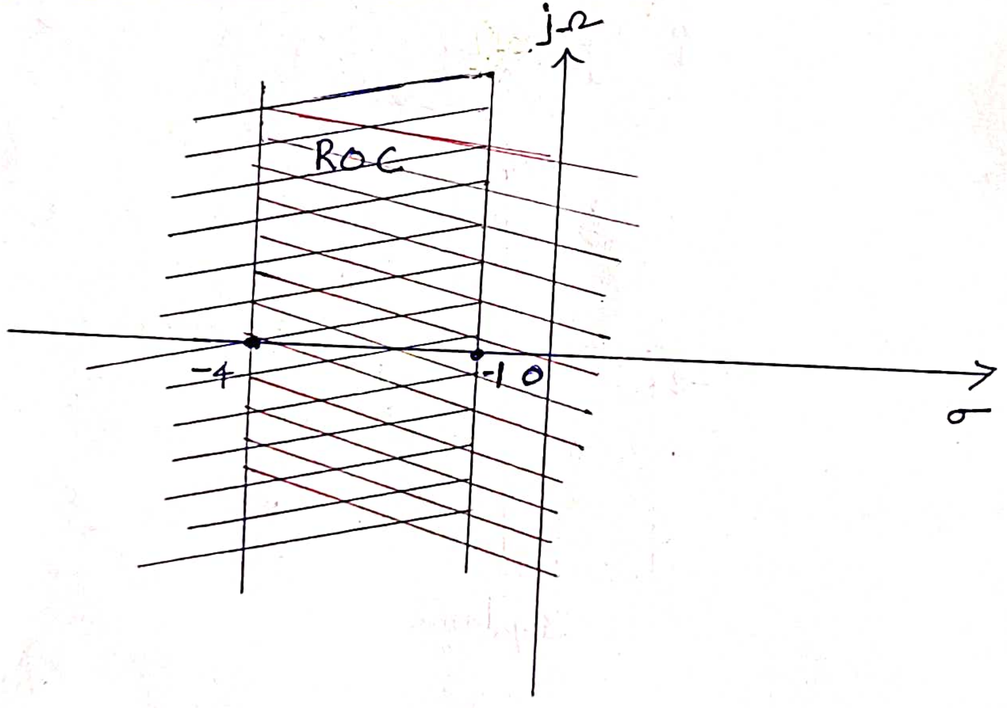
$$L.T\left[3e^{-t}u(-t)\right] = -3 \times \frac{1}{s+1} \quad ; \text{ Roc : } \quad Re(s) < -1$$

$$L.T\left[5e^{-4t}u(t)+3e^{-t}u(-t)\right] = \frac{5}{s+4} - \frac{3}{s+1} \; ; \text{ Roc : } -4 < Re(s) < -1$$

$$= \frac{5(s+1)-3(s+4)}{(s+4)(s+1)} \; ; \text{ Roc : } -4 < Re(s) < -1$$

$$= \frac{5s+5-3s-12}{s^2+s+4s+4} \; ; \text{ Roc : } -4 < Re(s) < -1$$

$$\boxed{LT\left[5e^{-4t}u(t)+3e^{-t}u(-t)\right] = \frac{2s-7}{s^2+5s+4} \; ; \text{ Roc : } -4 < Re(s) < -1}$$



s-plane

Group II

Problems based on Laplace Transformation:

1) Find the Laplace Transform of $A\cos\omega_0 t\, u(t)$.

Soln:

$$LT[x(t)] = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

$$LT[A\cos\omega_0 t\, u(t)] = \int_{-\infty}^{\infty} A\cos\omega_0 t\, u(t)\, e^{-st}\, dt$$

$$\boxed{u(t) = \begin{cases} 1 ; & t \geq 0 \\ 0 ; & t < 0 \end{cases}}$$

$$= A\int_{0}^{\infty} \cos\omega_0 t\; e^{-st}\, dt$$

$$= A\int_{0}^{\infty} \left[\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}\right] e^{-st}\, dt$$

$$= \frac{A}{2}\left[\int_{0}^{\infty} e^{j\omega_0 t}\, e^{-st}\, dt + \int_{0}^{\infty} e^{-j\omega_0 t}\, e^{-st}\, dt\right]$$

$$= \frac{A}{2}\left[\int_{0}^{\infty} e^{-(s-j\omega_0)t}\, dt + \int_{0}^{\infty} e^{-(s+j\omega_0)t}\, dt\right]$$

$$= \frac{A}{2}\left[\left(\frac{e^{-(s-j\omega_0)t}}{-(s-j\omega_0)}\right)_{0}^{\infty} + \left(\frac{e^{-(s+j\omega_0)t}}{-(s+j\omega_0)}\right)_{0}^{\infty}\right]$$

$$= -\frac{A}{2}\left[\frac{e^{-(s-j\omega_0)\infty} - e^{-(s-j\omega_0)\cdot 0}}{s-j\omega_0} + \frac{e^{-(s+j\omega_0)\infty} - e^{-(s+j\omega_0)\cdot 0}}{s+j\omega_0}\right]$$

$$= -\frac{A}{2}\left[\frac{e^{-\infty} - e^{0}}{s-j\omega_0} + \frac{e^{-\infty} - e^{0}}{s+j\omega_0}\right]$$

$$= -\frac{A}{2}\left[\frac{0-1}{s-j\omega_0} + \frac{0-1}{s+j\omega_0}\right] = -\frac{A}{2}\left[\frac{(-1)}{s-j\omega_0} + \frac{(-1)}{s+j\omega_0}\right]$$

$$= \frac{A}{2}\left[\frac{1}{s-j\omega_0} + \frac{1}{s+j\omega_0}\right] = \frac{A}{2}\left[\frac{s+j\omega_0+s-j\omega_0}{(s-j\omega_0)(s+j\omega_0)}\right]$$

$$= \frac{A}{2}\left[\frac{2s}{s^2-(j\omega_0)^2}\right] = A\cdot\frac{s}{s^2+\omega_0^2}$$

$$\boxed{LT[A \cos\omega_0 t \; u(t)] = A.\frac{s}{s^2+\omega_0^2}}$$

Note: $LT[5\cos 2t \, u(t)] = 5\times \frac{s}{s^2+4.} = \frac{5s}{s^2+4.}$

2) Find the Laplace Transform of $A \sin\omega_0 t \, u(t)$.

soln:

$$LT[x(t)] = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$LT[A \sin\omega_0 t \cdot u(t)] = \int_{-\infty}^{\infty} A \sin\omega_0 t \; u(t) \cdot e^{-st} dt$$

$$\boxed{u(t) = \begin{cases} 1 ; & t \geq 0 \\ 0 ; & t < 0 \end{cases}}$$

$$= A\int_0^{\infty} \sin\omega_0 t \; e^{-st} dt$$

$$= A \int_0^{\infty} \left[\frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j} \right] e^{-st} dt$$

$$= \frac{A}{2j}\left[\int_0^{\infty} e^{j\omega_0 t} \cdot e^{-st} dt - \int_0^{\infty} e^{-j\omega_0 t} e^{-st} dt \right]$$

$$= \frac{A}{2j}\left[\int_0^{\infty} e^{-(s-j\omega_0)t} dt - \int_0^{\infty} e^{-(s+j\omega_0)t} dt \right]$$

$$= \frac{A}{2j}\left[\left(\frac{e^{-(s-j\omega_0)t}}{-(s-j\omega_0)} \right)_0^{\infty} - \left(\frac{e^{-(s+j\omega_0)t}}{-(s+j\omega_0)t} \right)_0^{\infty} \right]$$

$$= \frac{-A}{2j}\left[\frac{e^{-\infty} - e^{0}}{s-j\omega_0} - \left(\frac{e^{-\infty} - e^{0}}{s+j\omega_0} \right) \right] = \frac{-A}{2j}\left[\frac{0-1}{s-j\omega_0} - \left(\frac{0-1}{s+j\omega_0} \right) \right]$$

$$= \frac{-A}{2j}\left[\frac{-1}{s-j\omega_0} + \frac{1}{s+j\omega_0} \right] = \frac{-A}{2j}\left[\frac{-s-j\omega_0 + s - j\omega_0}{(s-j\omega_0)(s+j\omega_0)} \right]$$

$$= \frac{-A}{2j} \times \left[\frac{-2j\omega_0}{s^2-(j\omega_0)^2} \right] = \frac{A\omega_0}{s^2+\omega_0^2}$$

$$\boxed{LT[A \sin\omega_0 t \, u(t)] = A \cdot \frac{\omega_0}{s^2+\omega_0^2}}$$

Note: $LT[7 \sin 5t \, u(t)] = 7 \cdot \frac{5}{s^2+25} = \frac{35}{s^2+25}$.

3) Find the Laplace Transform of $e^{-at} \cos\omega_0 t \, u(t)$.

Soln:

$$L \cdot T[\cos\omega_0 t \, u(t)] = \frac{s}{s^2+\omega_0^2}$$

Let $x(t) = \cos\omega_0 t \, u(t)$.

So, $X(s) = \frac{s}{s^2+\omega_0^2}. \longrightarrow ①$.

W.K.T,

Frequency shifting Property of LT,

$$L \cdot T[e^{-at} x(t)] = X(s+a).$$

$$\therefore LT[e^{-at} \cos\omega_0 t \, u(t)] = \frac{s+a}{(s+a)^2+\omega_0^2}. \quad [\text{using} \cdot \text{eqn} ①]$$

4) Find the laplace transform of $e^{-at} \sin\omega_0 t \, u(t)$.

Soln:

Let $x(t) = \sin\omega_0 t \, u(t)$.

$$L \cdot T[x(t)] = X(s) = LT[\sin\omega_0 t \, u(t)]$$

$$X(s) = \frac{\omega_0}{s^2+\omega_0^2} \longrightarrow ①$$

W.K.T,

Frequency shifting property of L.T,

$$L \cdot T[e^{-at} x(t)] = X(s+a)$$

$$LT[e^{-at} \sin\omega_0 t \, u(t)] = \frac{\omega_0}{(s+a)^2+\omega_0^2} \quad [\text{using eqn.} ①]$$

5) Find the laplace transform of $e^{at} \sin\omega_0 t\, u(t)$.

Soln:

Let, $x(t) = \sin\omega_0 t\, u(t)$.

$$L\cdot T[x(t)] = LT[\sin\omega_0 t\, u(t)]$$

$$X(s) = \frac{\omega_0}{s^2+\omega_0^2} \longrightarrow ①$$

W.K.T,

Frequency shifting property of L.T,

$$L.T[e^{at}\, x(t)] = X(s-a).$$

$$LT[e^{at} \sin\omega_0 t\, u(t)] = \frac{\omega_0}{(s-a)^2+\omega_0^2} \quad [\because \text{using eqn } ①]$$

6). Find the laplace transform of $e^{at} \cos\omega_0 t\, u(t)$.

Soln:

Let, $x(t) = \cos\omega_0 t\, u(t)$.

$$LT[x(t)] = LT[\cos\omega_0 t\, u(t)]$$

$$X(s) = \frac{s}{s^2+\omega_0^2} \longrightarrow ①$$

W.K.T,

Frequency shifting Property of L.T,

$$L.T[e^{at}\, x(t)] = X(s-a).$$

$$LT[e^{at} \cos\omega_0 t\, u(t)] = \frac{s-a}{(s-a)^2+\omega_0^2} \quad [\because \text{using eqn. } ①]$$

Group III ·

Problems based on Laplace Transform:

1) Find the Laplace Transform of coshbt u(t).

Soln:

$$L\cdot T[x(t)] = \int_{-\infty}^{\infty} x(t)\cdot e^{-st} \, dt$$

$$L\cdot T[\cosh bt\, u(t)] = \int_{-\infty}^{\infty} \cosh bt\, u(t)\, e^{-st} \, dt$$

$$\boxed{u(t) = \begin{cases} 1 ; & t \geq 0 \\ 0 ; & t < 0 \end{cases}}$$

$$= \int_{0}^{\infty} \cosh bt\, e^{-st} \, dt$$

$$= \int_{0}^{\infty} \left[\frac{e^{bt} + e^{-bt}}{2} \right] e^{-st} \, dt$$

$$= \frac{1}{2}\left[\int_{0}^{\infty} e^{bt} e^{-st} dt + \int_{0}^{\infty} e^{-bt} e^{-st} dt \right]$$

$$= \frac{1}{2}\left[\int_{0}^{\infty} e^{-(s-b)t} dt + \int_{0}^{\infty} e^{-(s+b)t} dt \right]$$

$$= \frac{1}{2}\left[\left(\frac{e^{-(s-b)t}}{-(s-b)} \right)_{0}^{\infty} + \left(\frac{e^{-(s+b)t}}{-(s+b)} \right)_{0}^{\infty} \right]$$

$$= -\frac{1}{2}\left\{ \left[\frac{e^{-\infty} - e^{0}}{(s-b)} \right] + \left[\frac{e^{-\infty} - e^{0}}{(s+b)} \right] \right\}$$

$$= -\frac{1}{2}\left\{ \left[\frac{0-1}{(s-b)} + \frac{0-1}{(s+b)} \right] \right\} = \frac{1}{2}\left[\frac{1}{s-b} + \frac{1}{s+b} \right]$$

$$= \frac{1}{2}\left[\frac{s+b+s-b}{(s-b)(s+b)} \right] = \frac{1}{2}\left[\frac{2s}{s^2-b^2} \right]$$

$$\boxed{LT[\cosh bt\, u(t)] = \frac{s}{s^2-b^2}}$$

2) Find the Laplace Transform of sinhbt u(t).

Soln:

$$L.T[x(t)] = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$L.T[\sinh bt\ u(t)] = \int_{-\infty}^{\infty} \sinh bt\ u(t)\ e^{-st}\ dt$$

$$\boxed{u(t) = \begin{cases} 1 ; & t \geq 0 \\ 0 ; & t < 0 \end{cases}}$$

$$= \int_{0}^{\infty} \sinh bt\ e^{-st}\ dt$$

$$= \int_{0}^{\infty} \left[\frac{e^{bt} - e^{-bt}}{2}\right] e^{-st}\ dt$$

$$= \frac{1}{2}\left[\int_{0}^{\infty} e^{bt} \cdot e^{-st}\ dt - \int_{0}^{\infty} e^{-bt} \cdot e^{-st}\ dt\right]$$

$$= \frac{1}{2}\left[\int_{0}^{\infty} e^{-(s-b)t}\ dt - \int_{0}^{\infty} e^{-(s+b)t}\ dt\right]$$

$$= \frac{1}{2}\left\{\left[\frac{e^{-(s-b)t}}{-(s-b)}\right]_{0}^{\infty} - \left[\frac{e^{-(s+b)t}}{-(s+b)}\right]_{0}^{\infty}\right\}$$

$$= -\frac{1}{2}\left\{\frac{e^{-\infty} - e^{0}}{(s-b)} - \frac{(e^{-\infty} - e^{0})}{(s+b)}\right\}$$

$$= -\frac{1}{2}\left\{\frac{0-1}{(s-b)} - \frac{(0-1)}{(s+b)}\right\} = -\frac{1}{2}\left\{\frac{-1}{(s-b)} + \frac{1}{s+b}\right\}$$

$$= -\frac{1}{2}\left\{\frac{-s-b+s-b}{(s-b)(s+b)}\right\} = -\frac{1}{2}\left[\frac{-2b}{s^2-b^2}\right]$$

$$\boxed{L.T[\sinh bt\ u(t)] = \frac{b}{s^2-b^2}}$$

3). Find the laplace transform of e^{-at} coshbt u(t).

Soln:

Let, $x(t) = \cosh bt\, u(t)$

$$L.T[x(t)] = L.T[\cosh bt\, u(t)]$$

$$X(s) = \frac{s}{s^2-b^2} \longrightarrow ①$$

W.K.T,

Frequency shifting property of L.T,

$$L.T[e^{-at} x(t)] = X(s+a).$$

So,

$$LT[e^{-at} \cosh bt\, u(t)] = \frac{s+a}{(s+a)^2-b^2} \quad [\because \text{using eqn } ①]$$

4) Find the laplace transform of e^{-at} sinhbt u(t).

soln:

Let, $x(t) = \sinh bt\, u(t)$

$$L.T[x(t)] = L.T[\sinh bt\, u(t)]$$

$$X(s) = \frac{b}{s^2-b^2} \longrightarrow ①.$$

W.K.T,

Frequency shifting property of L.T,

$$L.T[e^{-at} x(t)] = X(s+a)$$

So,

$$LT[e^{-at} \sinh bt\, u(t)] = \frac{b}{(s+a)^2-b^2} \quad [\because \text{using eqn } ①].$$

5) Find the laplace transform of e^{at} coshbt u(t).

Soln:

Let, $x(t) = \cosh bt\, u(t)$.

$$L.T[x(t)] = L.T[\cosh bt\, u(t)]$$

$$X(s) = \frac{s}{s^2 - b^2} \longrightarrow ①$$

W.k.T,

Frequency shifting property of L.T,

$$L.T[e^{at}\, x(t)] = X(s-a).$$

So,

$$L.T[e^{at} \cosh bt\, u(t)] = \frac{s-a}{(s-a)^2 - b^2} \quad [\because \text{using eqn } ①]$$

6) Find the laplace transform of e^{at} sinhbt u(t).

Soln:

Let, $x(t) = \sinh bt\, u(t)$

$$L.T[x(t)] = LT[\sinh bt\, u(t)]$$

$$X(s) = \frac{b}{s^2 - b^2}$$

W.k.T,

Frequency shifting property of L.T,

$$L.T[e^{at}\, x(t)] = X(s-a).$$

So,

$$L.T[e^{at} \sinh bt\, u(t)] = \frac{b}{(s-a)^2 - b^2} \quad [\because \text{using eqn } ①].$$

Group IV

Problems based on Laplace Transform.

* Note:

i) Differentiation in frequency domain property of L.T.

$$L.T\left[-t\,x(t)\right] = \frac{d}{ds}X(s)$$

$$L.T\left[(-1)^n t^n\,x(t)\right] = \frac{d^n}{ds^n}X(s).$$

ii) $L.T\left[t^{n-1}\,u(t)\right] = \frac{(n-1)!}{s^n}$ $\left[\,! \rightarrow \text{factorial}.\; L.T\left[u(t)\right] = \frac{1}{s}\right].$

iii) $LT\left[t^{n-1}\,e^{-at}\,u(t)\right] = \frac{(n-1)!}{(s+a)^n}$

U.Q

Problems:

1) Find the LT of $(t-2)^2\,u(t-2)$.

Soln:

W.k.T, $LT\left[t^{n-1}\,u(t)\right] = \frac{(n-1)!}{s^n}$

$$LT\left[t^2\,u(t)\right] = \frac{(3-1)!}{s^3} = \frac{2!}{s^3}\left[\because n=3\right] \quad n-1=2$$

$$LT\left[t^2 u(t)\right] = \frac{2}{s^3}. \rightarrow ①$$

W.k.T,

Time shifting property of L.T,

$$L.T\left[x(t-t_0)\right] = e^{-st_0}\,X(s)$$

So,

$$L.T\left[(t-2)^2\,u(t-2)\right] = e^{-2s} \cdot \frac{2}{s^3}. \quad \left[\because \text{using eqn } ①\right]$$

U.Q

2) Find the L.T of $(t-4)^3 u(t-4)$.

Soln:

Diff. prop. of L.T in frequency domain.

$$L.T[t^{n-1} u(t)] = \frac{(n-1)!}{s^n}.$$

$$LT[t^3 \cdot u(t)] = \frac{(4-1)!}{s^4} \quad \left[\because n-1 = 3 \atop n = 4 \right.$$

$$= \frac{3!}{s^4} \quad [3! = 3\times2\times1. = 6]$$

$$L.T[t^3 u(t)] = \frac{6}{s^4} \longrightarrow ①.$$

W.k.T,

Time shifting property of Laplace Transform,

$$L.T[x(t-t_0)] = e^{-st_0} X(s).$$

So,

$$L.T[(t-4)^3 u(t-4)] = e^{-4s} * \frac{6}{s^4}.$$

U.Q

3) Laplace Transform of $u(t)+u(t-2)$.

Soln:

$$LT[u(t)+u(t-2)] = LT[u(t)] + LT[u(t-2)]$$

$$= \frac{1}{s} + e^{-2s} \cdot \frac{1}{s} \quad \left[\because \text{using time shifting prop.} \atop L.T[x(t-t_0)] = e^{-st_0} X(s) \right]$$

U.Q

4) Laplace Transform of $\delta(t) + 2\delta(t-3)$.

Soln:

$$LT[\delta(t) + 2\delta(t-3)] = L.T[\delta(t)] + 2\, LT[\delta(t-3)]$$

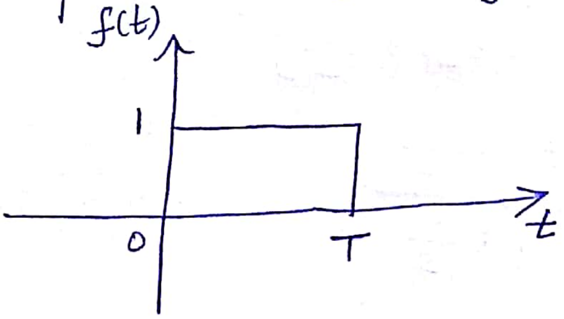
$$= 1 + 2 \cdot e^{-3s} \cdot 1$$

$$= 1 + 2e^{-3s}.$$

$$\left[\because \text{using time shifting prop.} \atop LT[x(t-t_0)] = e^{-st_0} X(s) \right].$$

V.Q

5) Find the Laplace Transform of the given signal.



Soln:

$$L.T\left[x(t)\right] = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

$$L.T\left[f(t)\right] = \int_{-\infty}^{\infty} f(t)\, e^{-st}\, dt$$

$$= \int_{0}^{T} 1.e^{-st}\, dt = \int_{0}^{T} e^{-st}\, dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_{0}^{T} = -\frac{1}{s}\left[e^{-sT} - e^{0}\right] = -\frac{1}{s}\left[e^{-sT} - 1\right]$$

$$\boxed{F(s) = \frac{1}{s}\left[1 - e^{-sT}\right]}$$

GATE 2016

6) Find the initial value and final value of the following function.

$$X(s) = \frac{5s+50}{s(s+5)}$$

Soln:

Initial value : $x(0) = \underset{s\to\infty}{Lt}\ s\, X(s)$

$$= \underset{s\to\infty}{Lt}\ s \times \frac{(5s+50)}{s(s+5)}$$

$$= \underset{s\to\infty}{Lt}\ \frac{5s+50}{s+5}$$

$$x(0) = \underset{s \to \infty}{Lt} \frac{s\left[5 + \frac{50}{s}\right]}{s\left[1 + \frac{5}{s}\right]}$$

$$= \underset{s \to \infty}{Lt} \frac{5 + \frac{50}{s}}{1 + \frac{5}{s}}$$

$$= \frac{5 + \frac{50}{\infty}}{1 + \frac{5}{\infty}}$$

$$= \frac{5+0}{1+0} = \frac{5}{1} \qquad \left[\because \frac{50}{\infty} = 0 \; , \frac{5}{\infty} = 0\right]$$

$$\boxed{x(0) = 5}$$

Final value : $x(\infty) = \underset{s \to 0}{Lt} \; s\,X(s)$

$$= \underset{s \to 0}{Lt} \; s \times \frac{(5s+50)}{s(s+5)}$$

$$= \underset{s \to 0}{Lt} \frac{5s+50}{s+5}$$

$$= \frac{5(0)+50}{0+5} = \frac{50}{5}$$

$$\boxed{x(\infty) = 10}$$

Unit III Linear Time Invariant Continuous Time Systems

→ Impulse Response

It is the response of the system to unit impulse signal $\delta(t)$. It is denoted by $h(t)$ and can be determined by taking inverse Laplace transform of the system transfer function.

* System transfer function :

$$H(s) = \frac{L.T[o/p \text{ of a CT system}]}{LT[I/p \text{ of a CT system}]} = \frac{LT[y(t)]}{LT[x(t)]}$$

$$H(s) = \frac{Y(s)}{X(s)}.$$

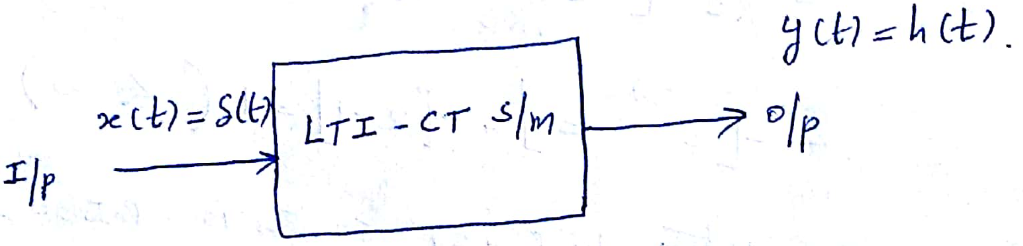
$$Inv.L.T[H(s)] = LT^{-1}[H(s)] = LT^{-1}\left[\frac{Y(s)}{X(s)}\right]$$

$$\boxed{h(t) = LT^{-1}\left[\frac{Y(s)}{X(s)}\right]}$$

↳ impulse response.

* Define a CT Linear Time Invariant system:

A continuous Time system which obeys superposition principal and Time invariant property is called Linear Time Invariant Continuous Time system.



U.Q *** Causality condition.**

A LTI - CT system is said to be causal if it satisfies the following condition:

$$\boxed{h(t) = 0 \quad ; \quad t < 0}$$

U.Q: **stability condition**

A LTI-CT system is said to be stable if it satisfies the following condition:

$$\boxed{\int_{-\infty}^{\infty} |h(t)|\, dt < \infty}$$

→ **Problems based on causality and stability.**

1. Verify whether the following system is BIBO stable or not.

$$h(t) = \begin{cases} \frac{1}{RC} e^{-t/RC} & ; t \geq 0 \\ 0 & ; t < 0. \end{cases}$$

Soln:

Condition for stability is $\int_{-\infty}^{\infty} |h(t)| dt < \infty$.

So,

$$\int_{-\infty}^{\infty} |h(t)|\, dt = \int_{0}^{\infty} \left|\frac{1}{RC} e^{-t/RC}\right| dt$$

$$= \frac{1}{RC}\left[\frac{e^{-t/RC}}{-1/RC}\right]_0^{\infty} = -\left[e^{-\frac{\infty}{RC}} - e^{0/RC}\right]$$

$$= -\left[e^{-\infty} - e^{0}\right] = -[0-1] = 1 \; (< \infty)$$

∴ The given ~~system~~ LTI - CT system is BIBO stable.

2. Verify whether the following system is BIBO stable or not.

$$h(t) = \begin{cases} \frac{1}{\sqrt{LC}} \sin\left(\frac{-t}{\sqrt{LC}}\right) & ; \quad t \geq 0 \\ 0 & ; \quad t < 0. \end{cases}$$

Soln:

Condition for stability is $\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$

So,

$$\int_{-\infty}^{\infty} |h(t)|\, dt = \int_{0}^{\infty} \left| \frac{1}{\sqrt{LC}} \sin\left(\frac{-t}{\sqrt{LC}}\right) \right| dt$$

$$= \frac{1}{\sqrt{LC}} \int_{0}^{\infty} \left| - \sin\left(\frac{t}{\sqrt{LC}}\right) \right| dt$$

$$= \frac{1}{\sqrt{LC}} \int_{0}^{\infty} \sin\left(\frac{t}{\sqrt{LC}}\right) dt = \frac{1}{\sqrt{LC}} \left[\frac{-\cos\left(\frac{t}{\sqrt{LC}}\right)}{\frac{1}{\sqrt{LC}}} \right]_{0}^{\infty}$$

$$= -\left[\cos(\infty) - \cos 0 \right] = -\left[0 - 1 \right] = 1 \ (< \infty).$$

∴ The given LTI-CT system is BIBO stable.

3. Find whether the following system whose impulse response is given, is causal and stable: $h(t) = e^{-2t} u(t-1)$.

Soln:

condition for causality is : $h(t) = 0 \ ; \ t < 0$.

$$h(t) = e^{-2t} u(t-1)$$

when $t = -1$; $h(-1) = e^{-2(-1)} . u(-1-1) = e^{2} u(-2) = e^{2} \times 0 = 0$

when $t = -2$; $h(-2) = e^{-2(-2)} u(-2-1) = e^{4} u(-3) = e^{4} \times 0 = 0$

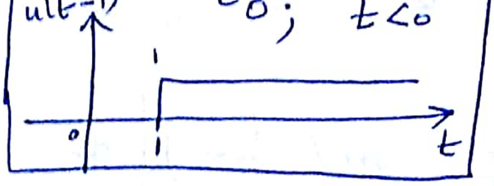
∴ The given system is causal.

For stable stable, $\int_{-\infty}^{\infty}|h(t)|\,dt < \infty$.

So,

$$\int_{-\infty}^{\infty}|h(t)|\,dt = \int_{-\infty}^{\infty}\left|e^{-2t}\,u(t-1)\right|dt .$$

$$u(t-1) = \begin{cases} 1 ; & t \geq 1 \\ 0 ; & t < 0 \end{cases}$$



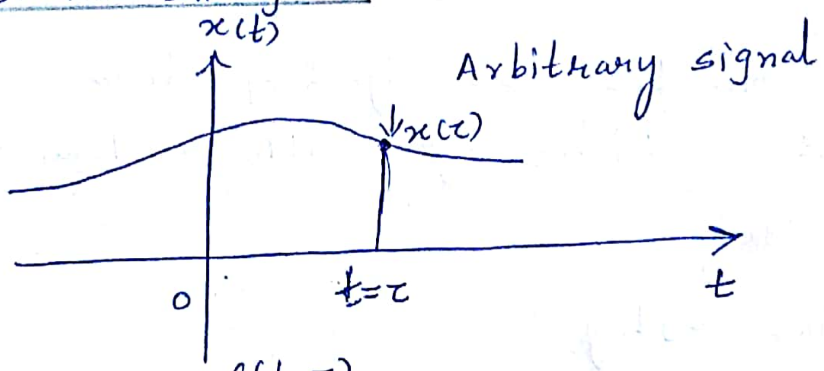
$$= \int_{1}^{\infty}\left|e^{-2t}\right|dt = \int_{1}^{\infty} e^{-2t}\,dt$$

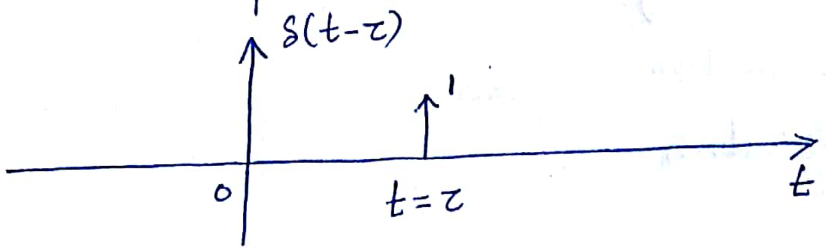
$$= \left[\frac{e^{-2t}}{-2}\right]_{1}^{\infty} = -\frac{1}{2}\left[e^{-\infty} - e^{-2}\right] = -\frac{1}{2}\left[0 - 0.14\right]$$

$$= 0.07 \;(< \infty)$$

$\therefore$ The given system is stable.

→ Convolution Integral :

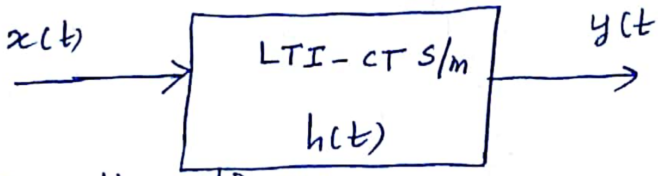




[∵ Impulse signal : $\delta(t)=\begin{cases}1 ; & t=0\\ 0 ; & t\neq 0\end{cases}$]

Now,

$$x(t)=\int_{-\infty}^{\infty} x(\tau)\cdot\delta(t-\tau)\,d\tau \longrightarrow ①$$



$y(t)=T[x(t)]=x(t)*h(t).$

From the diagram,

$$y(t)=T[x(t)]=T\left[\int_{-\infty}^{\infty} x(\tau)\cdot\delta(t-\tau)\,d\tau\right] \quad [\because \text{using eqn } ①]$$

$$=\int_{-\infty}^{\infty} x(\tau)\cdot T[\delta(t-\tau)]\,d\tau$$

$\delta(t)\rightarrow$ is shifted $[\delta(t-\tau)]$

$x(\tau)\rightarrow$ is kept stationary.

$T[\delta(t-\tau)]=h(t-\tau)$

$$y(t)=\int_{-\infty}^{\infty} x(\tau)\cdot h(t-\tau)\,d\tau$$

$$\boxed{y(t)= x(t)*h(t)=\int_{-\infty}^{\infty} x(\tau)\cdot h(t-\tau)\,d\tau}$$

↳ Convolution Integral formula.

Note:

1) What are the basic signal operations required to compute convolution by Continuous Time system?

a) Folding.

b) Time shifting

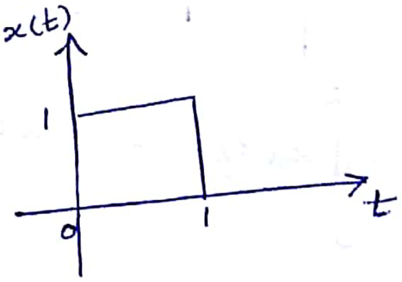
c) Multiplication

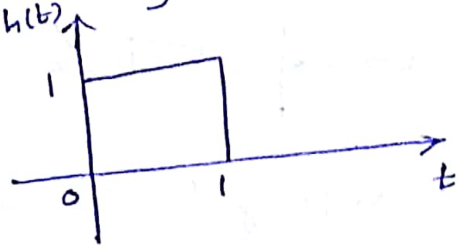
d) Integration.

→ Problems based on convolution

* GROUP I : convolution by graphical method:

1) Find the convolution of the given signals:

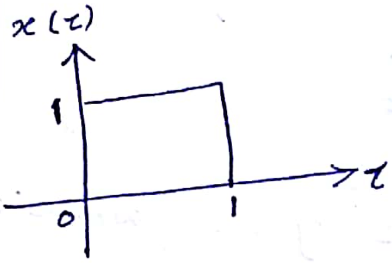


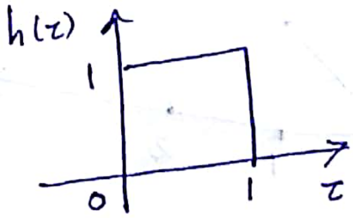


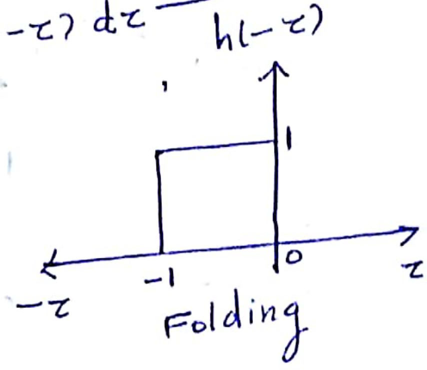
Soln:

convolution integral formula

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau \longrightarrow ①$$



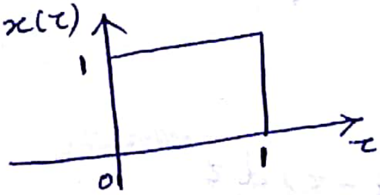


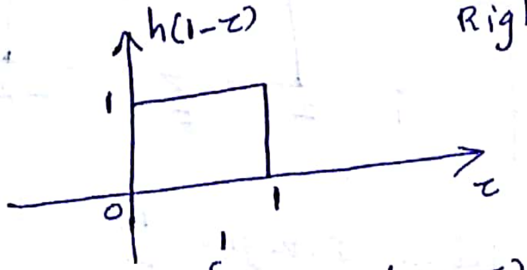


step 1 : $t=0$

①⇒ overlapping area: $y(t)\Big|_{t=0} = \int_{-1}^{1} x(\tau) \cdot h(-\tau)\, d\tau = 0.$

step 2 : $t=1$





Right shift by 1unit.

overlapping area : $y(t)\Big|_{t=1} = \int_0^1 x(\tau) .\, h(1-\tau)\, d\tau$

$$= \int_0^1 1 . 1 .\, d\tau = \int_0^1 d\tau = [\tau]_0^1$$

$$= [1-0] = 1$$

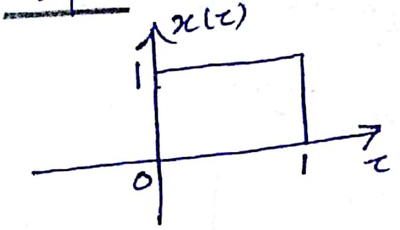
Note :

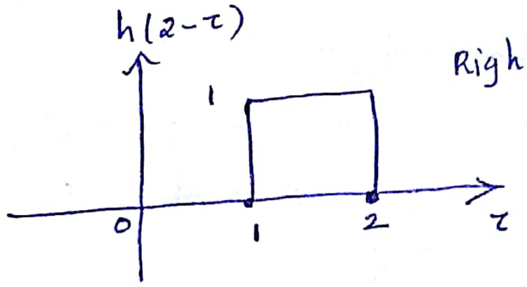
$x(\tau)\downarrow \quad \downarrow h(\tau)$

starting point : $0 + 0 = 0$

Ending point : $1 + 1 = 2$

Step 3 : $t = 2$

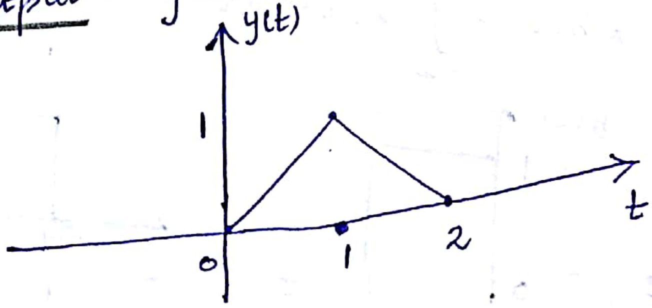




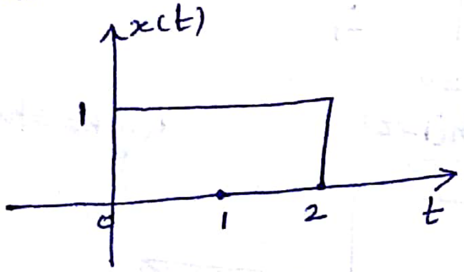
Right shift by 2units.

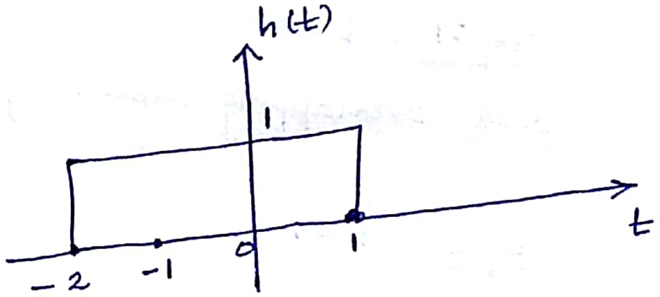
Overlapping Area : $y(t)\Big|_{t=2} = \int_{0}^{2} x(\tau)\cdot h(2-\tau)\, d\tau. = 0.$

output : $y(t) = x(t) * h(t)$



2) Find the response $y(t)$ of an LTI system whose $x(t)$ and $h(t)$ shown in figure.

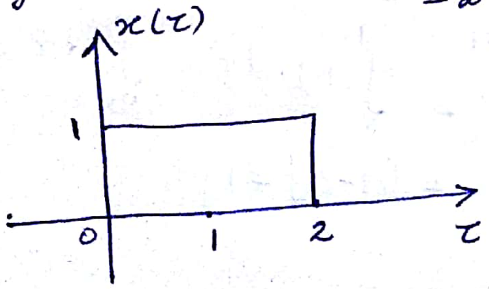


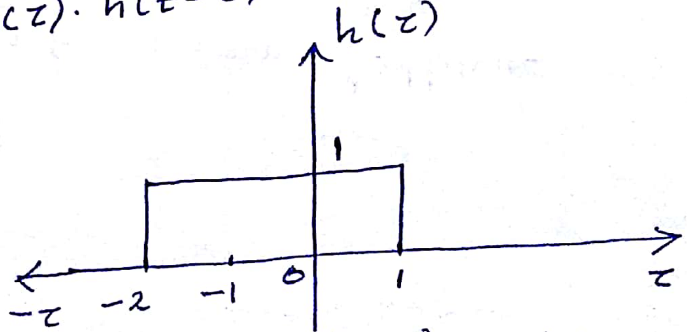


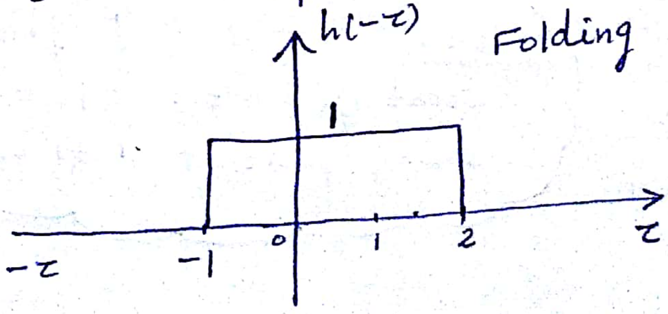
Soln:

Convolution Integral Formula

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\cdot h(t-\tau)\, d\tau$$







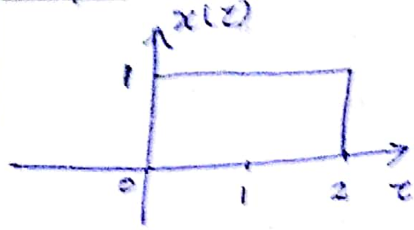
Folding

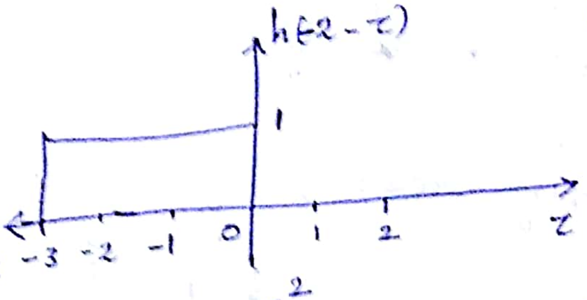
Note: starting point : $\overset{x(t)}{0} + \overset{h(t)}{(-2)} = -2$

Ending point : $2 + 1 = 3$

step 1 : $t = -2$

Left shift by 2 units.

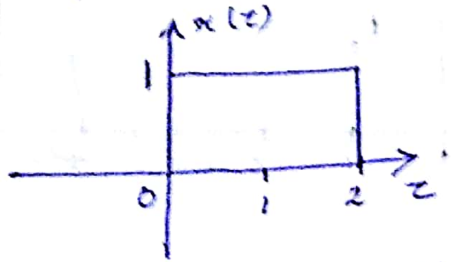


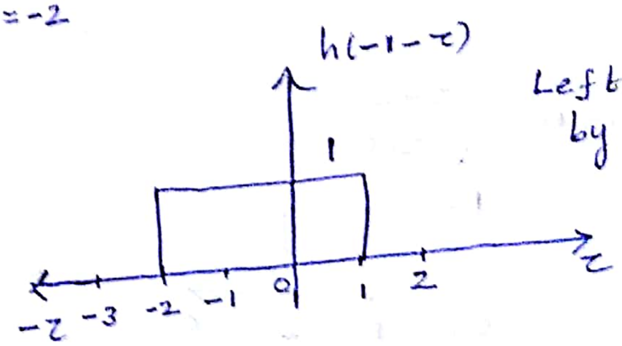


overlapping area : $y(t)\Big|_{t=-2} = \int_{-3}^{2} x(\tau).\,h(-2-\tau)\,d\tau = 0.$

step 2 : $t = -1$

Left shift by 1 unit.



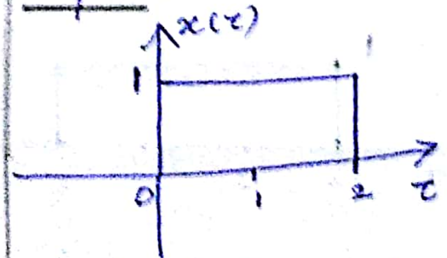


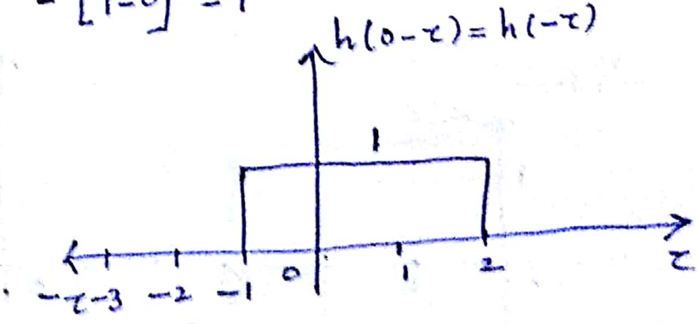
overlapping area : $y(t)\Big|_{t=-1} = \int_{-2}^{2} x(\tau)\, h(-1-\tau)\, d\tau.$

$= \int_{0}^{1} 1.1.\, d\tau = \int_{0}^{1} d\tau = [\tau]_{0}^{1}$

$= [1-0] = 1$

step 3 : $t = 0$



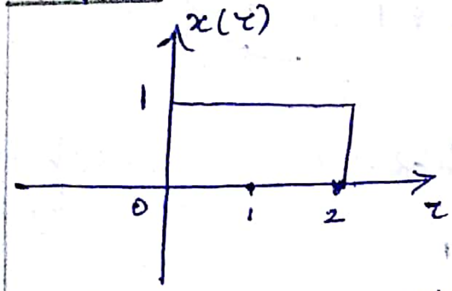


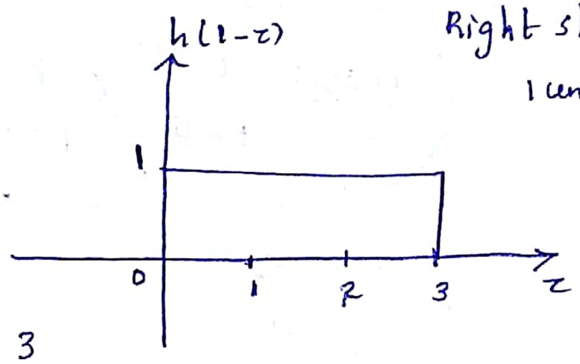
overlapping area : $y(t)\Big|_{t=0} = \int_{-1}^{2} x(\tau).\, h(-\tau)\, d\tau$

$= \int_{0}^{2} 1.1.\, d\tau = \int_{0}^{2} d\tau = [\tau]_{0}^{2} = 2-0$

$= 2$

Step 4 : $t=1$





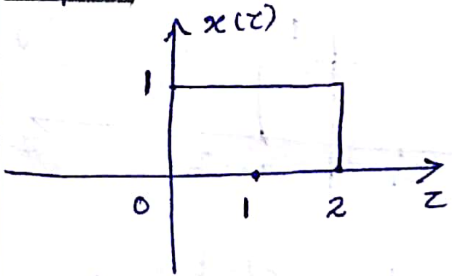
Right shift by 1 unit.

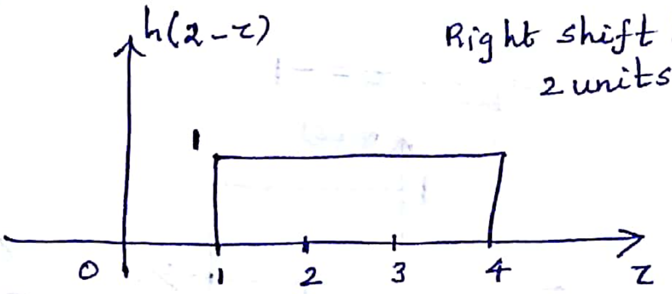
overlapping area : $y(t)\Big|_{t=1} = \int_0^3 x(\tau)\,h(1-\tau)\,d\tau$

$= \int_0^2 1.1.d\tau = \int_0^2 d\tau = [\tau]_0^2 = 2-0$

$=2$

Step 5 : $t=2$





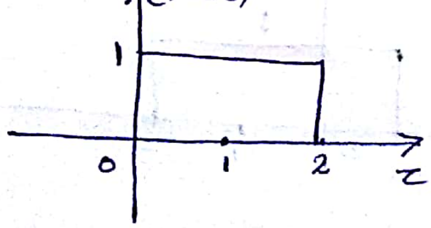
Right shift by 2 units.

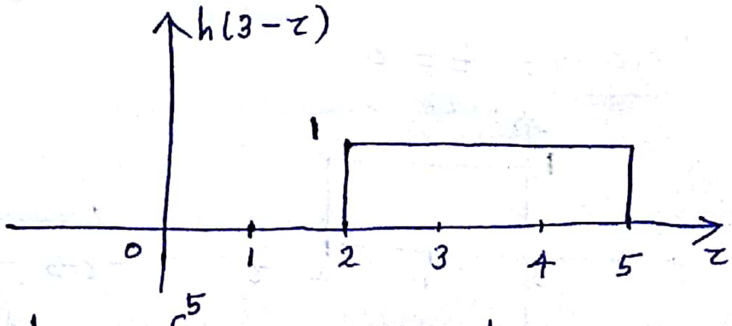
overlapping area : $y(t)\Big|_{t=2} = \int_0^4 x(\tau).\,h(2-\tau)\,d\tau.$

$= \int_1^2 1.1.\,d\tau = \int_1^2 d\tau = [\tau]_1^2 = 2-1$

$=1$

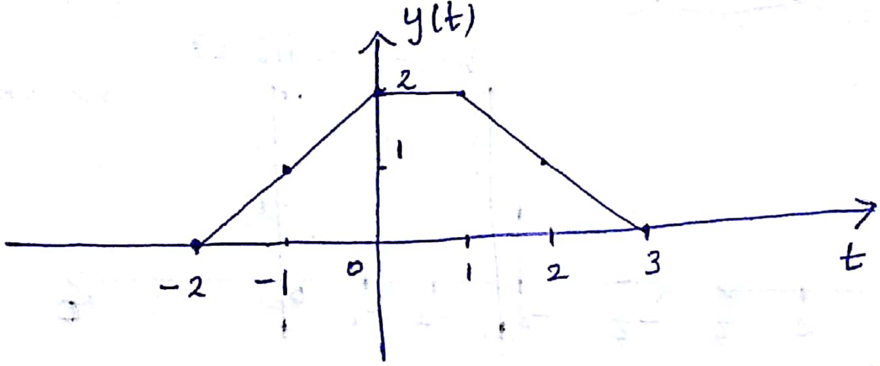
Step 6 : $t=3$.





overlapping area : $y(t)\Big|_{t=3} = \int_0^5 x(\tau).h(3-\tau)\,d\tau = 0.$

output : $y(t) = x(t) * h(t)$

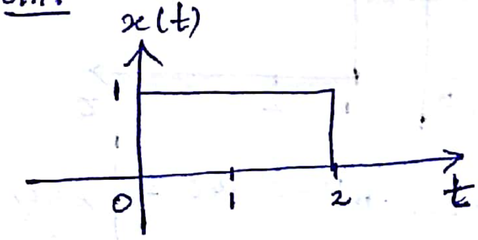


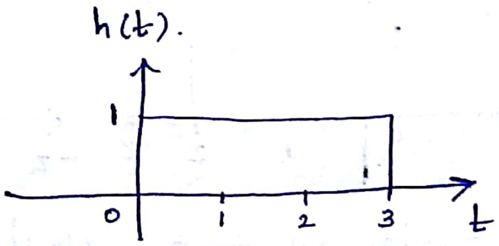
3) Find the convolution by graphical method :

$$x(t) = \begin{cases} 1 ; & 0 \leq t \leq 2 \\ 0 ; & \text{elsewhere} \end{cases}$$

$$h(t) = \begin{cases} 1 ; & 0 \leq t \leq 3 \\ 0 ; & \text{elsewhere} \end{cases}$$

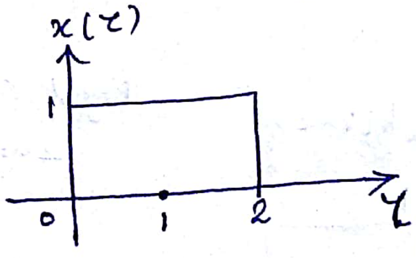
Soln :

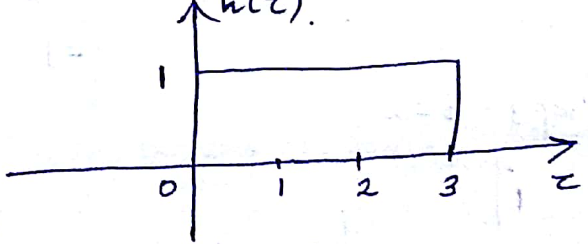


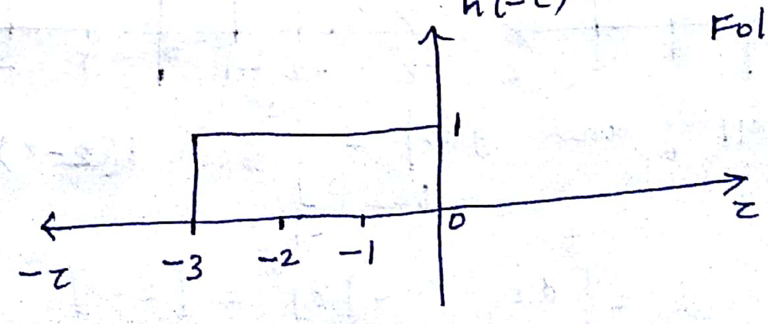


Convolution Integral formula :

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau) \cdot h(t-\tau)\, d\tau .$$







Folding.

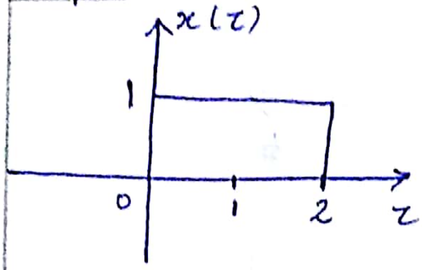
Note: starting point : $\overset{x(\tau)}{0} + \overset{h(\tau)}{0} = 0$

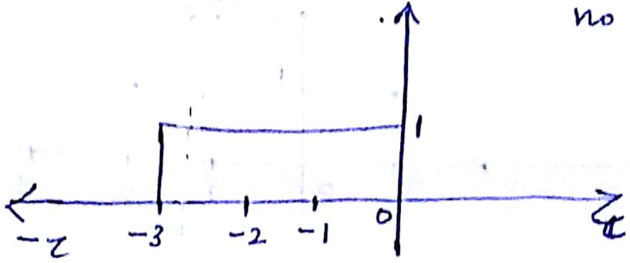
Ending point : $2+3=5$

Step 1: $t=0$

$h(0-\tau) = h(-\tau)$

no shift





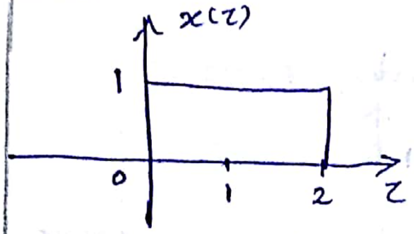
overlapping area:

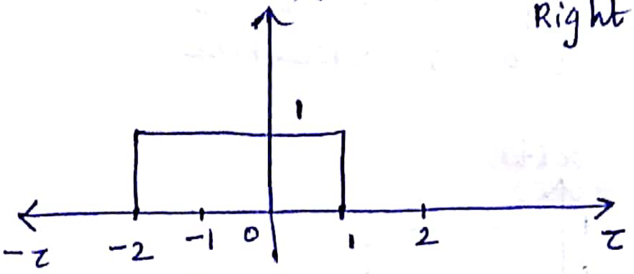
$$y(t)\Big|_{t=0} = \int_{-3}^{2} x(\tau)\cdot h(-\tau)\,d\tau = 0$$

Step 2: $t=1$

$h(1-\tau)$

Right shift by 1 unit





overlapping area: $y(t)\Big|_{t=1} = \int_{-2}^{2} x(\tau)\cdot h(1-\tau)\,d\tau$

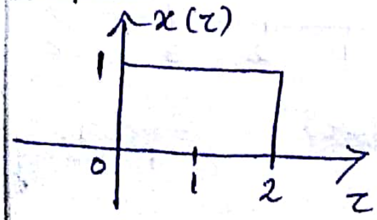
$$= \int_{0}^{1} 1\cdot 1\cdot d\tau = \int_{0}^{1} d\tau = [\tau]_0^1 = 1-0$$

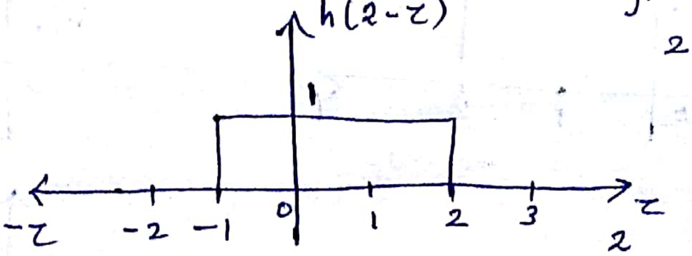
$=1$

Step 3: $t=2$

$h(2-\tau)$

Right shift by 2 units.



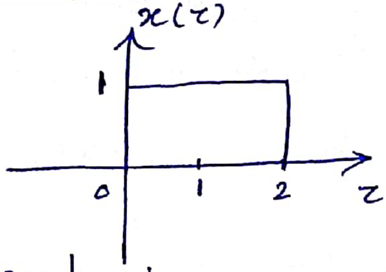


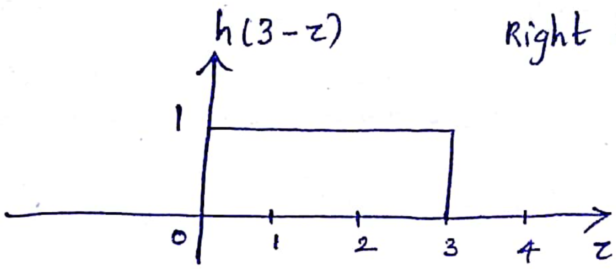
overlapping area: $y(t)\Big|_{t=2} = \int_{-1}^{2} x(\tau)\cdot h(2-\tau)\,d\tau = \int_{0}^{2} 1\cdot 1\,d\tau$

$$= \int_{0}^{2} d\tau = [\tau]_0^2 = [2-0]$$

$= 2$

Step 4 : $t = 3$.





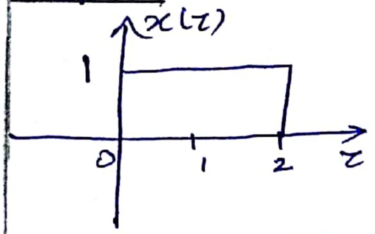
Right shift by 3units.

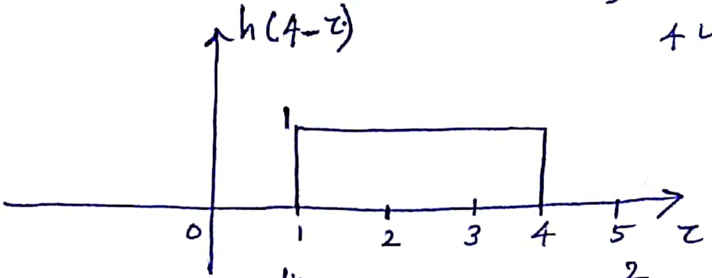
overlapping area : $y(t)\Big|_{t=3} = \int_0^3 x(\tau).\,h(3-\tau)\,d\tau.$

$$= \int_0^2 1.1.\,d\tau = \int_0^2 d\tau = [\tau]_0^2 = 2-0$$

$$= 2$$

Step 5 : $t=4$





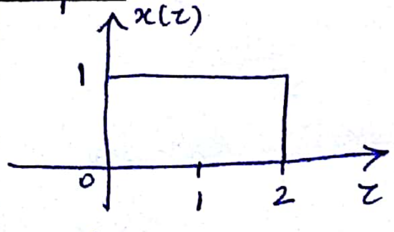
Right shift by 4 units.

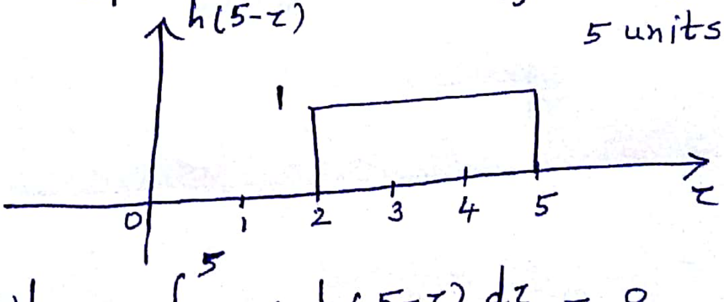
overlapping area : $y(t)\Big|_{t=4} = \int_0^4 x(\tau).\,h(4-\tau)\,d\tau = \int_1^2 1.1.\,d\tau$

$$= \int_1^2 d\tau = [\tau]_1^2 = 2-1$$

$$=1$$

Step 6 : $t=5$

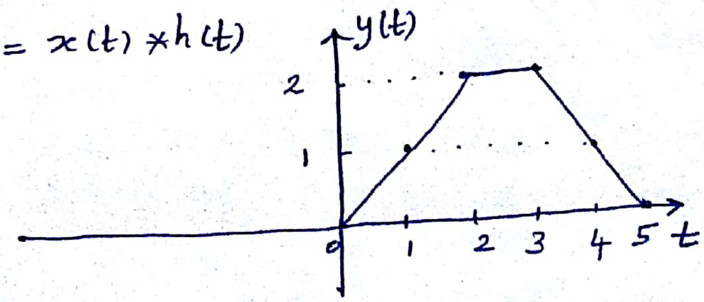




Right shift by 5 units

overlapping area : $y(t)\Big|_{t=5} = \int_0^5 x(\tau).\,h(5-\tau)\,d\tau = 0$

output: $y(t) = x(t) * h(t)$



→ Problems based on Convolution [Causal Signals].

Group II : Convolution by convolution integral Formula

1. Convolve the signal $x(t) = e^{-2t} u(t)$ with $h(t) = u(t)$.

Soln:

$$x(t) = e^{-2t} u(t)$$

$$h(t) = u(t).$$

The given signals are causal. So, integration limit varies from 0 to 't'.

$$y(t) = x(t) * h(t). = \int_0^t x(\tau) . h(t-\tau) \, d\tau$$

$$= \int_0^t e^{-2\tau} . 1 \, d\tau . = \int_0^t e^{-2\tau} \, d\tau = \left[\frac{e^{-2\tau}}{-2} \right]_0^t$$

$$= -\frac{1}{2} \left[e^{-2t} - e^{0} \right] = \frac{-1}{2} \left[e^{-2t} - 1 \right] = \frac{1}{2} \left[1 - e^{-2t} \right]$$

Ans: $y(t) = \frac{1}{2} \left[1 - e^{-2t} \right] u(t).$

2. Find $y(t)$ if $x(t) = \sin at \, u(t)$ and $h(t) = u(t)$.

Soln:

$$x(t) = \sin at \, u(t)$$

$$h(t) = u(t)$$

The given signals are causal. so, integration limit varies from '0' to 't'.

$$y(t) = x(t) * h(t) = \int_0^t x(\tau) . h(t-\tau) \, d\tau$$

$$= \int_0^t \sin a\tau . 1 \, d\tau = \int_0^t \sin a\tau \, d\tau = \left[\frac{-\cos a\tau}{a} \right]_0^t = \frac{-1}{a} \left[\cos at - \cos 0 \right]$$

$$= \frac{-1}{a} \left[\cos at - 1 \right] = \frac{1}{a} \left[1 - \cos at \right]$$

Ans:

$$y(t) = \frac{1}{a} \left[1 - \cos at \right] u(t).$$

3) Find $y(t)$ if $x(t) = t\,u(t)$ and $h(t) = u(t)$.

Soln:

$$x(t) = t\,u(t)$$

$$h(t) = u(t).$$

The given signals are causal. ∴ Integration limit varies from 0 to t.

$$y(t) = x(t) * h(t) = \int_0^t x(\tau) \cdot h(t-\tau)\, d\tau$$

$$y(t). = \int_0^t \tau \cdot 1 \cdot d\tau = \int_0^t \tau\, d\tau = \left[\frac{\tau^2}{2}\right]_0^t = \frac{1}{2}[t^2 - 0] = \frac{1}{2}t^2$$

Ans:

$$y(t) = \frac{1}{2}t^2 u(t).$$

4) Find $y(t)$ if $x(t) = u(t)$ and $h(t) = u(t)$.

Soln:

$$x(t) = u(t)$$

$$h(t) = u(t)$$

The given signals are causal. ∴ Integration limit varies from '0' to 't'.

$$y(t) = x(t) * h(t) = \int_0^t x(\tau) \cdot h(t-\tau)\, d\tau.$$

$$y(t) = \int_0^t 1 \cdot 1 \cdot d\tau = \int_0^t d\tau = [\tau]_0^t = t - 0 = t$$

Ans: $y(t) = t\,u(t).$

5). Find $y(t)$, if $x(t) = e^{at} u(t)$ and $h(t) = e^{at} u(t)$.

Soln:

$$x(t) = e^{at} u(t)$$

$$h(t) = e^{at} u(t)$$

The given signals are causal. ∴ Integration limit varies from '0' to 't'.

$$y(t) = x(t) * h(t) = \int_0^t x(\tau) \cdot h(t-\tau)\, d\tau$$

$$y(t) = \int_0^t e^{a\tau} \cdot e^{a(t-\tau)}\, d\tau = \int_0^t e^{a\tau} \cdot e^{at} \cdot e^{-a\tau}\, d\tau = e^{at}\int_0^t d\tau$$

$$= e^{at}[\tau]_0^t = e^{at}[t-0] = te^{at}.$$

Ans: $y(t) = te^{at} u(t).$

Group III : Convolution of Noncausal signals.

1) Convolve the following signals:

$x(t) = e^{-3t}u(t)$ and $h(t) = u(t+3)$

Soln:

$x(t) = e^{-3t}u(t)$

$h(t) = u(t+3)$ → Non causal signal

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau).h(t-\tau)\,d\tau = \int_{-\infty}^{\infty} h(\tau).x(t-\tau)\,d\tau$$

$$\boxed{\begin{aligned} h(\tau) &= u(\tau+3) \\ x(t-\tau) &= e^{-3(t-\tau)}\,u(t-\tau) \end{aligned}}$$

$$y(t) = \int_{-\infty}^{\infty} u(\tau+3).\ e^{-3(t-\tau)}\,u(t-\tau)\,d\tau$$

$$u(\tau+3) = \begin{cases} 1 ; & \tau \geqslant -3 \\ 0 ; & \text{elsewhere} \end{cases}$$

$$u(t-\tau) = \begin{cases} 1 ; & \tau \leq t \\ 0 ; & \text{elsewhere} \end{cases}$$

$$u(\tau+3).u(t-\tau) = \begin{cases} 1 ; & -3 \leq \tau \leq t \\ 0 ; & \text{elsewhere} \end{cases}$$

$\tau + 3 = 0$

$\boxed{\tau = -3}$

$\tau + 3 = 1$

$\tau = 1 - 3 = -2$ //

$t - \tau = 0$

$\tau = t$

$t - \tau = 1$

$\boxed{\tau = t-1}$

Now,

$$y(t) = \int_{-3}^{t} e^{-3(t-\tau)}\,d\tau = e^{-3t}\int_{-3}^{t} e^{3\tau}\,d\tau = e^{-3t}\left[\frac{e^{3\tau}}{3}\right]_{-3}^{t}$$

$$= e^{-3t} \times \frac{1}{3}\left[e^{3\tau}\right]_{-3}^{t} = \frac{e^{-3t}}{3}\left[e^{3t} - e^{-9}\right] = \frac{1}{3}\left[e^{0} - e^{-3t}.e^{-9}\right]$$

$$\boxed{y(t) = \frac{1}{3}\left[1 - e^{-3(t+3)}\right].}$$

2. Find $y(t)$, if $x(t) = u(t-3) - u(t-5)$ and $h(t) = e^{-3t} u(t)$.

Soln:

$$x(t) = u(t-3) - u(t-5)$$

$$h(t) = e^{-3t} u(t)$$

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau).h(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

$$y(t) = [u(t-3) - u(t-5)] * e^{-3t}u(t)$$

$$y(t) = \underbrace{[u(t-3) * e^{-3t}u(t)]}_{I} - \underbrace{[u(t-5) * e^{-3t}u(t)]}_{II.} \longrightarrow ①$$

Solution I :

$$u(t-3) * e^{-3t}u(t) = \int_{-\infty}^{\infty} u(\tau-3) . e^{-3(t-\tau)} u(t-\tau)\, d\tau$$

$u(\tau-3) = \begin{cases} 1 & ; \quad \tau \geq 3 \\ 0 & ; \quad \text{elsewhere} \end{cases}$	$\tau-3=0$ $\boxed{\tau=3}$ $\tau-3=1$ $\tau=1+3$ $\boxed{\tau=4}$
$u(t-\tau) = \begin{cases} 1 & ; \quad \tau \leq t \\ 0 & ; \quad \text{elsewhere} \end{cases}$ $u(\tau-3).u(t-\tau) = \begin{cases} 1 & ; \quad 3 \leq \tau \leq t \\ 0 & ; \quad \text{elsewhere} \end{cases}$	$t-\tau=0$ $\boxed{\tau=t}$ $t-\tau=1$ $\boxed{\tau=t-1}$

$$= \int_{3}^{t} e^{-3(t-\tau)} d\tau = \int_{3}^{t} e^{-3t} . e^{3\tau} d\tau$$

$$= e^{-3t} \int_{3}^{t} e^{3\tau} d\tau = e^{-3t} \left[\frac{e^{3\tau}}{3}\right]_{3}^{t}$$

$$= e^{-3t} \times \frac{1}{3} \times [e^{3t} - e^{9}] = \frac{1}{3}[e^{0} - e^{-3t}.e^{9}]$$

$$u(t-3) * e^{-3t} u(t) = \frac{1}{3}[1 - e^{-3(t-3)}] \rightarrow I$$

solution II :

$$u(t-5) * e^{-3t} u(t) = \int_{-\infty}^{\infty} u(\tau-5) \cdot e^{-3(t-\tau)} u(t-\tau)\, d\tau$$

$u(\tau-5) = \begin{cases} 1 & ; \tau \geq 5 \\ 0 & ; \text{elsewhere} \end{cases}$	$\tau - 5 = 0$ $\boxed{\tau = 5}$ $\tau - 5 = 1$ $\boxed{\tau = 6}$
$u(t-\tau) = \begin{cases} 1 & ; \tau \leq t \\ 0 & ; \text{elsewhere} \end{cases}$ $u(\tau-5) \cdot u(t-\tau) = \begin{cases} 1 & ; 5 \leq \tau \leq t \\ 0 & ; \text{elsewhere} \end{cases}$	$t - \tau = 0$ $\boxed{\tau = t}$ $t - \tau = 1$ $\tau = t-1$

$$= \int_{5}^{t} e^{-3(t-\tau)} d\tau = e^{-3t} \int_{5}^{t} e^{3\tau} d\tau$$

$$= e^{-3t} \left[\frac{e^{3\tau}}{3} \right]_{5}^{t} = \frac{e^{-3t}}{3} \left[e^{3t} - e^{15} \right]$$

$$= \frac{1}{3} \left[e^{0} - e^{-3t} \cdot e^{15} \right]$$

$$= \frac{1}{3} \left[1 - e^{-3(t-5)} \right] \longrightarrow \text{II} .$$

Now,

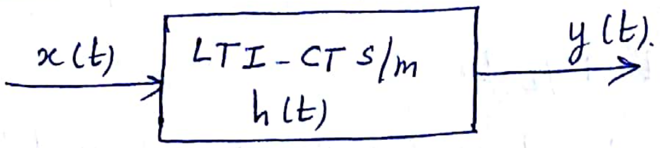
$$① \Rightarrow y(t) = \frac{1}{3} \left[1 - e^{-3(t-3)} \right] - \left[\frac{1}{3} \left[1 - e^{-3(t-5)} \right] \right]$$

$$y(t) = \frac{1}{3} - \frac{1}{3} \cdot e^{-3(t-3)} - \frac{1}{3} + \frac{1}{3} \cdot e^{-3(t-5)}$$

$$\boxed{y(t) = \frac{1}{3} \left[e^{-3(t-5)} - e^{-3(t-3)} \right]}$$

→ Properties of Convolution and Impulse Response connection:

1) Commutative :



$$y(t) = x(t) * h(t) = h(t) * x(t)$$

2) Associative

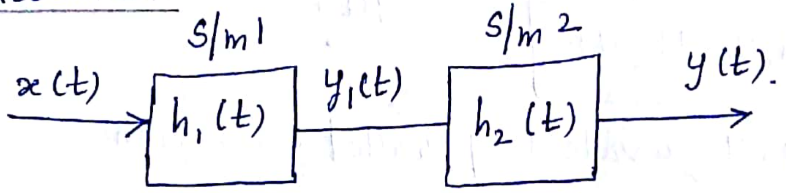


Figure: cascade (or) series connection .

From the diagram,

$$y(t) = y_1(t) * h_2(t)$$

$$y(t) = [x(t) * h_1(t)] * h_2(t)$$

(or)

$$y(t) = x(t) * [h_1(t) * h_2(t)]$$

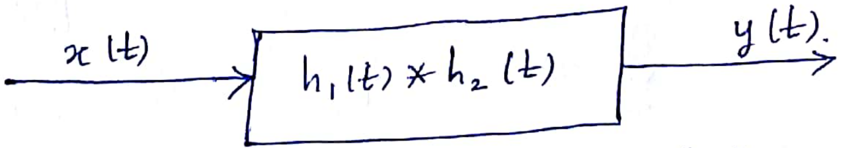


Fig: Equivalent cascade (or) series connection.

3) Distributive :

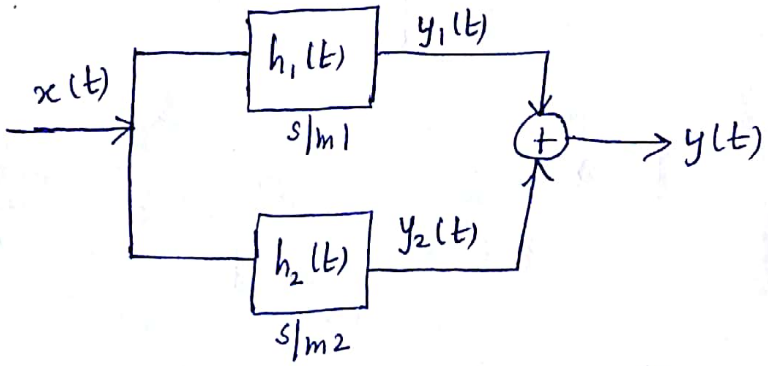


Figure : Parallel connection

From the figure,

$$y_1(t) = x(t) * h_1(t) \quad \& \quad y_2(t) = x(t) * h_2(t)$$

$$y(t) = y_1(t) + y_2(t)$$

$$y(t) = [x(t) * h_1(t)] + [x(t) * h_2(t)] \longrightarrow ①$$

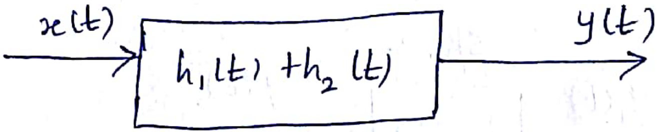


Fig: Equivalent parallel connection.

From the diagram,

$$y(t) = x(t) * [h_1(t) + h_2(t)] \longrightarrow Ⓐ$$

$$y(t) = [x(t) * h_1(t)] + [x(t) * h_2(t)] \longrightarrow ②.$$

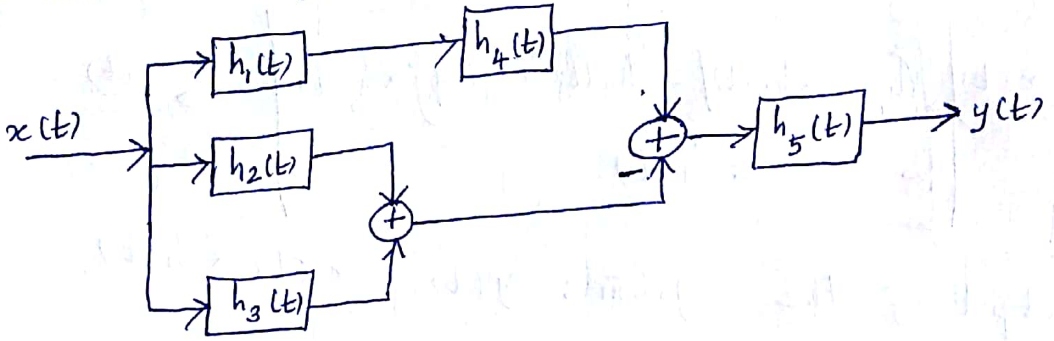
From ① & ②,

$$y(t) = [x(t) * h_1(t)] + [x(t) + h_2(t)] = x(t) * [h_1(t) + h_2(t)]$$

[∵ using eqn Ⓐ].

→ Problems based on Impulse Response Connection:

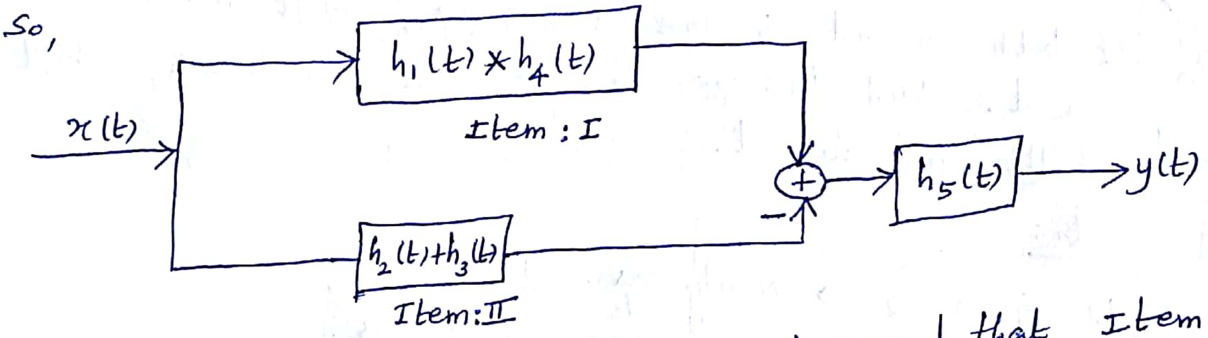
U.Q 1) For the interconnection of system shown in figure. Obtain the overall impulse response in terms of impulse responses of individual subsystems.



Soln:

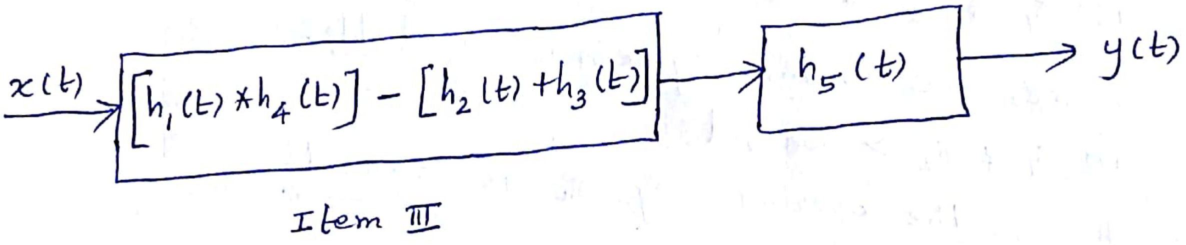
From the given block diagram, we found that $h_1(t)$ and $h_4(t)$ are connected in cascade & $h_2(t)$ and $h_3(t)$ are connected in parallel.

So,



From the above block diagram, we observed that Item : I and Item : II are connected in parallel.

So,



From the above block diagram, we found that Item: III and $h_5(t)$ are connected in cascade.

So,

$$x(t) \longrightarrow \boxed{\begin{array}{c}\left\{\left[h_1(t) * h_4(t)\right] - \left[h_2(t) + h_3(t)\right]\right\} * h_5(t) \\ = h(t)\end{array}} \longrightarrow y(t)$$

Output of the system : $y(t) = x(t) * h(t)$.

Where, $h(t) \rightarrow$ overall impulse response

$$\boxed{h(t) = \left\{\left[h_1(t) * h_4(t)\right] - \left[h_2(t) + h_3(t)\right]\right\} * h_5(t)}$$

U.Q 2) Consider two systems S_1 and S_2 are connected in series and if both S_1 and S_2 are non-linear, shift invariant, stable and causal. Will the cascade also be non-linear, shift invariant, stable and causal.

Soln:

i) S_1 & $S_2 \rightarrow$ Nonlinear.

Hence, the overall cascade system is linear.

ii) S_1 & $S_2 \rightarrow$ shift Invariant

Hence, the overall cascade system is also shift Invariant.

iii) S_1 & $S_2 \rightarrow$ stable.

Hence, the overall system is unstable.

iv) S_1 & $S_2 \rightarrow$ causal

Hence, the overall system is causal.

→ Properties of Impulse Response and Impulse Function interms of convolution.

i) Replication property

a) $\delta(t) * x(t) = x(t)$

b) $\delta(t) * h(t) = h(t)$.

ii) Time shifting property

a) $x(t) * \delta(t-t_0) = x(t-t_0)$

b) $h(t) * \delta(t-t_0) = h(t-t_0)$

iii) causality :

Impulse Response : $h(t) = 0 \;;\; t < 0$.

iv) stability :

$$\int_{-\infty}^{\infty} |h(t)|\, dt < \infty.$$

→ Problems based on properties $\delta(t)$ & $h(t)$ interms of convolution:

1) Find the output of LTI system with impulse response $h(t) = \delta(t-3)$ and $x(t) = \cos 4t + \cos 7t$.

soln:

$h(t) = \delta(t-3)$

$x(t) = \cos 4t + \cos 7t$.

output :

$$y(t) = x(t) * h(t)$$

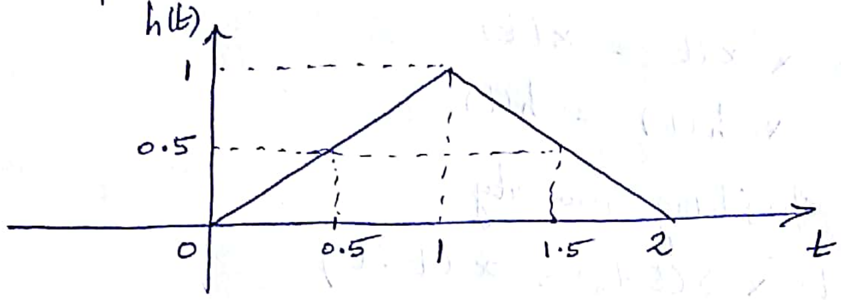
$$= [\cos 4t + \cos 7t] * \delta(t-3)$$

$$= \cos 4t * \delta(t-3) + \cos 7t * \delta(t-3)$$

$$\boxed{y(t) = \cos 4(t-3) + \cos 7(t-3)}$$

[∵ using time shifting property of impulse function]

2) Find the response of the system shown in figure for the input $x(t) = \delta(t) - \delta(t-1.5)$. Here, $h(t)$ is the impulse response of the system.



Soln:

To find the response (or) output of the given system:

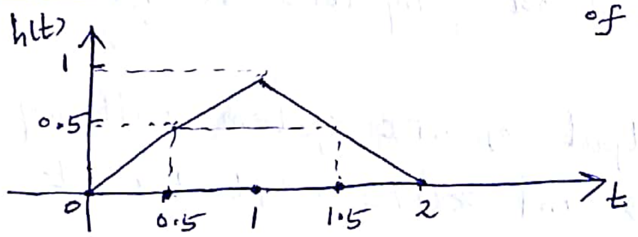
$$y(t) = x(t) * h(t).$$

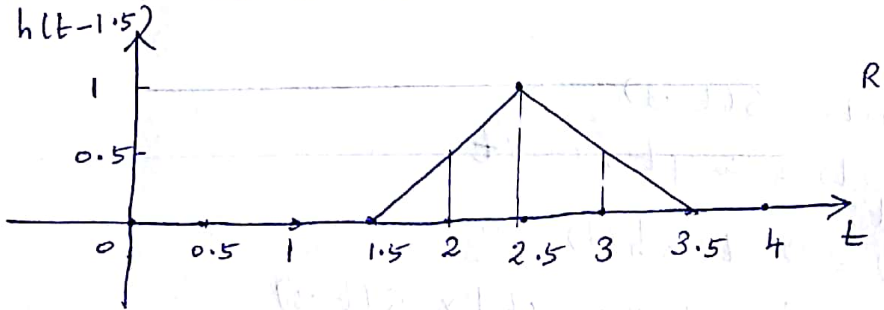
$$y(t) = [\delta(t) - \delta(t-1.5)] * h(t).$$

$$y(t) = \delta(t) * h(t) - [\delta(t-1.5) * h(t)]$$

$$\boxed{y(t) = h(t) - h(t-1.5)}$$

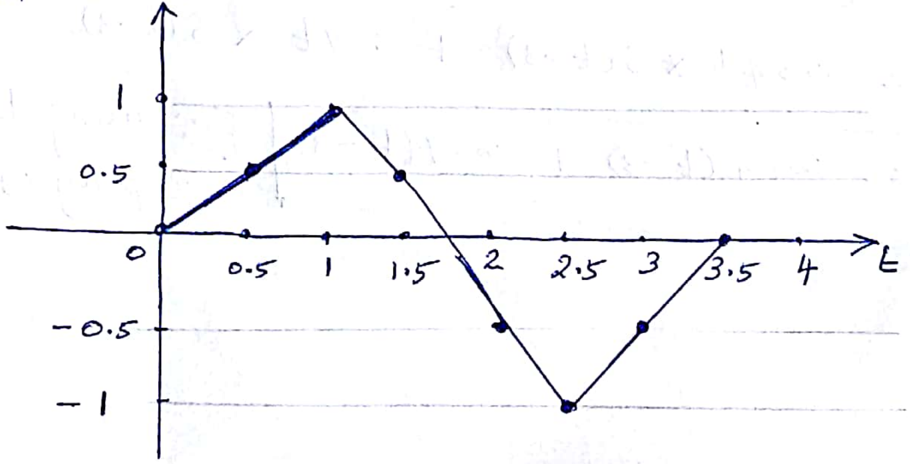
[∵ using Replication & Time shifting Properties of Impulse function]



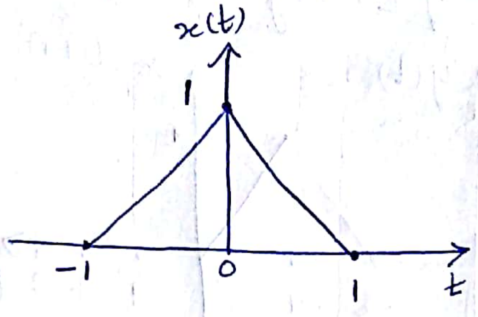


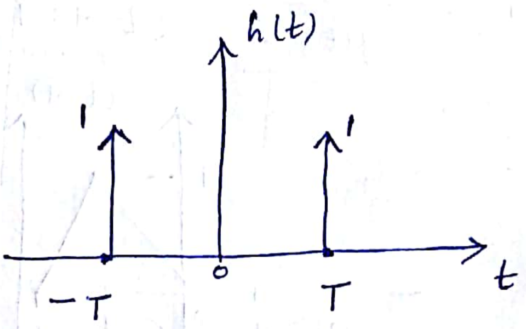
Right shift by 1.5 units.

y(t) = h(t) - h(t-1.5).



3) Find the convolution of $x(t)$ and $h(t)$ when $T = 2, 1, 0.5$.





Soln:

From the graph,

$$h(t) = \delta(t-T) + \delta(t+T)$$

* To find the convolution of $x(t)$ and $h(t)$:

$$y(t) = x(t) * h(t).$$

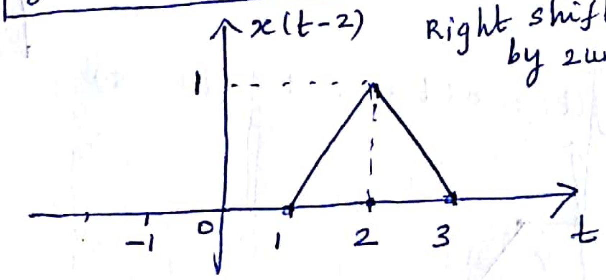
$$= x(t) * [\delta(t-T) + \delta(t+T)]$$

$$= [x(t) * \delta(t-T)] + [x(t) * \delta(t+T)]$$

$$\boxed{y(t) = x(t-T) + x(t+T)}$$ $\rightarrow$ ① [∵ using time shifting property of impulse function]

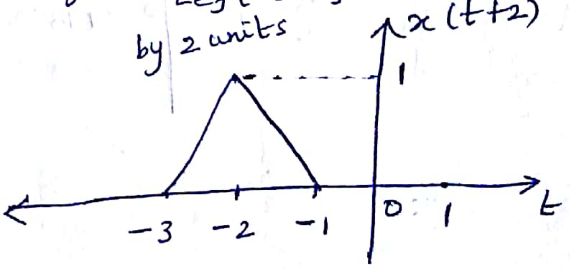
i) When $T = 2$

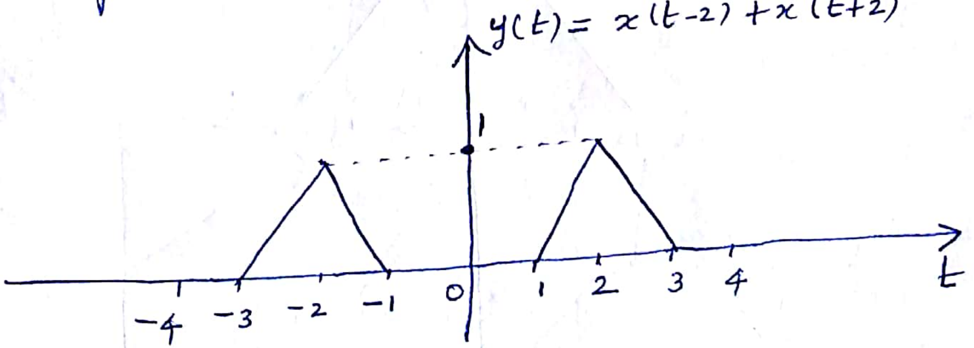
$$\boxed{y(t) = x(t-2) + x(t+2)}$$ [∵ using eqn ①]



Right shift by 2units

Left shift by 2 units

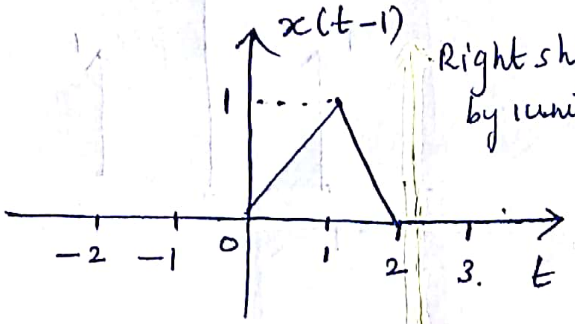




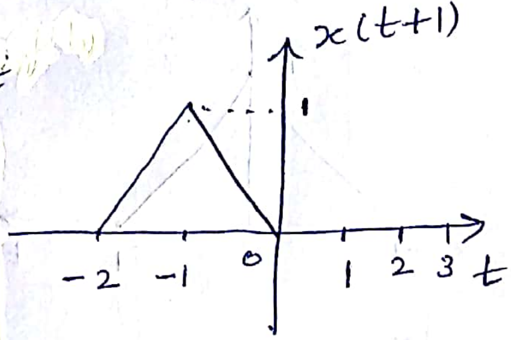
Scanned by CamScanner

ii) $T = 1$

$\boxed{y(t) = x(t-1) + x(t+1)}$ [$\because$ using eqn ①]

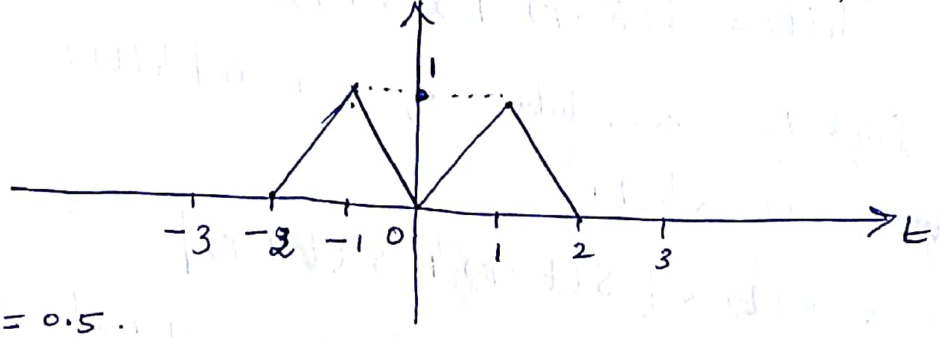


Right shift by 1 unit



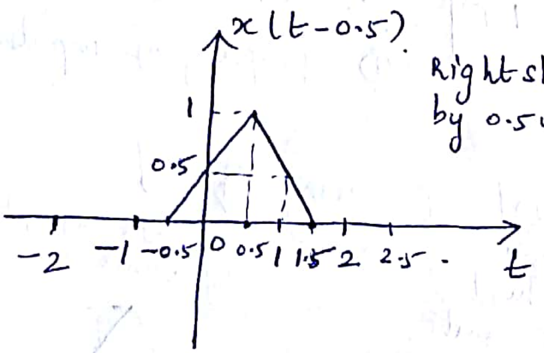
Left shift by 1 unit

$y(t) = x(t-1) + x(t+1)$



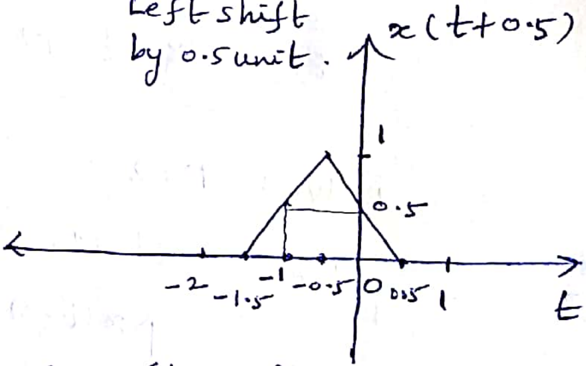
iii) $T = 0.5$.

$\boxed{y(t) = x(t-0.5) + x(t+0.5)}$ [$\because$ using eqn ①].

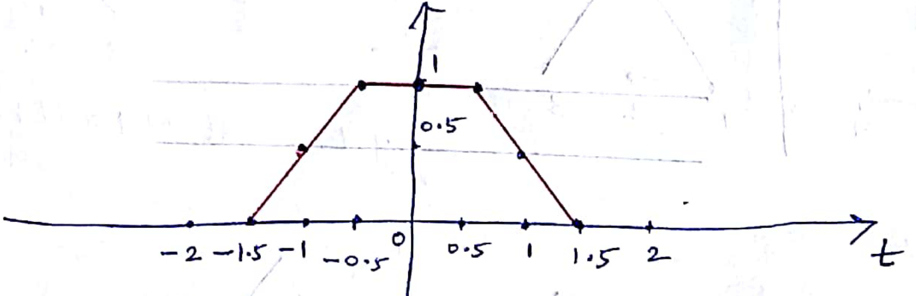


Right shift by 0.5 unit

Left shift by 0.5 unit.



$y(t) = x(t-0.5) + x(t+0.5)$



4) Find $y(t) = [2\delta(t+1) + \delta(t-5)] * u(t-1)$.

Soln:

$$y(t) = [2\delta(t+1) + \delta(t-5)] * x(t) \quad [\because \text{Let } x(t) = u(t-1)]$$

$$y(t) = [x(t) * 2\delta(t+1)] + [x(t) * \delta(t-5)]$$

$$y(t) = 2x(t+1) + x(t-5).$$

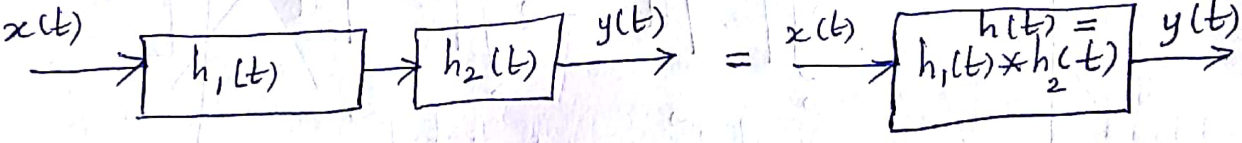
$$y(t) = 2u[t-1+1] + u[t-1-5] \quad [\because x(t) = u(t-1)]$$

$$\boxed{y(t) = 2u[t] + u[t-6]}$$

U.Q 5) The system shown below is formed by connecting two systems in cascade. Impulse Response response of the systems are given by $h_1(t) = e^{-2t}u(t)$ and $h_2(t) = 2e^{-t}u(t)$. Find the overall impulse response of the system and determine the overall system is BIBO stable.

Soln:

For cascade connection,



overall Impulse Response: $h(t) = h_1(t) * h_2(t)$

Given:

$$\left.\begin{aligned} h_1(t) &= e^{-2t}u(t) ; \\ h_2(t) &= 2e^{-t}u(t) \end{aligned}\right\}$$ The given signals are causal. So, the integral limit varies from '0' to 't'.

Now,

$$h(t) = e^{-2t}u(t) * 2e^{-t}u(t).$$

$$h(t) = \int_0^t e^{-2\tau} \cdot 2e^{-(t-\tau)} \, d\tau$$

$$= 2e^{-t}\int_0^t e^{-2\tau}.e^{\tau}\,d\tau = 2e^{-t}\int_0^t e^{-\tau}\,d\tau$$

$$= 2e^{-t}\left[\frac{e^{-\tau}}{-1}\right]_0^t = -2e^{-t}\left[e^{-t}-e^{0}\right]$$

$$= -2e^{-t}\left[e^{-t}-1\right]$$

$$h(t) = 2e^{-t}\left[1-e^{-t}\right]$$

<u>Ans:</u>

$$h(t) = 2e^{-t}\left[1-e^{-t}\right]u(t)$$

(or)

$$h(t) = 2e^{-t}u(t) - 2e^{-2t}u(t).$$

* To check stablity:

condition for stability : $\int_{-\infty}^{\infty} |h(t)|$ ...

$$\int_{-\infty}^{\infty} |h(t)|\,dt = \int_{-\infty}^{\infty} \left|2e^{-t}u(t) - 2e^{-2t}u(t)\right| dt.$$

$$= 2\int_0^{\infty} |e^{-t}|dt - 2\int_0^{\infty} |e^{-2t}|dt \qquad \left[\because u(t) = \begin{cases}1 & ; t \geq 0 \\ 0 & ; t < 0\end{cases}\right]$$

$$= 2\left[\frac{e^{-t}}{-1}\right]_0^{\infty} - 2\left[\frac{e^{-2t}}{-2}\right]_0^{\infty}$$

$$= -2\left[e^{-\infty} - e^{0}\right] + \left[e^{-\infty} - e^{0}\right]$$

$$= -2[0-1] + [0-1]$$

$$= 3 - 1 = 2 \quad [<\infty]$$

$\therefore$ The given system is BIBO stable.

→ Inverse Laplace Transform.

Type : simple poles.

Problems :

1) Find the inverse laplace transform of $X(s) = \frac{3s+7}{s^2-2s-3}$.

(i) $Re(s) \geq 3$.

(ii) $Re(s) \leq -1$

(iii) $-1 \leq Re(s) \leq 3$.

Soln:

$$X(s) = \frac{3s+7}{s^2-2s-3}$$

Factorization of Dr.

s^2-2s-3 : $a = 1, b = -2, c = -3$.

$$s = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{+2 \pm \sqrt{4+12}}{2} = \frac{2 \pm \sqrt{16}}{2}$$

$$s = \frac{2 \pm 4}{2} = \frac{6}{2}, -\frac{2}{2} = 3, -1.$$

$s = 3, -1$

Now,

$$X(s) = \frac{3s+7}{(s-3)(s+1)}$$

$$X(s) = \frac{K_0}{s-3} + \frac{K_1}{s+1} \longrightarrow ①$$

$$K_0 = X(s) \times (s-3)\Big|_{s=3} = \frac{3s+7}{(s-3)(s+1)} \times (s-3)\Big|_{s=3}$$

$$K_0 = \frac{3(3)+7}{(3+1)} = \frac{9+7}{4} = \frac{16}{4}$$

$$\boxed{K_0 = 4}$$

$$K_1 = X(s).(s+1)\Big|_{s=-1}$$

$$K_1 = \frac{3s+7}{(s-3)(s+1)} \times (s+1)\Bigg|_{s=-1}$$

$$K_1 = \frac{3(-1)+(7)}{(-1-3)} = \frac{-3+7}{-4} = \frac{4}{-4}$$

$$\boxed{K_1 = -1}$$

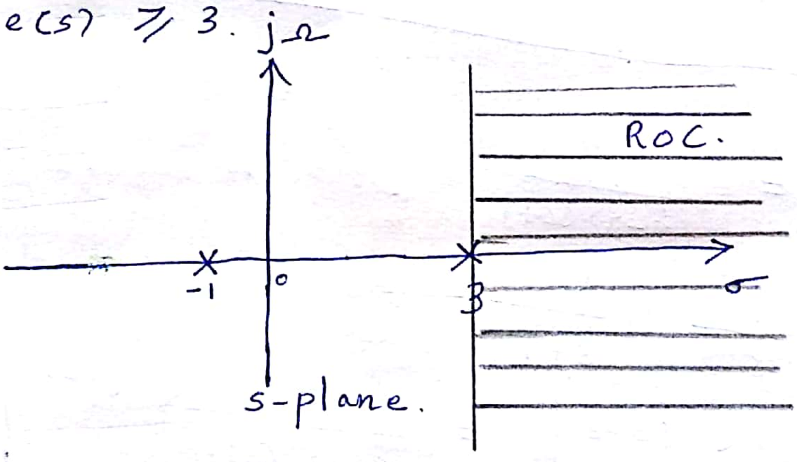
Now,

① ⟹ $X(s) = \frac{4}{s-3} - \frac{1}{s+1} \rightarrow$ ②.

without consideration of RoC:
Taking Inv. Lap. Transform of eqn ②,
$x(t) = 4 \cdot e^{3t} u(t) - e^{-t} u(t)$.

With the consideration of RoC:

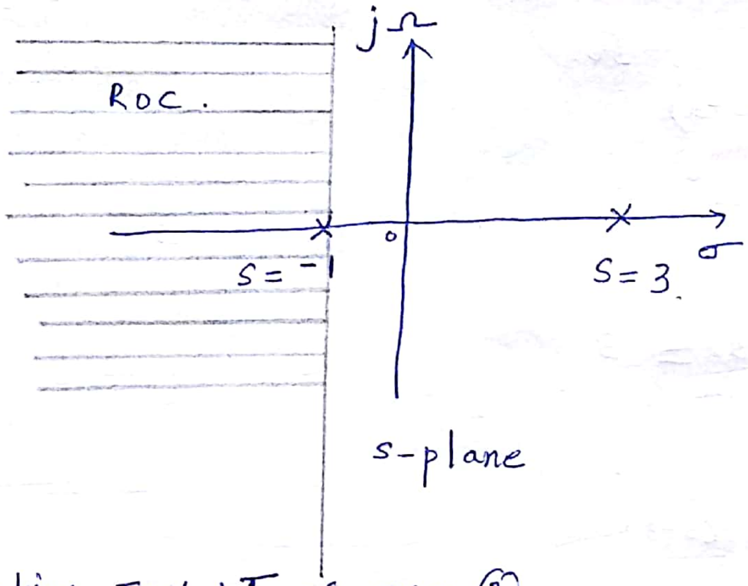
i) $Re(s) \geq 3$.



Taking Inv. Lap. Tran. of eqn ②,

$$\boxed{x(t) = 4e^{3t} u(t) - e^{-t} u(t)}$$

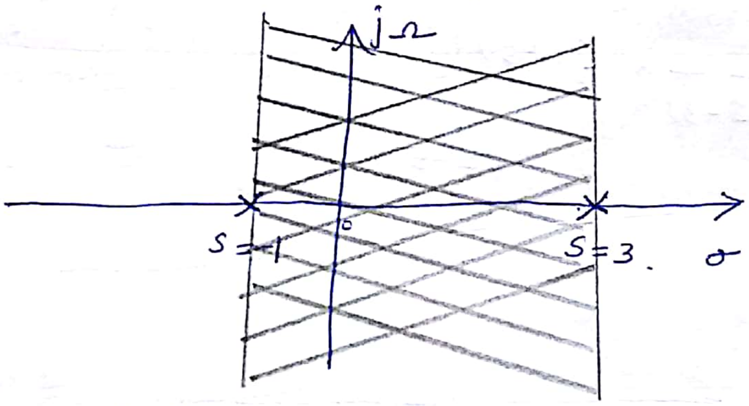
ii) $Re(s) \leq -1$



Taking Inv. LT of eqn ②.

$$\boxed{x(t) = 4e^{3t}u(t) + e^{-t}u(-t)}$$

iii) Roc : $-1 \leq Re(s) \leq 3$



Taking Inv. L.T of eqn ②,

$$\boxed{x(t) = -4e^{3t}u(-t) - e^{-t}u(t)}$$

2) Detect the inverse laplace transform of

$$X(s) = \frac{1}{(s+5)(s-3)}$$ for

i) Re(s) ≥ 3

ii) Re(s) ≤ -5

iii) -5 ≤ Re(s) ≤ 3.

Soln:

$$X(s) = \frac{1}{(s+5)(s-3)}$$

$$X(s) = \frac{k_0}{s+5} + \frac{K_1}{s-3} \longrightarrow ①$$

$$k_0 = X(s) \times (s+5)\Big|_{s=-5}$$

$$k_0 = \frac{1}{(s+5)(s-3)} \times (s+5)\Big|_{s=-5}$$

$$k_0 = \frac{1}{-5-3}$$

$$\boxed{k_0 = -\frac{1}{8}}$$

$$k_1 = X(s) \times (s-3)\Big|_{s=3}$$

$$K_1 = \frac{1}{(s+5)(s-3)} \times (s-3)\Big|_{s=3}$$

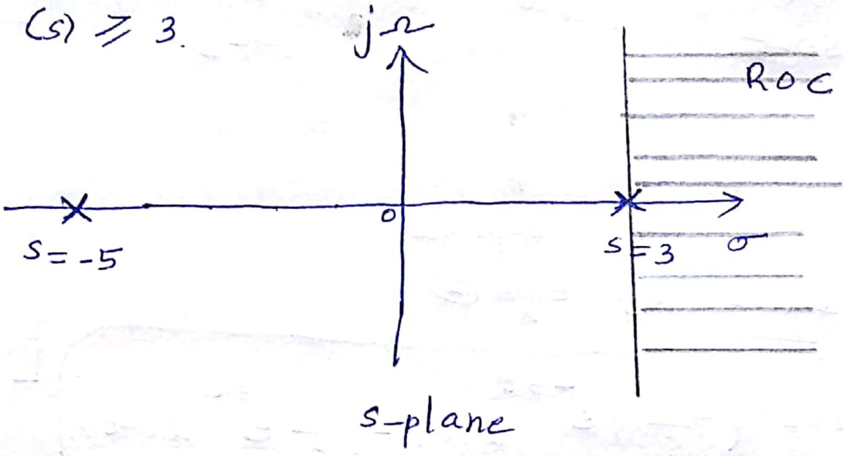
$$k_1 = \frac{1}{3+5}$$

$$\boxed{k_1 = \frac{1}{8}}$$

① ⟹ $X(s) = \frac{-1/8}{s+5} + \frac{1/8}{s-3}.$

$X(s) = \left(-\frac{1}{8}\right) \cdot \frac{1}{s+5} + \left(\frac{1}{8}\right) \cdot \frac{1}{s-3}. \rightarrow$ ②.

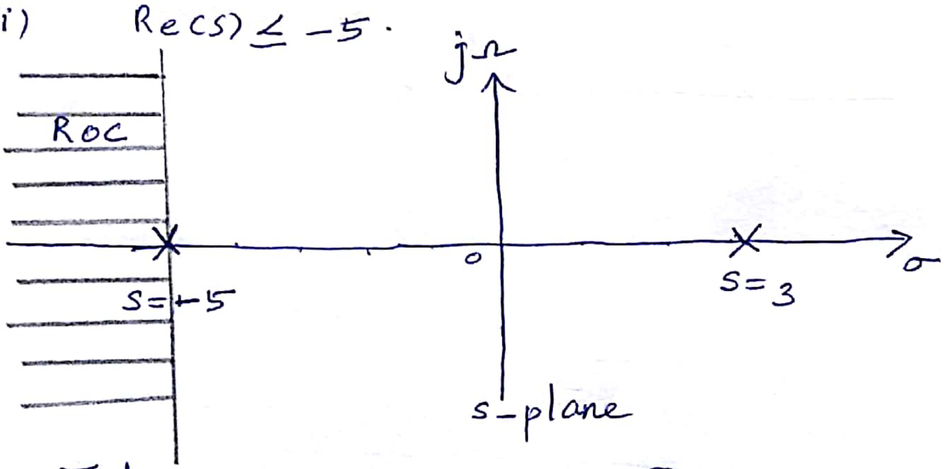
(i) Re (s) ≥ 3.



Taking Inv. LT of eqn ②,

$$\boxed{x(t) = \left(-\frac{1}{8}\right) e^{-5t} u(t) + \frac{1}{8} e^{3t} u(t)}$$

ii) Re(s) ≤ -5.

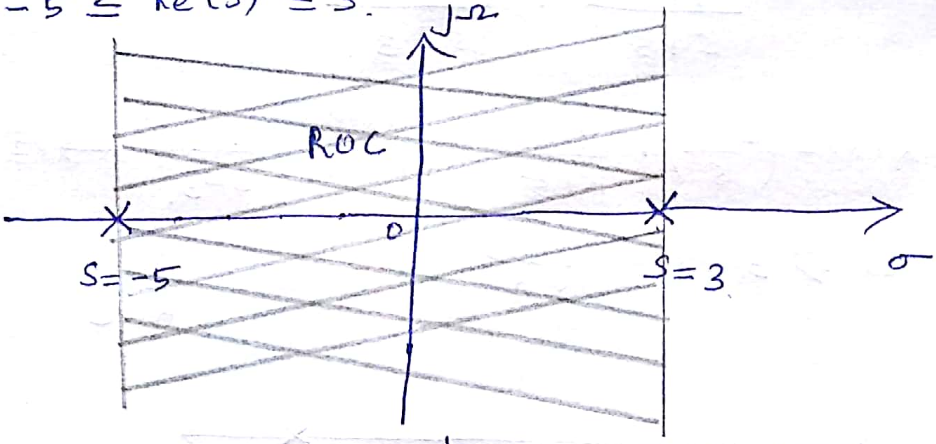


Taking Inv. LT. of eqn ②,

$x(t) = -\left(-\frac{1}{8}\right) e^{-5t} u(-t) + \frac{1}{8} e^{3t} u(t)$

$$\boxed{x(t) = \frac{1}{8} e^{-5t} u(-t) + \frac{1}{8} e^{3t} u(t).}$$

iii) $-5 \leq Re(s) \leq 3$.



s-plane.

Taking Inv. LT of eqn ②,

$$\left[\begin{array}{l} Re(s) \leq 3 \\ Re(s) \geq -5 \end{array}\right]$$

$$\boxed{x(t) = \left(-\frac{1}{8}\right) e^{-5t} u(t) - \frac{1}{8} e^{3t} u(-t).}$$

Type : complex poles.

Problems.

1) Find the Inverse Laplace Transform of $H(s) = \frac{s^2+2s+1}{(s+2)(s^2+4)}$.

soln:

$$H(s) = \frac{s^2+2s+1}{(s+2)(s^2+4)}$$

Dr. factorization:

$s^2+4 = 0$

$s^2 = -4$

$s = \sqrt{-4} = \pm j2$

$$H(s) = \frac{s^2+2s+1}{(s+2)(s-j2)(s+j2)}$$

$$H(s) = \frac{k_0}{s+2} + \frac{k_1}{s-j2} + \frac{k_2}{s+j2} \longrightarrow ①$$

$$k_0 = H(s) \times (s+2) \Big|_{s=-2}$$

$$k_0 = \frac{(s^2+2s+1) \times (s+2)}{(s+2)(s^2+4)} \Bigg|_{s=-2}$$

$$= \frac{(-2)^2+2(-2)+1}{((-2)^2+4)} = \frac{4-4+1}{4+4}$$

$$\boxed{k_0 = \frac{1}{8}}$$

$$K_1 = H(s) \times (s-j2)\Big|_{s=j2}$$

$$K_1 = \frac{s^2+2s+1}{(s+2)(s-j2)(s+j2)} \times (s-j2)\Bigg|_{s=j2}$$

$$K_1 = \frac{(j2)^2+2(j2)+1}{(j2+2)(j2+j2)} = \frac{-4+j4+1}{(j2+2)\ (j4)} = \frac{-3+j4}{-8+j8}$$

$$K_1 = \frac{-3+j4}{-8+j8} \times \frac{-8-j8}{-8-j8}$$

$$K_1 = \frac{24+j24-j32+32}{(-8)^2-(j8)^2} = \frac{56-j8}{64+64} = \frac{56-j8}{128} = \frac{8(7-j)}{128}$$

$$\boxed{K_1 = \frac{7-j}{16}}$$

$$K_2 = K_1^*$$

$$\boxed{K_2 = \frac{7+j}{16}}$$

Now, eqn① $\Rightarrow$ $H(s) = \left(\frac{1}{8}\right)\frac{1}{s+2} + \left(\frac{7-j}{16}\right)\cdot\frac{1}{s-j2} + \left(\frac{7+j}{16}\right)\cdot\frac{1}{s+j2}$

$$H(s) = \left(\frac{1}{8}\right)\frac{1}{s+2} + \frac{7}{16}\left[\frac{1}{s-j2} + \frac{1}{s+j2}\right] - \frac{j}{16}\left[\frac{1}{s-j2} - \frac{1}{s+j2}\right]$$

$$H(s) = \left(\frac{1}{8}\right)\frac{1}{s+2} + \frac{7}{16}\left[\frac{s+j2+s-j2}{(s-j2)(s+j2)}\right] - \frac{j}{16}\left[\frac{s+j2-s+j2}{(s-j2)(s+j2)}\right]$$

$$H(s) = \left(\frac{1}{8}\right)\frac{1}{s+2} + \frac{7}{16}\left[\frac{2s}{s^2+4}\right] - \frac{j}{16}\left[\frac{j4}{s^2+4}\right]$$

$$\boxed{\begin{aligned}(s+j2)(s-j2) &= s^2-(j2)^2 \\ &= s^2+4\end{aligned}}$$

w.k.T $j^2 = -1$

$$H(s) = \left(\frac{1}{8}\right)\frac{1}{s+2} + \frac{14}{16}\cdot\left(\frac{s}{s^2+4}\right) + \frac{2}{16}\left[\frac{2}{s^2+4}\right] \quad [\because j^2 = -1]$$

$$H(s) = \left(\frac{1}{8}\right)\frac{1}{s+2} + \frac{7}{8}\left[\frac{s}{s^2+2^2}\right] + \frac{1}{8}\left[\frac{2}{s^2+2^2}\right] \rightarrow ②.$$

Taking Inv. L.T of eqn ②,

$$\boxed{h(t) = \left(\frac{1}{8}\right)e^{-2t}u(t) + \left(\frac{7}{8}\right)\cos 2t\, u(t) + \left(\frac{1}{8}\right)\sin 2t\, u(t).}$$

→ Problems based on Inverse Laplace Transform:

Type : Multipoles.

Problems:

1) Find the inverse laplace transform of $X(s) = \frac{s-2}{s(s+1)^3}$

Soln:

$$X(s) = \frac{s-2}{s(s+1)^3}$$

$[(s+1)^3 \rightarrow$ Multipoles$]$
$s \rightarrow$ simple pole.

$$X(s) = \frac{A}{s} + \frac{K_0}{(s+1)^3} + \frac{K_1}{(s+1)^2} + \frac{K_2}{(s+1)} \longrightarrow ①$$

$$A = X(s).\, s \Big|_{s=0}$$

$$A = \frac{s-2}{s(s+1)^3} \times s \Big|_{s=0} = \frac{-2}{1^3} = \frac{-2}{1}$$

$$\boxed{A = -2}$$

Multipole Residue:

$$K_j = \frac{1}{j!} \frac{d^j}{ds^j} [X_1(s)] \Big|_{s=s_0}$$

where, $X_1(s) = X(s).(s-s_0)^n$

In our case, $X_1(s) = \frac{(s-2)}{s(s+1)^3} \times (s+1)^3 = \frac{s-2}{s}.$

Now,

$$K_0 = \frac{1}{0!} \frac{d^0}{ds^0} \left[\frac{s-2}{s}\right] \Bigg|_{s=-1} = \frac{s-2}{s}\Big|_{-1} = \frac{-1-2}{-1} = \frac{-3}{-1}$$

$$\boxed{K_0 = 3}$$

$[\because 0! = 1]$

$$K_1 = \frac{1}{1!}\frac{d}{ds}\left[\frac{s-2}{s}\right]\Bigg|_{s=-1} \qquad [1! = 1]$$

$$K_1 = \frac{d}{ds}\left[\frac{s-2}{s}\right]\Bigg|_{s=-1} = \frac{s(1-0)-(s-2)(1)}{s^2}\Bigg|_{s=-1}$$

$$K_1 = \frac{s-s+2}{s^2}\Bigg|_{s=-1} = \frac{2}{s^2}\Bigg|_{s=-1} = \frac{2}{(-1)^2} = \frac{2}{1}$$

$$\boxed{K_1 = 2}$$

$$K_2 = \frac{1}{2!}\frac{d^2}{ds^2}\left[\frac{s-2}{s}\right]\Bigg|_{s=-1}$$

$$[\because 2! = 2\times1 = 2]$$

$$K_2 = \frac{1}{2}\left[\frac{d}{ds}\left[\frac{s.(1-0)-(s-2)(1)}{s^2}\right]\right]\Bigg|_{s=-1}$$

$$K_2 = \frac{1}{2}\frac{d}{ds}\left[\frac{s-s+2}{s^2}\right]\Bigg|_{s=-1} = \frac{1}{2}\frac{d}{ds}\left[\frac{2}{s^2}\right]\Bigg|_{s=-1} = \frac{1}{2}\frac{d}{ds}\left[2s^{-2}\right]\Bigg|_{s=-1}$$

$$K_2 = \frac{1}{2}\times\left[2\times -2\times s^{-3}\right]\Bigg|_{-1} = \left[\frac{-2}{s^3}\right]\Bigg|_{-1} = \frac{-2}{(-1)^3} = \frac{-2}{-1}$$

$$\boxed{K_2 = 2}$$

Now,

$$① \Rightarrow X(s) = \frac{-2}{s} + \frac{3}{(s+1)^3} + \frac{2}{(s+1)^2} + \frac{2}{(s+1)} \longrightarrow ②.$$

$$\boxed{L.T[t^{n-1}e^{-at}u(t)] = \frac{(n-1)!}{(s+a)^n}}$$

$$\boxed{LT\left[\frac{t^{n-1}}{(n-1)!}e^{-at}u(t)\right] = \frac{1}{(s+a)^n}}$$

$$\boxed{LT[u(t)] = \frac{1}{s}}$$

$$\boxed{LT[e^{-at}u(t)] = \frac{1}{s+a}}$$

Taking Inverse LT of eqn②,

$$x(t) = -2u(t) + 3\times \frac{t^{3-1}}{(3-1)!} \cdot e^{-t} u(t) + 2\times \frac{t^{2-1}}{(2-1)!} e^{-t} u(t)$$

$$+ 2 . e^{-t} u(t).$$

$$\boxed{x(t) = -2u(t) + \frac{3}{2} t^2 e^{-t} u(t) + 2t\, e^{-t} u(t) + 2 e^{-t} u(t)}$$

$[\because 2! = 2\times1=2$

$1! = 1]$

2) Find the inverse laplace transform of $X(s) = \frac{s+3}{(s+1)(s+2)^2}$

Soln:

$$X(s) = \frac{s+3}{(s+1)(s+2)^2}$$

$(s+2)^2 \rightarrow$ Multipoles

$s+1 \rightarrow$ simple pole.

$$X(s) = \frac{A}{s+1} + \frac{k_0}{(s+2)^2} + \frac{k_1}{s+2} \rightarrow ①$$

$$A = X(s) . (s+1)\Big|_{s=-1}$$

$$A = \frac{(s+3)}{(s+1)(s+2)^2} . (s+1)\Big|_{s=-1} = \frac{s+3}{(s+2)^2}\Big|_{s=-1} = \frac{-1+3}{(-1+2)^2} = \frac{2}{1^2} = \frac{2}{1}$$

$$\boxed{A = 2}$$

$$k_j = \frac{1}{j!} \frac{d^n}{ds^n}\left[X_1(s)\right]\Big|_{s=s_0}$$

where,

$$X_1(s) = X(s) . (s-s_0)^n$$

Here, $X_1(s) = \frac{s+3}{(s+1)(s+2)^2} . (s+2)^2 = \frac{s+3}{s+1}.$

$$K_0 = \frac{1}{0!} \frac{d^0}{ds^0}\left[\frac{s+3}{s+1}\right]\Bigg|_{s=-2} = \left[\frac{s+3}{s+1}\right]\Bigg|_{s=-2} = \frac{-2+3}{-2+1} = \frac{1}{-1} = -1$$

$[\because 0! = 1]$

$$\boxed{K_0 = -1}$$

$$K_1 = \frac{1}{1!} \frac{d}{ds}\left[\frac{s+3}{s+1}\right]\Bigg|_{s=-2} = \frac{(s+1)(1+0)-(s+3)(1+0)}{(s+1)^2}\Bigg|_{s=-2}$$

$$K_1 = \frac{s+1-s-3}{(s+1)^2}\Bigg|_{s=-2} = \frac{-2}{(s+1)^2}\Bigg|_{s=-2} = \frac{-2}{(-2+1)^2} = \frac{-2}{(-1)^2} = \frac{-2}{1} = -2$$

$$\boxed{K_1 = -2}$$

Now, ① $\Rightarrow$ $X(s) = \frac{2}{s+1} - \frac{1}{(s+2)^2} - \frac{2}{s+2} \rightarrow$ ②

$$\boxed{\begin{aligned} LT\left[\frac{t^{n-1}}{(n-1)!} e^{-at} u(t)\right] &= \frac{1}{(s+a)^n} \\ LT\left[e^{-at} u(t)\right] &= \frac{1}{s+a} \end{aligned}}$$

Taking Inverse Laplace Transform of eqn ②,

$$x(t) = 2 \cdot e^{-t} u(t) - \frac{t^{2-1}}{(2-1)!} e^{-2t} u(t) - 2 \cdot e^{-2t} u(t)$$

$$\boxed{x(t) = 2e^{-t} u(t) - t\, e^{-2t} u(t) - 2e^{-2t} u(t).}$$ $[\because 1! = 1]$

→ Differential Equation.

The input and output characteristics of a continuous Time system is described by the differential equation.

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

a_k → output co-efficients

b_k → input co-efficients

N → order of differential equation for output $y(t)$.

M → order of differential equation for input $x(t)$.

→ System function (or) Transfer function:

i) System function:

$$H(s) = \frac{LT[y(t)]}{LT[x(t)]} = \frac{Y(s)}{X(s)}$$ [∵ using Laplace Transform]

$$H(j\Omega) = \frac{FT[y(t)]}{FT[x(t)]} = \frac{Y(j\Omega)}{X(j\Omega)}$$ [using Fourier Transform].

ii) Impulse Response : h(t).

$$h(t) = LT^{-1}[H(s)] = LT^{-1}\left[\frac{Y(s)}{X(s)}\right].$$

$$h(t) = FT^{-1}[H(j\Omega)] = FT^{-1}\left[\frac{Y(j\Omega)}{X(j\Omega)}\right]$$

iii) To find step response:

ie) To find $y(t)$ when $x(t) = u(t)$.

Procedure:

$$Y(s) = H(s) . X(s).$$

$$\boxed{y(t) = LT^{-1}[Y(s)] = LT^{-1}[H(s). X(s)]}$$

For stepresponse, $x(t) = u(t)$ & $X(s) = \frac{1}{s}$. [∵ $LT[u(t)] = \frac{1}{s}$]

→ Analysis of CT systems using Laplace Transform:

Concept:

i) $$LT\left[\frac{d^2y(t)}{dt^2}\right] = s^2Y(s) - s\,y(0^{\pm}) - \left.\frac{dy(t)}{dt}\right|_{t=0^{\pm}}$$

ii) $$LT\left[\frac{dy(t)}{dt}\right] = sY(s) - y(0^{\pm})$$

* Problems:

1) Deterime the system function for $\frac{dy(t)}{dt} + y(t) = 2x(t)$.

Soln:

$$\frac{dy(t)}{dt} + y(t) = 2x(t)$$

Taking L.T,

$$L.T\left[\frac{dy(t)}{dt} + y(t)\right] = LT\left[2x(t)\right]$$

$$sY(s) + Y(s) = 2X(s)$$

$$Y(s)[s+1] = 2X(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{2}{s+1}$$

$$\boxed{\text{System Function: } H(s) = \frac{2}{s+1}}$$

2) Determine the transfer function and impulse response for the LTI system:

a) $$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = x(t)$$

b) $$\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 8y(t) = \frac{dx(t)}{dt} + 4x(t)$$

Soln:

a) $$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = x(t).$$

$$LT\left[\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t)\right] = LT[x(t)]$$

$$s^2Y(s) + 3sY(s) + 2Y(s) = X(s)$$

$$Y(s)[s^2+3s+2] = X(s)$$

$$\boxed{H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s^2+3s+2}} \rightarrow \text{system function}$$

Factorizing the Dr. Polynomial:

s^2+3s+2: $a=1$, $b=3$, $c=2$

$$s = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-3 \pm \sqrt{3^2-4\times1\times2}}{2\times1} = \frac{-3\pm\sqrt{9-8}}{2}$$

$$s = \frac{-3\pm1}{2} = \frac{-3+1}{2}, \frac{-3-1}{2} = \frac{-2}{2}, \frac{-4}{2} = -1, -2.$$

$s=-1$ and $s=-2$ (simple poles)

$$H(s) = \frac{1}{(s+1)(s+2)}$$

$$H(s) = \frac{K_0}{s+1} + \frac{K_1}{s+2} \rightarrow ①$$

$$K_0 = H(s)\cdot(s+1)\Big|_{s=-1}$$

$$K_0 = \frac{1}{(s+1)(s+2)} \times (s+1)\Big|_{s=-1} = \frac{1}{(s+2)}\Big|_{s=-1} = \frac{1}{(-1+2)} = \frac{1}{1}$$

$$\boxed{K_0 = 1}$$

$$k_1 = H(s) \cdot (s+2)\Big|_{s=-2}$$

$$k_1 = \frac{1}{(s+1)(s+2)} \times (s+2)\Big|_{s=-2} = \frac{1}{(s+1)}\Big|_{s=-2} = \frac{1}{(-2+1)} = \frac{1}{-1}$$

$$\boxed{k_1 = -1}$$

Now,

$$① \Rightarrow H(s) = \frac{1}{s+1} - \frac{1}{s+2}$$

$$LT^{-1}[H(s)] = LT^{-1}\left[\frac{1}{s+1} - \frac{1}{s+2}\right]$$

$$\boxed{h(t) = e^{-t} u(t) - e^{-2t} u(t)} \rightarrow \text{Impulse Response}$$

b) Soln:

$$\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 8y(t) = \frac{dx(t)}{dt} + 4x(t)$$

$$LT\left[\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 8y(t)\right] = LT\left[\frac{dx(t)}{dt} + 4x(t)\right]$$

$$s^2 Y(s) + 6sY(s) + 8Y(s) = sX(s) + 4X(s)$$

$$Y(s)[s^2 + 6s + 8] = X(s)[s+4]$$

$$\boxed{H(s) = \frac{Y(s)}{X(s)} = \frac{s+4}{s^2+6s+8}} \rightarrow \text{system Function}$$

Factorizing the Dr. Polynomial:

s^2+6s+8: $a=1$, $b=6$, $c=8$

$$s=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-6\pm\sqrt{6^2-4\times1\times8}}{2\times1}=\frac{-6\pm\sqrt{36-32}}{2}$$

$$s=\frac{-6\pm\sqrt{4}}{2}=\frac{-6\pm2}{2}=\frac{-6+2}{2},\frac{-6-2}{2}=\frac{-4}{2},\frac{-8}{2}$$

$\boxed{s=-2 \text{ and } s=-4}$ → simple poles.

Now,

$$H(s)=\frac{s+4}{(s+2)(s+4)}$$

$$\boxed{H(s)=\frac{1}{s+2}}$$

$$LT^{-1}[H(s)]=LT^{-1}\left[\frac{1}{s+2}\right]$$

$\boxed{h(t)=e^{-2t}u(t)}$ → Impulse Response

3) Determine the impulse response for the following differential equation:

$$\frac{d^2y(t)}{dt^2}+4\frac{dy(t)}{dt}+3y(t)=\frac{dx(t)}{dt}+2x(t)$$

Soln:

$$LT\left[\frac{d^2y(t)}{dt^2}+4\frac{dy(t)}{dt}+3y(t)\right]=LT\left[\frac{dx(t)}{dt}+2x(t)\right]$$

$$s^2 Y(s) + 4s Y(s) + 3Y(s) = s X(s) + 2X(s)$$

$$Y(s)[s^2 + 4s + 3] = X(s)[s+2]$$

$$\boxed{H(s) = \frac{Y(s)}{X(s)} = \frac{s+2}{s^2+4s+3}} \rightarrow \text{system function}$$

Factorizing Dr. polynomial

$$s = \frac{-4 \pm \sqrt{4^2 - 4\times1\times3}}{2\times1} = \frac{-4 \pm \sqrt{16-12}}{2}$$

$$s = \frac{-4 \pm \sqrt{4}}{2} = \frac{-4\pm2}{2} = \frac{-4+2}{2}, \frac{-4-2}{2} = \frac{-2}{2}, \frac{-6}{2}$$

$\boxed{s = -1 \text{ and } s = -3} \rightarrow$ simple poles.

Now,

$$H(s) = \frac{s+2}{(s+1)(s+3)}.$$

$$H(s) = \frac{K_0}{s+1} + \frac{K_1}{s+3} \rightarrow ①$$

$$K_0 = H(s)\cdot(s+1)\Big|_{s=-1}$$

$$K_0 = \frac{(s+2)}{(s+1)(s+3)} \times (s+1)\Big|_{s=-1} = \frac{(s+2)}{(s+3)}\Big|_{s=-1} = \frac{-1+2}{-1+3} = \frac{1}{2}$$

$$\boxed{K_0 = \frac{1}{2}}$$

$$K_1 = H(s).(s+3)\Big|_{s=-3}$$

$$K_1 = \frac{s+2}{(s+1)(s+3)} \times (s+3)\Big|_{s=-3} = \frac{(s+2)}{(s+1)}\Big|_{s=-3} = \frac{-3+2}{-3+1} = \frac{-1}{-2}$$

$$\boxed{K_1 = \frac{1}{2}}$$

Now,

$$① \Rightarrow H(s) = \left(\frac{1}{2}\right)\frac{1}{(s+1)} + \left(\frac{1}{2}\right)\frac{1}{(s+3)}$$

$$LT^{-1}[H(s)] = LT^{-1}\left[\left(\frac{1}{2}\right)\frac{1}{s+1} + \left(\frac{1}{2}\right)\frac{1}{s+3}\right]$$

$$\boxed{h(t) = \frac{1}{2}e^{-t}u(t) + \frac{1}{2}e^{-3t}u(t).}$$ → Impulse Response.

4) Determine the impulse response of the following differential equation:

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 6y(t) = x(t).$$

Soln:

$$LT\left[\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 6y(t)\right] = LT[x(t)]$$

$$s^2Y(s) + 3sY(s) + 6Y(s) = X(s)$$

$$Y(s)[s^2 + 3s + 6] = X(s)$$

$$\boxed{H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s^2+3s+6}}$$ → system function

Factorizing Dr. Polynomial:

$$s = \frac{-3 \pm \sqrt{3^2 - 4\times1\times6}}{2\times1} = \frac{-3 \pm \sqrt{9-24}}{2}$$

$$s = \frac{-3 \pm \sqrt{-15}}{2} = \frac{-3 \pm j\sqrt{15}}{2} = \frac{-3}{2} + j\frac{\sqrt{15}}{2}, \frac{-3}{2} - j\frac{\sqrt{15}}{2}$$

[Complex poles]

Now,

$$H(s) = \frac{1}{\left(s+\frac{3}{2}-j\frac{\sqrt{15}}{2}\right)\left(s+\frac{3}{2}+j\frac{\sqrt{15}}{2}\right)}$$

$$H(s) = \frac{K_0}{\left(s+\frac{3}{2}-j\frac{\sqrt{15}}{2}\right)} + \frac{K_1}{\left(s+\frac{3}{2}+j\frac{\sqrt{15}}{2}\right)} \longrightarrow ①.$$

$$K_0 = \left.\frac{\left(s+\frac{3}{2}-j\frac{\sqrt{15}}{2}\right)}{\left(s+\frac{3}{2}-j\frac{\sqrt{15}}{2}\right)\left(s+\frac{3}{2}+j\frac{\sqrt{15}}{2}\right)}\right|_{s=-\frac{3}{2}+j\frac{\sqrt{15}}{2}}$$

$$K_0 = \left.\frac{1}{\left(s+\frac{3}{2}+j\frac{\sqrt{15}}{2}\right)}\right|_{s=\frac{-3}{2}+j\frac{\sqrt{15}}{2}} = \frac{1}{\left[\frac{-3}{2}+j\frac{\sqrt{15}}{2}+\frac{3}{2}+j\frac{\sqrt{15}}{2}\right]}$$

$$K_o = \frac{1}{2 \times j \frac{\sqrt{15}}{2}} = \frac{1}{j\sqrt{15}}$$

$$K_o = \frac{1}{j\sqrt{15}} \times \frac{-j}{-j} = \frac{-j}{15} \qquad [\because j \times j = j^2 = -1]$$

$$\boxed{K_o = \frac{-j}{\sqrt{15}}} \quad \text{and} \quad \boxed{K_1 = k_o^* = \frac{j}{\sqrt{15}}}$$

Now,

$$① \Rightarrow \quad H(s) = \left(\frac{-j}{\sqrt{15}}\right) \cdot \frac{1}{s+\frac{3}{2}-j\frac{\sqrt{15}}{2}} + \left(\frac{j}{\sqrt{15}}\right) \cdot \frac{1}{s+\frac{3}{2}+j\frac{\sqrt{15}}{2}}$$

$$H(s) = \frac{\left(\frac{-j}{\sqrt{15}}\right)\left[s+\frac{3}{2}+j\frac{\sqrt{15}}{2}\right] + \left[\frac{j}{\sqrt{15}}\right]\left[s+\frac{3}{2}-j\frac{\sqrt{15}}{2}\right]}{\left(\underbrace{s+\frac{3}{2}}_{a} \underbrace{-}_{-} j\underbrace{\frac{\sqrt{15}}{2}}_{b}\right)\left(\underbrace{s+\frac{3}{2}}_{a} \underbrace{+}_{+} j\underbrace{\frac{\sqrt{15}}{2}}_{b}\right)}$$

$$H(s) = \frac{-j\frac{s}{\sqrt{15}} - j\frac{3}{2\sqrt{15}} + \frac{1}{2} + j\frac{s}{\sqrt{15}} + j\frac{3}{2\sqrt{15}} + \frac{1}{2}}{\left(s+\frac{3}{2}\right)^2 - \left(j\frac{\sqrt{15}}{2}\right)^2}$$

$$H(s) = \frac{1}{\left(s+\frac{3}{2}\right)^2 + \left(\frac{\sqrt{15}}{2}\right)^2} \qquad [\because j^2 = -1]$$

$$H(s) = \frac{2}{\sqrt{15}} \times \frac{\frac{\sqrt{15}}{2}}{\left(s+\frac{3}{2}\right)^2 + \left(\frac{\sqrt{15}}{2}\right)^2}$$

$$\boxed{LT\left[e^{-at} \sin w_o t \, u(t)\right] = \frac{w_o}{(s+a)^2 + w_o^2}}$$

$$LT^{-1}[H(s)] = LT^{-1}\left[\frac{2}{\sqrt{15}} \times \frac{\frac{\sqrt{15}}{2}}{(s+\frac{3}{2})^2+(\frac{\sqrt{15}}{2})^2}\right]$$

$$\boxed{h(t) = \frac{2}{\sqrt{15}} \times e^{-\frac{3}{2}t} \sin(\frac{\sqrt{15}}{2})t \ u(t)} \rightarrow \text{Impulse Response.}$$

Tutorial work:

5) Determine the impulse response of the following differential equation:

$$\frac{d^2y(t)}{dt^2} + \frac{1}{6}\frac{dy(t)}{dt} - \frac{1}{6}y(t) = x(t).$$

Ans:

$$h(t) = \frac{6}{5}e^{\frac{1}{3}t}u(t) - \frac{6}{5}e^{-\frac{1}{2}t}u(t).$$

[Roots: $s = \frac{1}{3}$ and $s = \frac{-1}{2}$]

→ Analysis of CT systems using Laplace Transform:

1. Find the response $y(t)$ of a CT system using Laplace Transform with transfer function $H(S) = \frac{1}{(S+2)(S+3)}$ for an input $x(t) = e^{-t}u(t)$.

Soln:

$$H(S) = \frac{1}{(s+2)(s+3)}$$

$$x(t) = e^{-t}u(t)$$

$$LT[x(t)] = LT[e^{-t}u(t)]$$

$$X(s) = \frac{1}{s+1}.$$

w.k.T

$$H(S) = \frac{Y(s)}{X(s)}$$

$$Y(s) = H(s)\cdot X(s).$$

$$Y(s) = \frac{1}{(s+2)(s+3)}\cdot\frac{1}{(s+1)} = \frac{1}{(s+2)(s+3)(s+1)}.$$

$$Y(s) = \frac{K_0}{s+2} + \frac{K_1}{s+3} + \frac{K_2}{s+1} \longrightarrow ①.$$

$$K_0 = Y(s)\cdot(s+2)\Big|_{s=-2}$$

$$K_0 = \frac{1}{\cancel{(s+2)}(s+3)(s+1)} \times \cancel{(s+2)}\Big|_{s=-2}$$

$$K_0 = \frac{1}{(-2+3)(-2+1)} = \frac{1}{1(-1)} = \frac{1}{-1}$$

$$\boxed{K_0 = -1}$$

$$K_1 = Y(s).(s+3)\Big|_{s=-3}$$

$$K_1 = \frac{1}{(s+2)\cancel{(s+3)}(s+1)} \times \cancel{(s+3)}\Big|_{s=-3} = \frac{1}{(-3+2)(-3+1)} = \frac{1}{(-1)(-2)}$$

$$\boxed{K_1 = \frac{1}{2}}$$

$$k_2 = Y(s) \cdot (s+1) \Big|_{s=-1}$$

$$k_2 = \frac{1}{(s+2)(s+3)\cancel{(s+1)}} \times \cancel{(s+1)} \Big|_{s=-1}$$

$$k_2 = \frac{1}{(-1+2)(-1+3)} = \frac{1}{1(2)} = \frac{1}{2}$$

$$\boxed{k_2 = \frac{1}{2}}$$

Sub. k_0, k_1 and k_2 values in eqn. ①,

$$① \Rightarrow Y(s) = \frac{-1}{s+2} + \left(\frac{1}{2}\right) \cdot \frac{1}{s+3} + \left(\frac{1}{2}\right) \cdot \frac{1}{s+1}$$

Taking Inv. L.T,

$$LT^{-1}[Y(s)] = LT^{-1}\left[-\frac{1}{s+2} + \left(\frac{1}{2}\right)\frac{1}{s+3} + \left(\frac{1}{2}\right) \cdot \frac{1}{s+1}\right]$$

$$\boxed{y(t) = -e^{-2t}u(t) + \frac{1}{2}e^{-3t}u(t) + \frac{1}{2}e^{-t}u(t)}$$

2) By using Laplace Transform, solve the following differential equation: $\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = \frac{dx(t)}{dt}$ where

$y(0^-) = 2$; $\frac{dy(t)}{dt}\Big|_{t=0^-} = 1$ and $x(t) = e^{-t}u(t)$.

Soln:

$$x(t) = e^{-t}u(t)$$

$$LT[x(t)] = LT[e^{-t}u(t)]$$

$$X(s) = \frac{1}{s+1} \rightarrow ①$$

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = \frac{dx(t)}{dt}$$

$$LT\left[\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t)\right] = LT\left[\frac{dx(t)}{dt}\right]$$

$$s^2Y(s) - sy(0^-) - \left.\frac{dy(t)}{dt}\right|_{t=0^-} + 3\left[sY(s) - y(0^-)\right] + 2Y(s)$$

$$= sX(s)$$

$$s^2Y(s) - 2s - 1 + 3sY(s) - 3(2) + 2Y(s) = sX(s).$$

$$Y(s)\left[s^2 + 3s + 2\right] - 2s - 7 = sX(s)$$

$$Y(s)\left[s^2 + 3s + 2\right] = sX(s) + 2s + 7$$

$$Y(s)\left[s^2 + 3s + 2\right] = \frac{s}{(s+1)} + 2s + 7 \quad [\because \text{using eqn } ①]$$

$$Y(s)\left[s^2 + 3s + 2\right] = \frac{s + 2s(s+1) + 7(s+1)}{(s+1)}$$

$$Y(s)\left[s^2 + 3s + 2\right] = \frac{s + 2s^2 + 2s + 7s + 7}{(s+1)}$$

$$Y(s)\left[s^2 + 3s + 2\right] = \frac{2s^2 + 10s + 7}{(s+1)}$$

$$\boxed{Y(s) = \frac{2s^2 + 10s + 7}{(s+1)(s^2 + 3s + 2)}} \longrightarrow ②.$$

Factorizing the Dr. Polynomial:

$s^2 + 3s + 2$: $\quad a = 1, \; b = 3, \; c = 2$

$$s = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-3 \pm \sqrt{3^2 - 4 \times 1 \times 2}}{2 \times 1} = \frac{-3 \pm \sqrt{9-8}}{2}$$

$$s = \frac{-3 \pm \sqrt{1}}{2} = \frac{-3 \pm 1}{2} = \frac{-3+1}{2}, \frac{-3-1}{2} = -\frac{2}{2}, -\frac{4}{2} = -1, -2$$

$s = -1$ and $s = -2$.

② ⇒ $Y(s) = \dfrac{2s^2 + 10s + 7}{(s+1)(s+1)(s+2)}$

$Y(s) = \dfrac{2s^2+10s+7}{(s+2)(s+1)^2}$ [simple pole & Multipoles type]

$Y(s) = \dfrac{A}{s+2} + \dfrac{k_0}{(s+1)^2} + \dfrac{k_1}{s+1} \longrightarrow$ ③.

$A = Y(s) \cdot (s+2) \Big|_{s=-2}$

$A = \dfrac{2s^2+10s+7}{\cancel{(s+2)}(s+1)^2} \times \cancel{(s+2)} \Big|_{s=-2}$

$A = \dfrac{2(-2)^2+10(-2)+7}{(-2+1)^2} = \dfrac{2(4)-20+7}{(-1)^2} = \dfrac{8-20+7}{1} = -\dfrac{5}{1}$

$\boxed{A = -5}$

Multipole type residue:

$K_j = \dfrac{1}{j!} \dfrac{d^j}{ds^j}\left(Y_1(s)\right)\Big|_{s=s_0}$

$Y_1(s) = Y(s) \cdot (s-s_0)^n$

Here,

$Y_1(s) = \dfrac{(2s^2+10s+7)}{(s+2)(s+1)^2} \times (s+1)^2$

$Y_1(s) = \dfrac{2s^2+10s+7}{s+2}$

Now,

$$k_0 = \frac{1}{0!}\frac{d^0}{ds^0}\left[\frac{2s^2+10s+7}{s+2}\right]\Bigg|_{s=-1} \qquad \left[\because 0! = 1 \quad \frac{d^0}{ds^0} = 1\right]$$

$$k_0 = \frac{2(-1)^2+10(-1)+7}{(-1+2)}$$

$$k_0 = \frac{2(1)-10+7}{1} = 2-10+7 = -1$$

$$\boxed{k_0 = -1}$$

$$k_1 = \frac{1}{1!}\frac{d^1}{ds^1}\left[\frac{2s^2+10s+7}{s+2}\right]\Bigg|_{s=-1}$$

$$k_1 = \frac{(s+2)[4s+10]-[2s^2+10s+7][1+0]}{(s+2)^2}\Bigg|_{s=-1}$$

$$k_1 = \frac{(-1+2)[4(-1)+10]-[2(-1)^2+10(-1)+7]}{(-1+2)^2}$$

$$k_1 = \frac{1[6]-[2-10+7]}{(1)^2} = \frac{6+1}{1} = 7$$

$\boxed{k_1 = 7}$ sub. A, k_0 and k_1 values in eqn ③,

$$③ \Rightarrow Y(s) = (-5)\frac{1}{s+2} - \frac{1}{(s+1)^2} + 7 \cdot \frac{1}{s+1}.$$

Taking Inv. L.T,

$$\boxed{y(t) = -5 \cdot e^{-2t} u(t) - t e^{-t} u(t) + 7 \cdot e^{-t} u(t).}$$

$$\left[\because LT\left[\frac{t^{n-1}}{(n-1)!} e^{-at} u(t)\right] = \frac{1}{(s+a)^n}\right]$$

3) Find the step response of the system whose impulse response is given as: $h(t) = u(t+1) - u(t-1)$].

Soln:

$$h(t) = u(t+1) - u(t-1).$$

$$LT[h(t)] = LT[u(t+1) - u(t-1)]$$

$$H(s) = e^{s} \cdot \frac{1}{s} - e^{-s} \cdot \frac{1}{s} \cdot$$ [∵ using time shifting prop. of Laplace Transform]

$$\boxed{H(s) = \frac{e^{s}}{s} - \frac{e^{-s}}{s}} \rightarrow ①$$

For step response, the i/p $x(t)$ is $u(t)$.

$$x(t) = u(t)$$

$$LT[x(t)] = LT[u(t)]$$

$$\boxed{X(s) = \frac{1}{s}} \rightarrow ②$$

W.k.T, $H(s) = \frac{Y(s)}{X(s)}$

$$Y(s) = H(s) \cdot X(s).$$

$$Y(s) = \left[\frac{e^{s}}{s} - \frac{e^{-s}}{s}\right] \cdot \left[\frac{1}{s}\right]$$ [∵ using eqn ① & ②]

$$\boxed{Y(s) = \frac{e^{s}}{s^2} - \frac{e^{-s}}{s^2}}$$

Taking Inv. L.T,

$$LT^{-1}[Y(s)] = LT^{-1}\left[\frac{e^{s}}{s^2} - \frac{e^{-s}}{s^2}\right]$$

$LT\left[\frac{t^{n-1}}{(n-1)!} u(t)\right] = \frac{1}{s^n}$ Time shifting prop. $LT[x(t-t_0)] = e^{-st_0} X(s)$	Here, $x(t) = u(t)$ $X(s) = \frac{1}{s}$

$$\boxed{y(t) = (t+1)u(t+1) - (t-1)u(t-1)}$$

4) Find the step response of the system whose impulse response is given by

$h(t) = u(t+2) - u(t-2)$

Soln:

$$h(t) = u(t+2) - u(t-2)$$

$$LT[h(t)] = LT[u(t+2) - u(t-2)]$$

$$\boxed{H(s) = \frac{e^{2s}}{s} - \frac{e^{-2s}}{s}} \rightarrow ①$$ [∵ Time shifting prop. of L.T]

For step response, the input $x(t)$ is $u(t)$.

$$x(t) = u(t)$$

$$LT[x(t)] = LT[u(t)]$$

$$\boxed{X(s) = \frac{1}{s}} \rightarrow ②$$

W.k.T, $H(s) = \frac{Y(s)}{X(s)}$

$$Y(s) = H(s) \cdot X(s)$$

$$Y(s) = \left[\frac{e^{2s}}{s} - \frac{e^{-2s}}{s}\right] \cdot \left[\frac{1}{s}\right]$$ [∵ using eqn ① & ②]

$$\boxed{Y(s) = \frac{e^{2s}}{s^2} - \frac{e^{-2s}}{s^2}}$$

Taking Inv-LT,

$$LT^{-1}[Y(s)] = LT^{-1}\left[\frac{e^{2s}}{s^2} - \frac{e^{-2s}}{s^2}\right].$$

$$\boxed{y(t) = (t+2)u(t+2) - (t-2)u(t-2)}$$

→ Analysis of CT systems using Laplace Transform

Problems:

V.Q 1) The LTI system is described by the differential equation $\frac{d^2y(t)}{dt^2} - \frac{dy(t)}{dt} - 2y(t) = x(t)$. obtain the impulse response for the following.

i) causal system ii) stable system iii) Neither causal nor stable system.

Soln:

$$\frac{d^2y(t)}{dt^2} - \frac{dy(t)}{dt} - 2y(t) = x(t)$$

Taking LT of the above eqn.,

$$LT\left[\frac{d^2y(t)}{dt^2} - \frac{dy(t)}{dt} - 2y(t)\right] = LT[x(t)]$$

$$s^2Y(s) - sY(s) - 2Y(s) = X(s)$$

$$Y(s)[s^2 - s - 2] = X(s)$$

$$\boxed{H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s^2 - s - 2}} \longrightarrow ①$$

Factorizing the Dr. polynomial

$s^2 - s - 2$: $a = 1$, $b = -1$, $c = -2$

$$s = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{1 \pm \sqrt{(-1)^2 - 4 \times 1 \times -2}}{2 \times 1} = \frac{1 \pm \sqrt{1+8}}{2} = \frac{1 \pm \sqrt{9}}{2}$$

$$s = \frac{1 \pm 3}{2} = \frac{1+3}{2}, \frac{1-3}{2} = \frac{4}{2}, \frac{-2}{2} = 2, -1$$

$s = 2$ and $s = -1$

① ⇒ $H(s) = \frac{1}{(s-2)(s+1)}$

$H(s) = \frac{k_0}{s-2} + \frac{k_1}{s+1} \longrightarrow ②$

$k_0 = H(s)\cdot(s-2)\Big|_{s=2}$

$k_0 = \frac{1}{\cancel{(s-2)}(s+1)} \times \cancel{(s-2)}\Big|_{s=2} = \frac{1}{2+1} = \frac{1}{3}$

$\boxed{k_0 = \frac{1}{3}}$

$k_1 = H(s)\cdot(s+1)\Big|_{s=-1}$

$k_1 = \frac{1}{(s-2)\cancel{(s+1)}} \times \cancel{(s+1)}\Big|_{s=-1} = \frac{1}{-1-2} = -\frac{1}{3}$

$\boxed{k_1 = -\frac{1}{3}}$

sub. ko & k_1 values in eqn ②,

② ⇒ $H(s) = \left(\frac{1}{3}\right)\cdot\frac{1}{s-2} - \left(\frac{1}{3}\right)\cdot\frac{1}{s+1} \longrightarrow ③$.

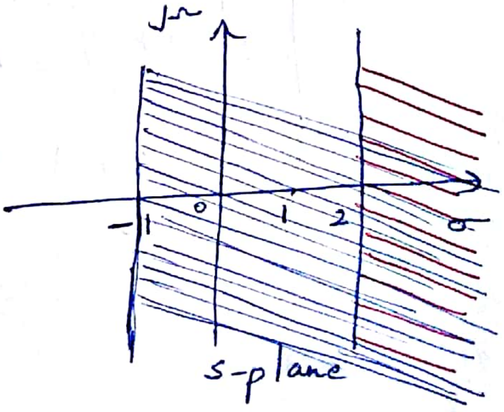
i) causal system :

For the causal system, Roc must lie on Right Hand side of the poles.

i.e, Required Roc : $Re(s) > 2$ & $Re(s) > -1$.

Now, Taking Inv. L.T of eqn ③,

$$\boxed{h(t) = \frac{1}{3}\cdot e^{2t}\,u(t) - \frac{1}{3}e^{-t}\,u(t)}$$



[common area (or) overlappled area lies on RHS of s-plane]

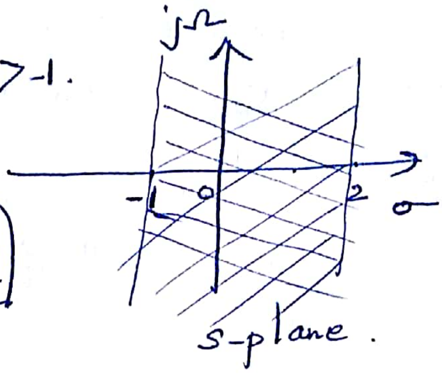
ii) Stable System:

For stable system, ROC must include the 'jΩ' axis.

∴ Required Roc: Re(S) < 2, Re(S) > -1.

Now, Taking Inv. L.T of eqn ③,

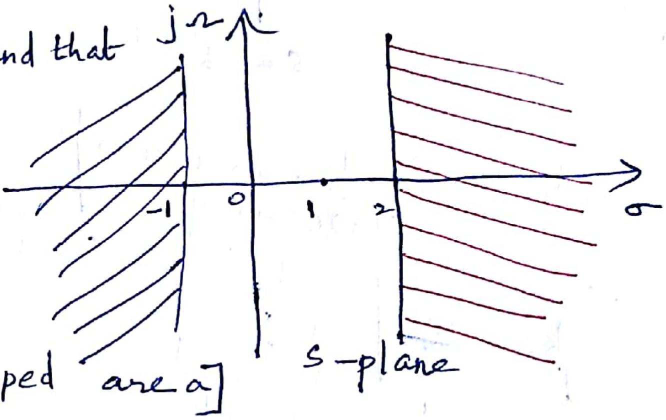
$$\boxed{h(t) = -\left(\frac{1}{3}\right) e^{+2t} u(-t) - \left(\frac{1}{3}\right) e^{-t} u(t)}$$



iii) Neither causal nor stable stable system:

∴ Required Roc: Re(S) > 2 & Re(s) < -1

From the diagram, we found that Roc does not include jΩ axis. Hence, the given system is not stable. Also, Roc does not lies on RHS of s-plane. [no overlapped area]



Taking Inv. LT of eqn ③,

$$\boxed{h(t) = \left(\frac{1}{3}\right) e^{2t} u(t) + \left(\frac{1}{3}\right) e^{-t} u(-t)}$$

U.Q 2) A system has transfer function $H(s) = \frac{3s-1}{(s+3)(s-1)}$. Find the impulse response, assuming the system is stable and the system is causal.

Soln:

$$H(s) = \frac{3s-1}{(s+3)(s-1)}$$

$$H(s) = \frac{k_0}{s+3} + \frac{k_1}{s-1} \longrightarrow ①$$

$$k_0 = H(s)\cdot(s+3)\Big|_{s=-3}$$

$$k_0 = \frac{3s-1}{\cancel{(s+3)}(s-1)} \times \cancel{(s+3)}\Big|_{s=-3} = \frac{3(-3)-1}{(-3-1)} = \frac{-9-1}{-4} = \frac{-10}{-4}$$

$$\boxed{k_0 = \frac{5}{2}.}$$

$$k_1 = H(s)\cdot(s-1)\Big|_{s=1}$$

$$k_1 = \frac{3s-1}{(s+3)\cancel{(s-1)}} \times \cancel{(s-1)}\Big|_{s=1} = \frac{3(1)-1}{(1+3)} = \frac{3-1}{4} = \frac{2}{4} = \frac{1}{2}$$

$$\boxed{k_1 = \frac{1}{2}}$$

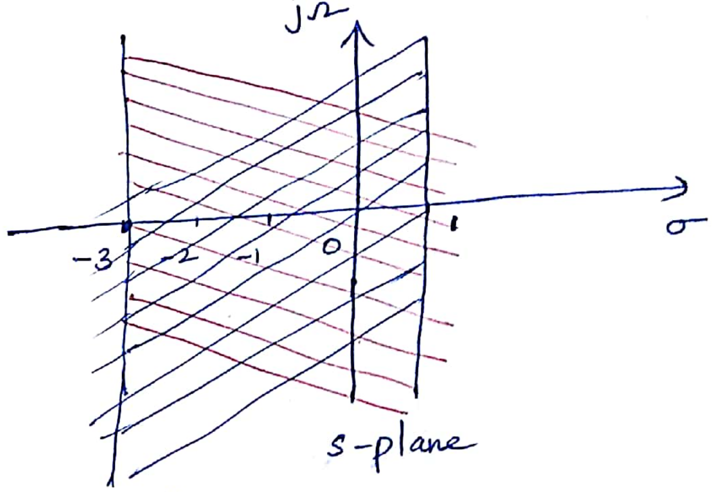
Sub. k_0 and k_1 values in eqn. ①,

$$① \Rightarrow H(s) = \left(\frac{5}{2}\right)\cdot\frac{1}{s+3} + \left(\frac{1}{2}\right)\frac{1}{s-1} \longrightarrow ②.$$

i) For stable system:

RoC must include '$j\Omega$' axis in the s-plane.

∴ Required RoC : $Re(s) > -3$ & $Re(s) < 1$

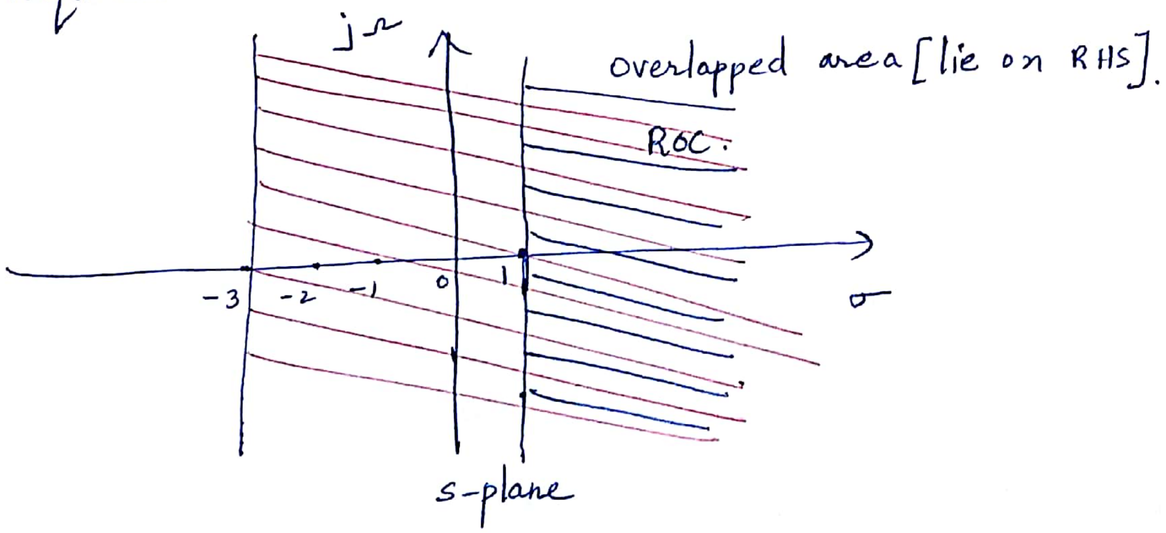


Now, taking Inv. L.T of eqn ②,

$$\boxed{h(t) = \left(\frac{5}{2}\right) e^{-3t} u(t) - \frac{1}{2} e^{t} u(-t)}$$

ii) For causal system,
Ros must lie on Right Hand side of 's'-plane.

∴ Required RoC : Re(s) > -3 & Re(s) > 1



Taking Inv. L.T of eqn. ②,

② ⇒ $$\boxed{h(t) = \frac{5}{2} e^{-3t} u(t) + \left(\frac{1}{2}\right) e^{t} u(t)}$$

U.Q
Part C

3) consider an LTI system with input $x(t) = e^{-t} u(t)$ and impulse response $h(t) = e^{-2t} u(t)$.

i) Determine L.T of $x(t)$ & $h(t)$.

ii) Verify convolution property. Determine the output $Y(s)$.

iii) From L.T of $y(t)$ as obtained in part (ii) determine $y(t)$.

iv) Verify your result in part (ii) by explicitly convolving $x(t)$ and $h(t)$.

Soln:

$x(t) = e^{-t} u(t) \rightarrow$ causal signal

$h(t) = e^{-2t} u(t). \rightarrow$ causal signal.

i) $LT[x(t)] = LT[e^{-t} u(t)]$

$$\boxed{X(s) = \frac{1}{s+1}}$$

$LT[h(t)] = LT[e^{-2t} u(t)]$

$$\boxed{H(s) = \frac{1}{s+2}}$$

ii) $LT[y(t)] = LT[x(t) * h(t)]$

$$Y(s) = X(s) \cdot H(s)$$

$$\boxed{Y(s) = \frac{1}{(s+1)(s+2)}}$$

iii) $$Y(s) = \frac{k_0}{s+1} + \frac{k_1}{s+2} \rightarrow ①$$

$$k_0 = Y(s) \cdot (s+1) \Big|_{s=-1}$$

$$K_0 = \frac{1}{\cancel{(s+1)}(s+2)} \times \cancel{(s+1)}\Big|_{s=-1} = \frac{1}{(-1+2)} = \frac{1}{1}$$

$$\boxed{K_0 = 1}$$

$$K_1 = Y(s) \times (s+2)\Big|_{s=-2}$$

$$k_1 = \frac{1}{(s+1)\cancel{(s+2)}} \times \cancel{(s+2)}\Big|_{s=-2} = \frac{1}{(-2+1)} = \frac{1}{-1} = -1$$

$$\boxed{k_1 = -1}$$

Now,

$$① \Rightarrow Y(s) = \frac{1}{s+1} - \frac{1}{s+2}$$

Taking Inv. Lap. Transf.,

$$\boxed{y(t) = e^{-t} u(t) - e^{-2t} u(t)} \longrightarrow ②.$$

iv) Verifying the eqn ② using convolution integral formula:

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau.$$

Here, Both the signals are causal. So, the integration limit is '0' to 't'.

$$y(t) = \int_{0}^{t} x(\tau) . h(t-\tau)\, d\tau.$$

$$\boxed{\begin{aligned} x(\tau) &= e^{-\tau} u(\tau) \\ h(t-\tau) &= e^{-2(t-\tau)} u(t-\tau) \end{aligned}}$$

Now,

$$y(t) = \int_0^t e^{-\tau} \cdot e^{-2(t-\tau)} \, d\tau$$

$$= e^{-2t} \int_0^t e^{-\tau} \cdot e^{2\tau} \, d\tau$$

$$= e^{-2t} \int_0^t e^{\tau} \, d\tau$$

$$y(t) = e^{-2t} \left[\frac{e^{\tau}}{1} \right]_0^t = e^{-2t} \left(e^{t} - e^{0} \right) = e^{-2t} (e^{t} - 1)$$

$$y(t) = e^{-t} - e^{-2t}$$

Ans:

$$y(t) = e^{-t} u(t) - e^{-2t} u(t) \rightarrow ③$$

Thus the results ② and ③ are verified.

4) An LTI system is represented by $\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + by(t) = \frac{dx(t)}{dt}$ with initial conditions $y(0)=1$, $y'(0)=3$. conclude the output of the system, when the input is $x(t)= u(t)$.

Soln:

$$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + by(t) = \frac{dx(t)}{dt}$$

Taking L.T,

$$s^2Y(s) - s\,y(0^-) - \left.\frac{dy(t)}{dt}\right|_{t=0^-} + 5[sY(s) - y(0^-)] + bY(s) = sX(s)$$

$$s^2Y(s) - s.1 - 3 + 5[sY(s) - 1] + 6Y(s) = sX(s).$$

$$Y(s)[s^2+5s+6] - s - 3 - 5 = sX(s).$$

$$Y(s)[s^2+5s+6] = sX(s) + s + 8 \longrightarrow ①$$

given

$x(t) = u(t)$

$X(s) = \frac{1}{s}$

$$Y(s)[s^2+5s+6] = s.\left(\frac{1}{s}\right) + s + 8$$

$$Y(s)[s^2+5s+6] = s+9$$

$$Y(s) = \frac{s+9}{s^2+5s+6}$$

$$y(s) = \frac{s+9}{(s+2)(s+3)}$$

$$y(s) = \frac{K_0}{s+2} + \frac{K_1}{s+3} \longrightarrow ①$$

$$K_0 = y(s).(s+2)\Big|_{s=-2} = \frac{s+9}{s+3}\Big|_{s=-2} = \frac{-2+9}{-2+3}$$

$$K_0 = \frac{7}{1} = 7$$

$$\boxed{K_0 = 7}$$

$$K_1 = y(s).(s+3)\Big|_{s=-3} = \frac{s+9}{s+2}\Big|_{s=-3} = \frac{-3+9}{-3+2} = \frac{6}{-1} = -6.$$

$$\boxed{K_1 = -6}$$

Eqn. ① $\Rightarrow$ $y(s) = \frac{7}{s+2} - \frac{6}{s+3}.$

Taking Inv. L.T,

$$\boxed{y(t) = 7 \cdot e^{-2t} u(t) - 6 e^{-3t} u(t)}$$

Fourier Transform Analysis in Continuous Time system:

Problems:

V.Q 1) A causal LTI system having a frequency response $H(j\Omega) = \frac{1}{j\Omega+3}$ is producing an output $y(t) = e^{-3t}u(t) - e^{-4t}u(t)$ for particular input $x(t)$. Justify $x(t)$.

Soln:

$$y(t) = e^{-3t}u(t) - e^{-4t}u(t)$$

Taking Fourier Transform,

$$Y(j\Omega) = \frac{1}{j\Omega+3} - \frac{1}{j\Omega+4}$$

$$Y(j\Omega) = \frac{j\Omega+4-j\Omega-3}{(j\Omega+3)(j\Omega+4)}$$

$$\boxed{Y(j\Omega) = \frac{1}{(j\Omega+3)(j\Omega+4)}}$$

Given: $H(j\Omega) = \frac{1}{j\Omega+3}$

W.K.T,

$$H(j\Omega) = \frac{Y(j\Omega)}{X(j\Omega)}$$

$$X(j\Omega) = \frac{Y(j\Omega)}{H(j\Omega)} = \frac{1}{(j\Omega+3)(j\Omega+4)} \times (j\Omega+3).$$

$$\boxed{X(j\Omega) = \frac{1}{j\Omega+4}.}$$

Taking Inverse Fourier Transform,

$$\boxed{x(t) = e^{-4t}u(t)}$$

2) Find the magnitude spectrum of Fourier Transform and Plot it, where $H(j\omega) = \frac{(1+2e^{-j\omega})}{(1+\frac{1}{2}e^{-j\omega})}$

V.Q

Soln:

$$H(j\omega) = \frac{(1+2e^{-j\omega})}{(1+0.5\,e^{-j\omega})}$$

$$= \frac{1+2\,[\cos\omega - j\sin\omega]}{1+0.5\,[\cos\omega - j\sin\omega]}$$

$$= \frac{1+2\cos\omega - j\,2\sin\omega}{1+0.5\cos\omega - j\,0.5\sin\omega} \times \frac{1+0.5\cos\omega + j\,0.5\sin\omega}{1+0.5\cos\omega + j\,0.5\sin\omega}$$

$$= \frac{1+0.5\cos\omega + j0.5\sin\omega + 2\cos\omega + \cos^2\omega + \cancel{j\cos\omega\sin\omega} - j2\sin\omega - \cancel{j\cos\omega\sin\omega} + \sin^2\omega}{(1+0.5\cos\omega)^2 - (j\,0.5\sin\omega)^2}$$

$$= \frac{2+2.5\cos\omega - j\,1.5\sin\omega}{1+\cos\omega + 0.25\cos^2\omega + 0.25\sin^2\omega}$$

$$= \frac{2+2.5\cos\omega - j\,1.5\sin\omega}{1.25+\cos\omega}$$

$$H(j\omega) = \frac{2+2.5\cos\omega}{1.25+\cos\omega} - j\,\frac{1.5\sin\omega}{1.25+\cos\omega}.$$

* Magnitude Spectrum:

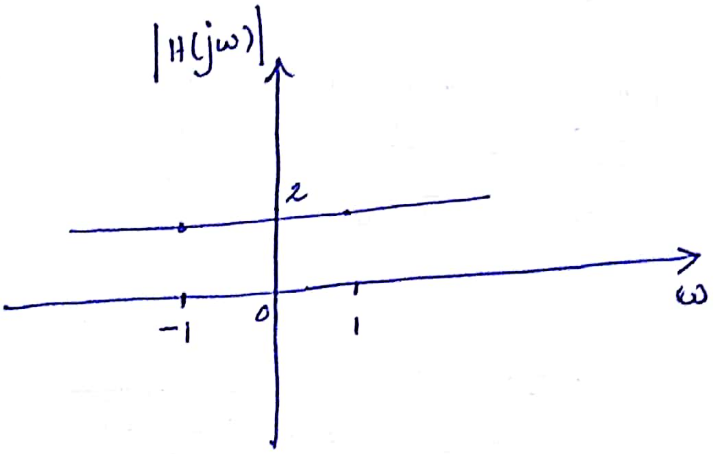
$|H(j\omega)|$ Vs ω

$$|H(j\omega)| = \left|\frac{2+2.5\cos\omega}{1.25+\cos\omega} - j\,\frac{1.5\sin\omega}{1.25+\cos\omega}\right|$$

$$= \sqrt{\left(\frac{2+2.5\cos w}{1.25+\cos w}\right)^2 + \left(\frac{-1.5\sin w}{1.25+\cos w}\right)^2}$$

$$|H(j\omega)| = \frac{\sqrt{(2+2.5\cos w)^2 + (1.5\sin w)^2}}{1.25+\cos w}$$

ω	$\lvert H(j\omega)\rvert$
-1	2
0	2
1	2



3) Find and plot the magnitude spectrum of the transfer function $H(j\omega) = \dfrac{e^{j\omega} + \alpha}{e^{j\omega} + \frac{1}{\alpha}}$.

Soln:

$$H(j\omega) = \frac{e^{j\omega} + \alpha}{e^{j\omega} + \frac{1}{\alpha}}$$

$$= \frac{\cos\omega + j\sin\omega + \alpha}{\cos\omega + j\sin\omega + \frac{1}{\alpha}}$$

$$= \frac{\alpha + \cos\omega + j\sin\omega}{\left(\frac{1}{\alpha} + \cos\omega\right) + j\sin\omega} \times \frac{\left(\frac{1}{\alpha} + \cos\omega\right) - j\sin\omega}{\left(\frac{1}{\alpha} + \cos\omega\right) - j\sin\omega}$$

$$= \frac{1 + \alpha\cos\omega - j\alpha\sin\omega + \frac{1}{\alpha}\cos\omega + \cos^2\omega \cancel{- j\sin\omega\cos\omega} + j\frac{1}{\alpha}\sin\omega \cancel{+ j\sin\omega\cos\omega} + \sin^2\omega}{\left(\frac{1}{\alpha} + \cos\omega\right)^2 - (j\sin\omega)^2}$$

$$= \frac{2 + \left(\alpha + \frac{1}{\alpha}\right)\cos\omega - j\left(\alpha - \frac{1}{\alpha}\right)\sin\omega}{\frac{1}{\alpha^2} + \frac{2}{\alpha}\cos\omega + \underbrace{\cos^2\omega + \sin^2\omega}_{1}}$$

$$H(j\omega) = \frac{2 + \left(\alpha + \frac{1}{\alpha}\right)\cos\omega}{1 + \frac{1}{\alpha^2} + \frac{2}{\alpha}\cos\omega} - j\,\frac{\left(\alpha - \frac{1}{\alpha}\right)\sin\omega}{1 + \frac{1}{\alpha^2} + \frac{2}{\alpha}\cos\omega}$$

Let $\alpha = 2$

$$H(j\omega) = \frac{2 + \frac{5}{2}\cos\omega}{1 + \frac{1}{4} + \cos\omega} - j\,\frac{\frac{3}{2}\sin\omega}{1 + \frac{1}{4} + \cos\omega}$$

$$H(j\omega) = \frac{2 + 2.5\cos\omega}{1.25 + \cos\omega} - j\,\frac{1.5\sin\omega}{1.25 + \cos\omega}$$

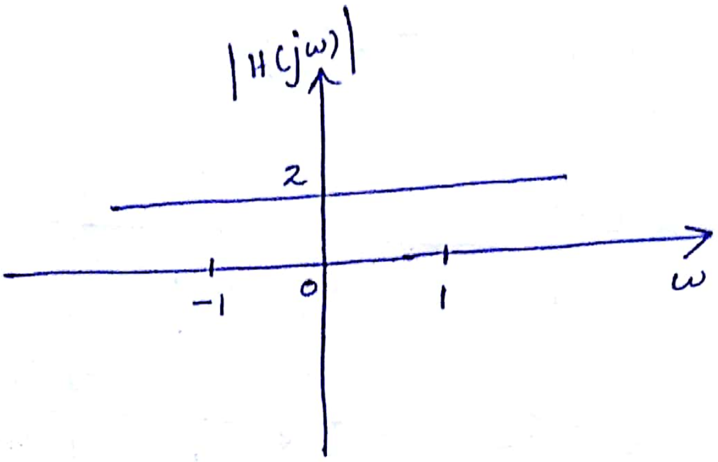
* Magnitude Spectrum:

$|H(j\omega)|$ Vs ω

$$|H(j\omega)| = \sqrt{\left(\frac{2+2.5\cos\omega}{1.25+\cos\omega}\right)^2 + \left(\frac{-1.5\sin\omega}{1.25+\cos\omega}\right)^2}$$

$$|H(j\omega)| = \frac{\sqrt{(2+2.5\cos\omega)^2 + (1.5\sin\omega)^2}}{1.25\cos\omega}$$

ω	$\lvert H(j\omega)\rvert$
-1	2
0	2
1	2



Placement question:

1. Which among the following is a memory less system?
   - a) capacitor.
   - b) Resistor
   - c) counter
   - d) FlipFlops

Ans: c) Resistor

⇒Problems based on Analysis of CT systems using Fourier Transform:

1) The input and output of a causal LTI system are related by the differential equation $\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 8y(t) = 2x(t)$. Using Fourier Transform.

i) Find the impulse response of the system

ii) Find the response of the system if $x(t) = te^{-2t}u(t)$.

Soln:

$$\frac{d^2y(t)}{dt} + 6\frac{dy(t)}{dt} + 8y(t) = 2x(t)$$

Taking Fourier Transform,

$$(j\Omega)^2 Y(j\Omega) + 6\cdot(j\Omega)Y(j\Omega) + 8Y(j\Omega) = 2X(j\Omega)$$

$$Y(j\Omega)[(j\Omega)^2 + 6j\Omega + 8] = 2X(j\Omega).$$

$$H(j\Omega) = \frac{Y(j\Omega)}{X(j\Omega)} = \frac{2}{(j\Omega)^2 + 6j\Omega + 8}$$

Factorizing the Dr. Polynomial:

$$H(j\Omega) = \frac{2}{(j\Omega+4)(j\Omega+2)} \longrightarrow Ⓐ$$

$$H(j\Omega) = \frac{K_0}{j\Omega+4} + \frac{K_1}{j\Omega+2} \longrightarrow ①$$

$$K_0 = H(j\Omega) \times (j\Omega+4)\Big|_{j\Omega=-4}$$

$$K_0 = \frac{2}{\cancel{(j\Omega+4)}(j\Omega+2)} \times \cancel{(j\Omega+4)}\Big|_{j\Omega=-4} = \frac{2}{-4+2} = \frac{2}{-2} = -1$$

$$\boxed{K_0 = -1}$$

$$k_1 = H(j\Omega) \times (j\Omega+2) \Big|_{j\Omega=-2}$$

$$= \frac{2}{(j\Omega+4)\cancel{(j\Omega+2)}} \times \cancel{(j\Omega+2)} \Big|_{j\Omega=-2}$$

$$= \frac{2}{-2+4} = \frac{2}{2} = 1$$

$$\boxed{k_1 = 1}$$

Sub. k_0 and k_1 values in eqn ①,

$$① \Rightarrow H(j\Omega) = -\frac{1}{j\Omega+4} + \frac{1}{j\Omega+2}.$$

Taking Inverse Fourier Transform,

$$\boxed{h(t) = -e^{-4t}u(t) + e^{-2t}u(t)} \rightarrow \text{Impulse Response.}$$

* To find the response $y(t)$ for the input $x(t) = te^{-2t}u(t)$.

$$x(t) = te^{-2t}u(t)$$

$$FT[x(t)] = F.T[te^{-2t}u(t)].$$

$$\boxed{\frac{t^{n-1}}{(n-1)!} e^{-at} u(t) \xrightarrow{FT} \frac{1}{(j\Omega+a)^n}}$$

$$X(j\Omega) = \frac{1}{(j\Omega+2)^2} \longrightarrow Ⓑ$$

W.K.T,

$$H(j\Omega) = \frac{Y(j\Omega)}{X(j\Omega)}$$

$$Y(j\Omega) = H(j\Omega) \cdot X(j\Omega)$$

$$Y(j\Omega) = \frac{2}{(j\Omega+4)(j\Omega+2)} \cdot \frac{1}{(j\Omega+2)^2}$$ [using eqn. Ⓐ & Ⓑ]

$$Y(j\Omega) = \frac{2}{(j\Omega+4)(j\Omega+2)^3} \longrightarrow \text{Ⓒ}.$$

$$Y(j\Omega) = \frac{A}{j\Omega+4} + \frac{k_0}{(j\Omega+2)^3} + \frac{k_1}{(j\Omega+2)^2} + \frac{k_2}{(j\Omega+2)} \longrightarrow \text{②}.$$

$$A = Y(j\Omega) \times (j\Omega+4) \Big|_{j\Omega=-4}$$

$$A = \frac{2}{\cancel{(j\Omega+4)}(j\Omega+2)^3} \times \cancel{(j\Omega+4)} \Big|_{j\Omega=-4}$$

$$A = \frac{2}{(-4+2)^3} = \frac{2}{(-2)^3} = \frac{2}{-8} = -\frac{1}{4}.$$

$$\boxed{A = -\frac{1}{4}}$$

Multipole residue:

$$k_j = \frac{1}{j!} \frac{d^j}{d(j\Omega)^j} \left[Y_1(j\Omega) \right] \Big|_{j\Omega = j\Omega_0}$$

where,

$$Y_1(j\Omega) = Y(j\Omega) \cdot (j\Omega - j\Omega_0)^n$$

Here,

$$Y_1(j\Omega) = \frac{2}{j\Omega+4}.$$

Now,

$$k_0 = \frac{1}{0!} \frac{d^0}{d(j\Omega)^0} \cdot \left[\frac{2}{j\Omega+4}\right] \Big|_{j\Omega=-2} = \frac{2}{j\Omega+4} \Big|_{j\Omega=-2} = \frac{2}{-2+4} = \frac{2}{2}$$

$$\boxed{k_0 = 1}$$

$$k_1 = \frac{1}{1!}\frac{d^1}{d(j\Omega)^1}\left[\frac{2}{j\Omega+4}\right]\Bigg|_{j\Omega=-2} = \frac{d}{d(j\Omega)}\left[\frac{2}{j\Omega+4}\right]\Bigg|_{j\Omega=-2}$$

$$k_1 = \frac{(j\Omega+4)[0]-2[1+0]}{(j\Omega+4)^2}\Bigg|_{j\Omega=-2} = \frac{-2}{(-2+4)^2} = \frac{-2}{4} = -\frac{1}{2}$$

$$\boxed{k_1 = -\frac{1}{2}}$$

$$k_2 = \frac{1}{2!}\frac{d^2}{d(j\Omega)^2}\left[\frac{2}{j\Omega+4}\right]\Bigg|_{j\Omega=-2} = \frac{1}{2}\frac{d}{d(j\Omega)}\left[\frac{(j\Omega+4)(0)-2(1+0)}{(j\Omega+4)^2}\right]\Bigg|_{j\Omega=-2}$$

$$k_2 = \frac{1}{\cancel{2}}\frac{d}{d(j\Omega)}\left[\frac{-\cancel{2}}{(j\Omega+4)^2}\right]\Bigg|_{j\Omega=-2} = -\frac{d}{d(j\Omega)}\left[(j\Omega+4)^{-2}\right]\Bigg|_{j\Omega=-2}$$

$$k_2 = -\left[-2 \times (j\Omega+4)^{-3} \times (1+0)\right]\Bigg|_{j\Omega=-2}$$

$$k_2 = \frac{2}{(j\Omega+4)^3}\Bigg|_{j\Omega=-2} = \frac{2}{(-2+4)^3} = \frac{2}{2^3} = \frac{2}{8} = \frac{1}{4}$$

$$\boxed{k_2 = \frac{1}{4}}$$

Sub. A, k_0, k_1 and k_2 values in eqn. ②,

$$② \Rightarrow Y(j\Omega) = \left(-\frac{1}{4}\right)\cdot\frac{1}{j\Omega+4} + \frac{1}{(j\Omega+2)^3} - \left(\frac{1}{2}\right)\cdot\frac{1}{(j\Omega+2)^2} + \left(\frac{1}{4}\right)\cdot\frac{1}{(j\Omega+2)}$$

Taking Inverse Fourier Transform,

$$\boxed{y(t) = \left(-\frac{1}{4}\right)e^{-4t}u(t) + t^2 e^{-2t}u(t) - \left(\frac{1}{2}\right)te^{-2t}u(t) + \left(\frac{1}{4}\right)e^{-2t}u(t).}$$

2) consider an LTI system described by $y'(t)+2y(t)=x(t)$. Using Fourier Transform find the output $y(t)$ to each of the following signals:

i) $x(t)=e^{-t}u(t)$

ii) $x(t)=u(t)$.

Soln:

$$y'(t)+2y(t)=x(t)$$

$$\frac{dy(t)}{dt}+2y(t)=x(t)$$

Taking Fourier Transform,

$$(j\Omega)Y(j\Omega)+2Y(j\Omega)=X(j\Omega)$$

$$Y(j\Omega)[j\Omega+2]=X(j\Omega)$$

$$H(j\Omega)=\frac{Y(j\Omega)}{X(j\Omega)}=\frac{1}{j\Omega+2}\longrightarrow ①$$

i) To find $y(t)$ for the input $x(t)=e^{-t}u(t)$.

$$x(t)=e^{-t}u(t)$$

$$F\cdot T[x(t)]=F\cdot T[e^{-t}u(t)]$$

$$X(j\Omega)=\frac{1}{j\Omega+1}\longrightarrow ②.$$

w.k.T,

$$H(j\Omega)=\frac{Y(j\Omega)}{X(j\Omega)}$$

$$Y(j\Omega)=H(j\Omega)\cdot X(j\Omega)$$

$$Y(j\Omega)=\frac{1}{(j\Omega+2)}\cdot\frac{1}{(j\Omega+1)}=\frac{1}{(j\Omega+2)(j\Omega+1)}$$ [∵ using eqns ① & ②]

$$Y(j\Omega)=\frac{k_0}{j\Omega+2}+\frac{k_1}{j\Omega+1}\longrightarrow ③$$

$$k_0 = Y(j\Omega) \times (j\Omega+2)\Big|_{j\Omega=-2}$$

$$k_0 = \frac{1}{\cancel{(j\Omega+2)}(j\Omega+1)} \times \cancel{(j\Omega+2)}\Big|_{j\Omega=-2} = \frac{1}{(-2+1)} = \frac{1}{-1}$$

$$\boxed{k_0 = -1}$$

$$k_1 = Y(j\Omega) \times (j\Omega+1)\Big|_{j\Omega=-1}$$

$$k_1 = \frac{1}{(j\Omega+2)\cancel{(j\Omega+1)}} \times \cancel{(j\Omega+1)}\Big|_{j\Omega=-1} = \frac{1}{(-1+2)} = \frac{1}{1}.$$

$$\boxed{k_1 = 1}$$

Sub. k_0 and k_1 values in eqn ③,

$$③ \Rightarrow \quad Y(j\Omega) = -\frac{1}{j\Omega+2} + \frac{1}{j\Omega+1}$$

Taking Inv. F.T,

$$\boxed{y(t) = -e^{-2t}u(t) + e^{-t}u(t)}$$

(ii) To find y(t) for the input $x(t) = u(t)$.

$$x(t) = u(t)$$

$$F.T[x(t)] = F.T[u(t)]$$

$$X(j\Omega) = \frac{1}{j\Omega} + \pi\,\delta(\Omega) \approx \frac{1}{j\Omega} \xrightarrow{} ④.$$

considered [For. Analysis].

W.K.T,

$$H(j\Omega) = \frac{Y(j\Omega)}{X(j\Omega)}$$

$$Y(j\Omega) = H(j\Omega) \cdot X(j\Omega)$$

$$Y(j\Omega) = \frac{1}{(j\Omega+2)} \cdot \frac{1}{(j\Omega)} = \frac{1}{(j\Omega+2)(j\Omega)}$$ [∵ using eqn ① & ④]

$$Y(j\Omega) = \frac{k_0}{j\Omega+2} + \frac{k_1}{j\Omega} \rightarrow ⑤.$$

$$k_0 = Y(j\Omega) \times (j\Omega+2)\Big|_{j\Omega=-2}$$

$$k_0 = \frac{1}{\cancel{(j\Omega+2)}(j\Omega)} \times \cancel{(j\Omega+2)}\Big|_{j\Omega=-2}$$

$$\boxed{k_0 = \frac{1}{-2}}$$

$$k_1 = \frac{1}{(j\Omega+2)\cancel{(j\Omega)}} \times \cancel{j\Omega}\Big|_{j\Omega=0} = \frac{1}{0+2} = \frac{1}{2}$$

$$\boxed{k_1 = \frac{1}{2}}$$

sub. k_0 and k_1 values in eqn. ⑤,

$$Y(j\Omega) = \left(-\frac{1}{2}\right)\frac{1}{j\Omega+2} + \left(\frac{1}{2}\right)\frac{1}{j\Omega}$$

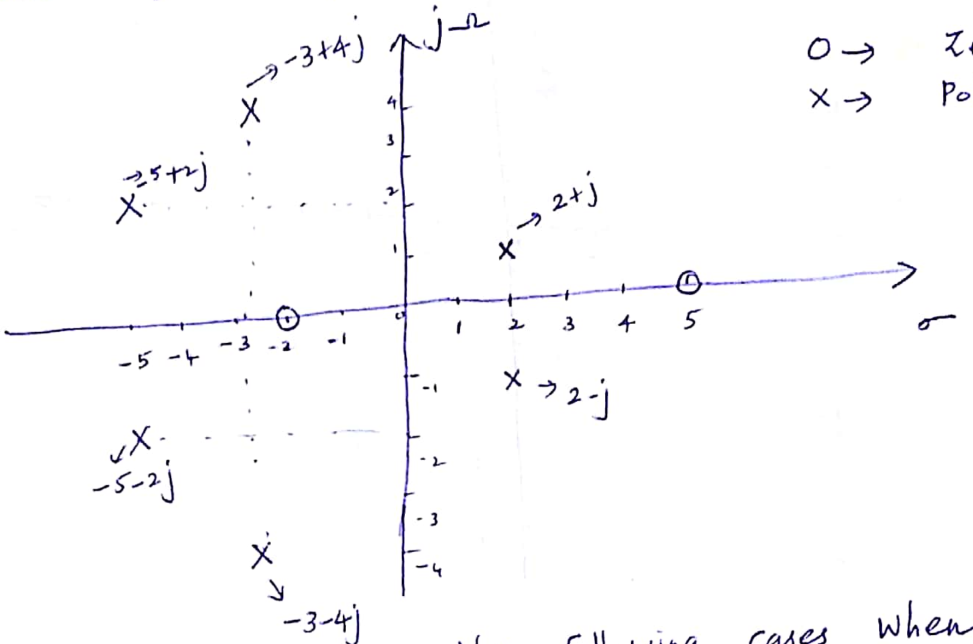
Taking. Inverse Fourier Transform

$$\boxed{y(t) = \left(-\frac{1}{2}\right) e^{-2t} u(t) + \left(\frac{1}{2}\right) \cdot u(t)},$$

Unit III

Part A : (Anna University Qn: Nov/Dec 2018)

1) The pole zero of the transfer function H(s) of a LTI system is given below.



Plot the ROC for the following cases when

(a) The system is causal.

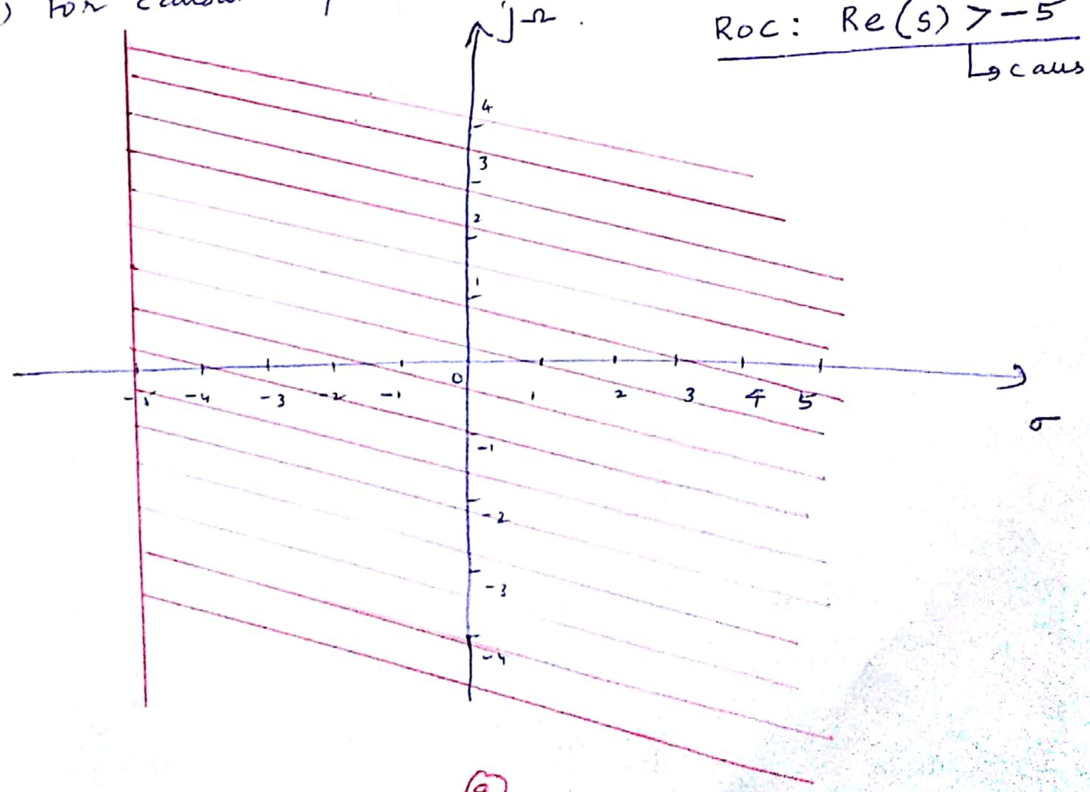
(b) The system is stable.

Soln:

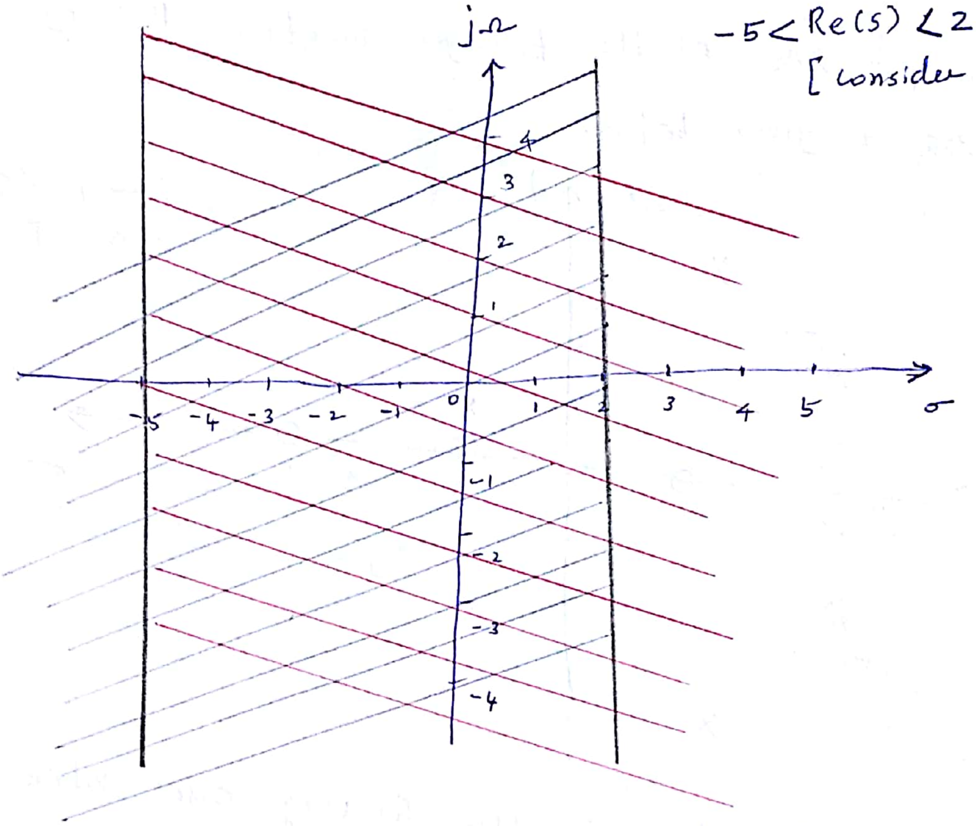
(a) For causal s/m ROC lies right hand side of poles.

ROC: Re(s) > -5

↳ causal.



(b) For stable system, ROC must include jΩ axis.



$-5 < Re(s) < 2$
[consider pole location] only.

2) Examine the initial and final value of the signal

U.Q

$x(t) = \sin 4t \, u(t)$.

Soln:

$$x(t) = \sin 4t \, u(t)$$

$$X(s) = \frac{4}{s^2+4^2}$$

Initial value:

$$x(0) = \underset{s \to \infty}{Lt} \; s \, X(s)$$

$$= \underset{s \to \infty}{Lt} \; s \times \frac{4}{s^2+4^2} = \underset{s \to \infty}{Lt} \; s \times \frac{4}{s^2\left[1+\frac{4^2}{s^2}\right]}$$

$$= \underset{s \to \infty}{Lt} \frac{4/s}{1+\frac{4^2}{s^2}} = \frac{\frac{4}{\infty}}{1+\frac{4^2}{\infty}}$$

$$= \frac{0}{1+0}$$

$$\boxed{x(0) = 0}$$

The result shows that the initial theorem gives the correct result. [$x(0) = \sin 0 \cdot u(0) = 0 \cdot 1 = 0$]

Final Value:

$$x(\infty) = \underset{s \to 0}{Lt} \; s \, X(s)$$

$$x(\infty) = \underset{s \to 0}{Lt} \; s \times \frac{4}{s^2+4^2} = \frac{0}{0+4^2} = \frac{0}{16}$$

$$\boxed{x(\infty) = 0}$$

The result shows that the final theorem gives an incorrect result. Because, the signal $x(t) = \sin 4t \, u(t)$ oscillates between +1 and -1 as $t \to \infty$. so it doesnot have a final value.

U.Q 3) Conclude the laplace transform of the following signal with their Region of Convergence.

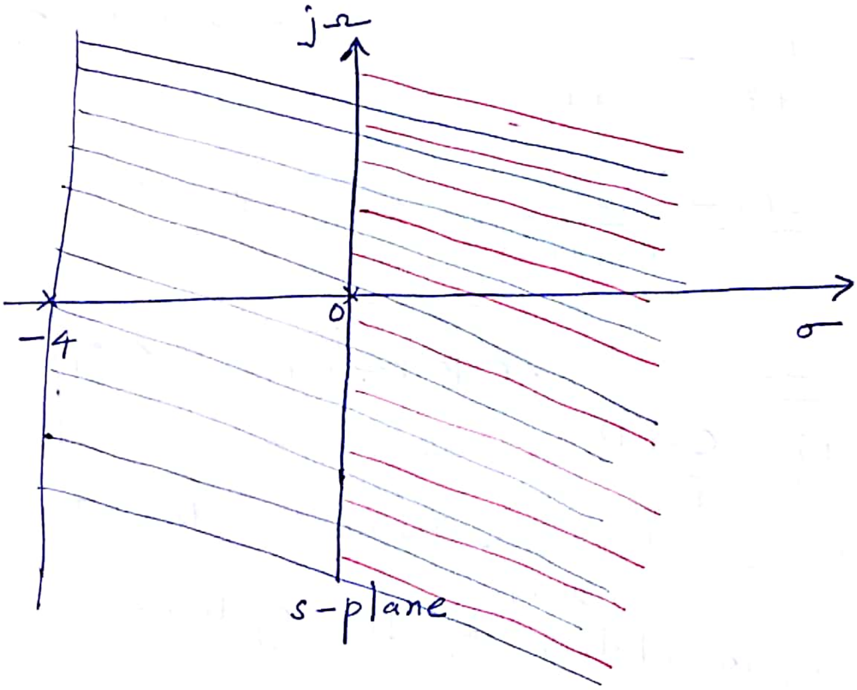
a) $x(t) = [3 - 6e^{-4t}]u(t)$

Soln:

$x(t) = 3u(t) - 6e^{-4t}u(t)$

Taking Laplace Transform,

$X(s) = \frac{3}{s} - \frac{6}{s+4}$ [Roc: Re(s) > 0 & Re(s) > -4]



So,

$X(s) = \frac{3}{s} - \frac{6}{s+4}$; Roc: Re(s) > 0

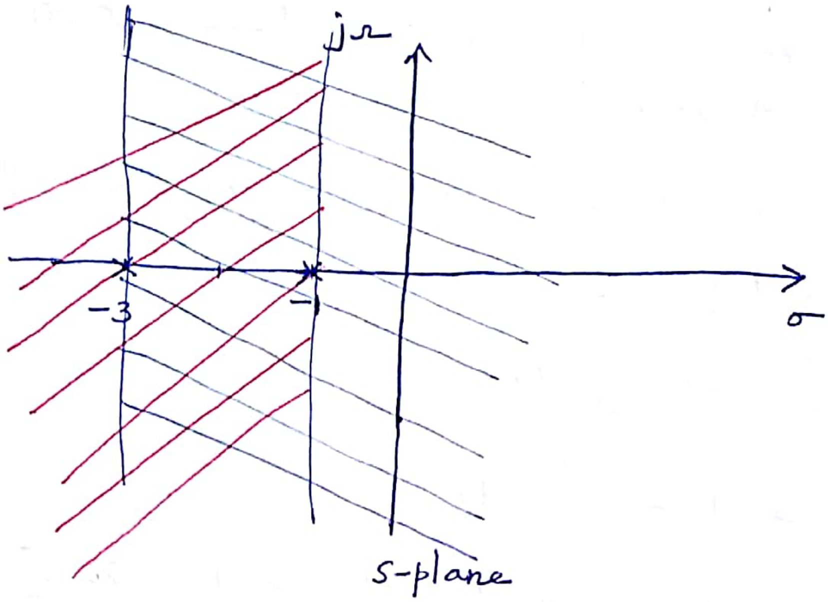
b) $e^{-3t}u(t) + e^{-t}u(-t)$.

Soln:

$x(t) = e^{-3t}u(t) - (-e^{-t}u(-t))$

Taking Laplace Transform,

$X(s) = \frac{1}{s+3} - \frac{1}{s+1}$ [Roc: Re(s) > -3 & Re(s) < -1]



So,

$$X(s) = \frac{1}{s+3} - \frac{1}{s+1} \; ; \quad ROC : -3 < Re(s) < -1$$

$$X(s) = \frac{s+1-s-3}{(s+3)(s+1)} \; ; \quad ROC : -3 < Re(s) < -1$$

$$\boxed{X(s) = \frac{-2}{(s+3)(s+1)} \; ; \; ROC : -3 < Re(s) < -1}$$

U.Q 4) Given $H(s) = \frac{1}{s^2+2s+1}$. Express the differential equation representation of the system.

Soln.

$$H(s) = \frac{1}{s^2+2s+1}$$

$$\frac{Y(s)}{X(s)} = \frac{1}{s^2+2s+1}$$

$$s^2 Y(s) + 2s Y(s) + Y(s) = X(s)$$

Taking Inverse Laplace Transform,

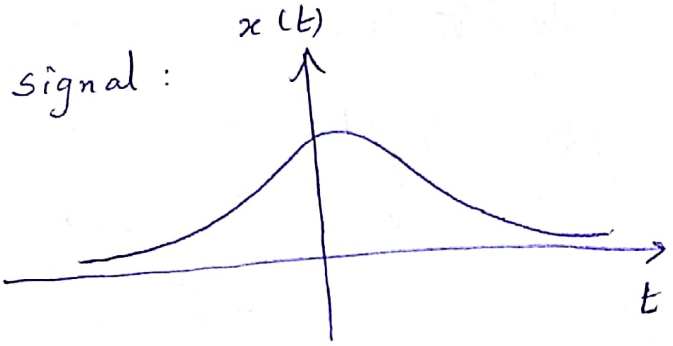
$$\boxed{\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + y(t) = x(t)}$$

Unit II : GATE question.

5) Find the fourier transform of Gaussian signal:

Soln:

Gaussian signal :



$$x(t) = e^{-at^2}$$

where , $a \rightarrow$ +ve constant $(a>0)$.

$$FT[x(t)] = \int_{-\infty}^{\infty} x(t)\, e^{-j\Omega t}\, dt$$

$$FT[e^{-at^2}] = \int_{-\infty}^{\infty} e^{-at^2}\, e^{-j\Omega t}\, dt$$

$$= \int_{-\infty}^{\infty} e^{-[at^2 + j\Omega t]}\, dt$$

$$at^2 + j\Omega t = \left(t\sqrt{a} + \frac{j\Omega}{2\sqrt{a}}\right)^2 + \frac{\Omega^2}{4a}$$

$$= t^2 a + 2.t\sqrt{a}\cdot\frac{j\Omega}{2\sqrt{a}} - \frac{\Omega^2}{4a} + \frac{\Omega^2}{4a} \qquad (\because j^2 = -1)$$

$$= at^2 + j\Omega t$$

Now

$$FT[e^{-at^2}] = \int_{-\infty}^{\infty} e^{-\left[\left(t\sqrt{a} + \frac{j\Omega}{2\sqrt{a}}\right)^2 + \frac{\Omega^2}{4a}\right]}\, dt.$$

$$= e^{-\frac{\Omega^2}{4a}} \int_{-\infty}^{\infty} e^{-\left(t\sqrt{a} + \frac{j\Omega}{2\sqrt{a}}\right)^2}\, dt$$

Let

$$p = t\sqrt{a} + \frac{j\Omega}{2\sqrt{a}}$$

$$\frac{dp}{dt} = \sqrt{a} + 0$$

$$dp = \sqrt{a}\, dt$$

$$dt = \frac{dp}{\sqrt{a}}$$

$$F.T\left[e^{-at^2}\right] = X(j\Omega) = e^{-\frac{\Omega^2}{4a}} \int_{-\infty}^{\infty} e^{-p^2} \frac{dp}{\sqrt{a}}.$$

$$X(j\Omega) = \frac{e^{-\frac{\Omega^2}{4a}}}{\sqrt{a}} \cdot \sqrt{\pi}$$

$$\boxed{\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}.}$$

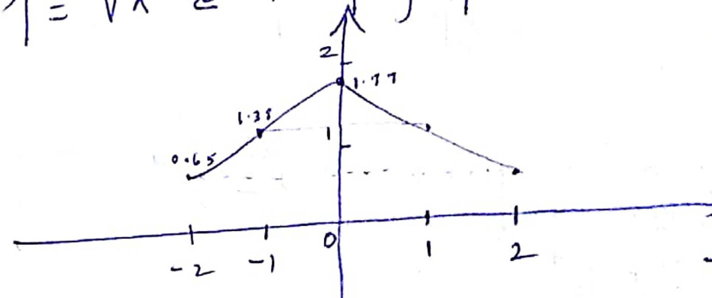
Ans:

$$X(j\Omega) = \sqrt{\frac{\pi}{a}} \cdot e^{-\frac{\Omega^2}{4a}}$$

$$F.T\left[e^{-at^2}\right] = \sqrt{\frac{\pi}{a}}\, e^{-\frac{\Omega^2}{4a}}$$

Spectrum: (Amplitude) $|X(j\Omega)|$ Vs Ω ; Choose $a = 1$.

$$|X(j\Omega)| = \sqrt{\pi}\, e^{-\frac{\Omega^2}{4}}$$



Ω	$\lvert X(j\Omega)\rvert$
-2	$0.36 \times \sqrt{\pi} = 0.65$
-1	$0.77 \times \sqrt{\pi} = 1.38$
0	$1 \cdot \sqrt{\pi} = 1.77$
1	1.38
2	0.65

Note:

Fouries series for Gaussian signal is not exist.

Since, Gaussian signal is aperiodic signal.

Fourier series is applicable for periodic signal. But, Fourier transform is applicable for aperiodic signal.

Unit IV - Analysis of Discrete Time signals

→ Introduction:

Discrete Time Fourier series - Analysis of Discrete Time Periodic signals.

Discrete Time Fourier Transform - Analysis of Discrete Time Aperiodic signal.

Z-Transform - Analysis of both periodic and aperiodic signals.

→ Discrete Time Fourier Transform (DTFT)

* Analysis Equation:

$$DTFT[x(n)] = X(\omega) = X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

* Synthesis Equation:

$$IDTFT[X(\omega)] = x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) . e^{j\omega n} d\omega .$$

→ Properties of DTFT:

1) Linearity

$$DTFT[a x_1(n) + b x_2(n)] = a X_1(\omega) + b X_2(\omega)$$

Soln:

$$DTFT[x(n)] = X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

$$DTFT[a x_1(n) + b x_2(n)] = \sum_{n=-\infty}^{\infty} [a x_1(n) + b x_2(n)] e^{-j\omega n}$$

$$= a \sum_{n=-\infty}^{\infty} x_1(n) e^{-j\omega n} + b \sum_{n=-\infty}^{\infty} x_2(n) e^{-j\omega n}$$

$$DTFT[a x_1(n) + b x_2(n)] = a X_1(\omega) + b X_2(\omega).$$

Hence, proved.

2) Time shifting Property:

$DTFT[x(n-n_o)] = e^{-jwn_o} X(\omega)$

Soln:

$$DTFT[x(n)] = \sum_{n=-\infty}^{\infty} x(n) e^{-jwn}$$

$$DTFT[x(n-n_o)] = \sum_{n=-\infty}^{\infty} x(n-n_o) e^{-jwn}$$

Let
$m = n-n_o$
$n = m+n_o$

$$= \sum_{m=-\infty}^{\infty} x(m) e^{-jw(m+n_o)}$$

$$DTFT[x(n-n_o)] = e^{-jwn_o} \sum_{m=-\infty}^{\infty} x(m) . e^{-jwm}$$

$$\boxed{DTFT[x(n-n_o)] = e^{-jwn_o} X(\omega).}$$

Hence proved.

Note:

$DTFT[x(n+n_o)] = e^{jwn_o} X(\omega).$

3) Frequency shifting Property:

$DTFT[e^{jw_on} x(n)] = X(\omega-\omega_o)$

Soln:

$$DTFT[x(n)] = \sum_{n=-\infty}^{\infty} x(n) e^{-jwn}$$

$$DTFT[e^{jw_on} x(n)] = \sum_{n=-\infty}^{\infty} e^{jw_on} x(n) e^{-jwn}$$

$$= \sum_{n=-\infty}^{\infty} x(n) e^{-j(\omega-\omega_o)n}$$

$$DTFT[e^{jw_on} x(n)] = X(\omega-\omega_o)$$

Hence proved.

4) Scaling Property:

$$DTFT[x(an)] = X\left(\frac{w}{a}\right)$$

Soln:

$$DTFT[x(n)] = \sum_{n=-\infty}^{\infty} x(n) e^{-jwn}$$

$$DTFT[x(an)] = \sum_{n=-\infty}^{\infty} x(an)\, e^{-jwn}$$

$$\boxed{\begin{aligned} \text{Let } m &= an \\ n &= \frac{m}{a} \end{aligned}}$$

$$= \sum_{m=-\infty}^{\infty} x(m) e^{-jw\cdot\frac{m}{a}}$$

$$DTFT[x(an)] = \sum_{m=-\infty}^{\infty} x(m)\, e^{-j\left(\frac{w}{a}\right)m}$$

$$\boxed{DTFT[x(an)] = X\left(\frac{w}{a}\right)}$$

Hence Proved.

5) Differentiation in Frequency domain:

$$DTFT[-jnx(n)] = \frac{dX(w)}{dw}$$

Soln:

$$DTFT[x(n)] = X(w) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-jwn}$$

$$\frac{dX(w)}{dw} = \sum_{n=-\infty}^{\infty} x(n) \frac{d}{dw}\left[e^{-jwn}\right]$$

$$= \sum_{n=-\infty}^{\infty} x(n) \quad e^{-jwn} \cdot (-jn)$$

$$= \sum_{n=-\infty}^{\infty} -jnx(n) \quad e^{-jwn}$$

$$\frac{dX(w)}{dw} = DTFT[-jnx(n)]$$

Hence proved.

6) <u>Periodicity:</u>

$$X(\omega+2\pi k) = X(\omega) \quad ; \quad k = 1, 2, 3, \ldots$$

<u>Soln:</u>

$$DTFT[x(n)] = X(\omega) = \sum_{n=-\infty}^{\infty} x(n) \cdot e^{-j\omega n}$$

$$X(\omega+2\pi k) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j(\omega+2\pi k)n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n} \cdot e^{-j2\pi k}$$

$$\boxed{\begin{aligned} e^{-j2\pi k} &= \cos 2\pi k - j\sin 2\pi k \;;\; k=1,2,3,\ldots \\ &= 1 \qquad ; k=1,2,3,\ldots \end{aligned}}$$

$$X(\omega+2\pi k) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

$$X(\omega+2\pi k) = X(\omega)$$

Hence proved.

7) <u>Convolution Property:</u>

$$DTFT[x_1(n) * x_2(n)] = X_1(\omega) \cdot X_2(\omega)$$

<u>Proof:</u>

$$DTFT[x(n)] = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

$$DTFT[x_1(n) * x_2(n)] = \sum_{n=-\infty}^{\infty} [x_1(n) * x_2(n)]\, e^{-j\omega n}.$$

Convolution Summation Formula

$$\boxed{x_1(n) * x_2(n) = \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k)}$$

$$= \sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k) \right] e^{-j\omega n}$$

Rearranging the terms:

$$= \sum_{k=-\infty}^{\infty} x_1(k) \cdot \sum_{n=-\infty}^{\infty} x_2(n-k)\, e^{-j\omega n}$$

$$= \underbrace{\sum_{k=-\infty}^{\infty} x_1(k) \cdot e^{-j\omega k}} \cdot X_2(\omega) \quad [\because \text{using time shifting prop.}].$$

$$\boxed{DTFT[x_1(n) * x_2(n)] = X_1(\omega) \cdot X_2(\omega)}$$

Hence proved.

8) Parseval's theorem for Discrete Time signals using DTFT:

Energy of the signal in time domain = Energy of the signal in frequency domain.

i.e., $E = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(\omega)|^2 d\omega.$

Proof:

$$IDTFT[X(\omega)] = x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega \rightarrow ①$$

$$L.H.S = E = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \sum_{n=-\infty}^{\infty} x(n) \cdot x^*(n)$$

$$= \sum_{n=-\infty}^{\infty} x(n) \cdot \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega \right]^* \quad [\because \text{using eqn①}]$$

$$= \sum_{n=-\infty}^{\infty} x(n) \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} X^*(\omega) e^{-j\omega n} d\omega.$$

Re-arranging the terms,

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} X^*(\omega).\ d\omega.\ \underbrace{\sum_{n=-\infty}^{\infty} x(n)\ e^{-j\omega n}}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} X^*(\omega).\ d\omega.\ X(\omega).$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\omega)|^2\ d\omega.\quad \left[\because X^*(\omega).\ X(\omega) = |X(\omega)|^2\right]$$

= R.H.S.

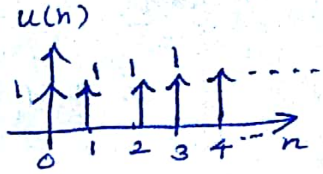
Hence proved.

→ Problems based on Discrete Time Fourier Transform.

1) Find the DTFT of unit step signal.

Soln:

unit step signal : $u(n) = \begin{cases} 1 & ; n \geq 0 \\ 0 & ; n < 0 \end{cases}$.



$$DTFT\,[x(n)] = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

$$DTFT\,[u(n)] = \sum_{n=-\infty}^{\infty} u(n)\, e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} e^{-j\omega n} = \sum_{n=0}^{\infty} [e^{-j\omega}]^{n}$$

Geometrical summation Formula (GSF)

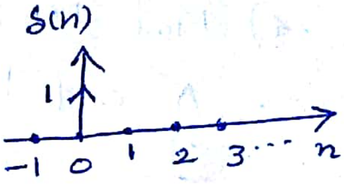
$$\sum_{n=0}^{\infty} a^{n} = \frac{1}{1-a} \quad ; \quad |a| < 1$$

$$\boxed{DTFT\,[u(n)] = \frac{1}{1-e^{-j\omega}}}$$

2) Find the DTFT of unit impulse signal.

Soln:

unit impulse signal : $\delta(n) = \begin{cases} 1 & ; n = 0 \\ 0 & ; n \neq 0 \end{cases}$



$$DTFT\,[x(n)] = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

$$DTFT\,[\delta(n)] = \sum_{n=-\infty}^{\infty} \delta(n) \cdot e^{-j\omega n}$$

$$= \delta(0) . e^{-j\omega(0)} = \delta(0) . e^{0} = 1 \cdot 1 = 1$$

$$\boxed{DTFT[\delta(n)] = 1}$$

3) Find the DTFT of $(\frac{1}{2})^n u(n)$:

Soln:

$$DTFT[x(n)] = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

$$DTFT\left[(\tfrac{1}{2})^n u(n)\right] = \sum_{n=-\infty}^{\infty} (\tfrac{1}{2})^n u(n) \cdot e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (\tfrac{1}{2})^n e^{-j\omega n} \qquad \left[\because u(n) = \begin{cases} 1 & ; n \geq 0 \\ 0 & ; n < 0 \end{cases} \right]$$

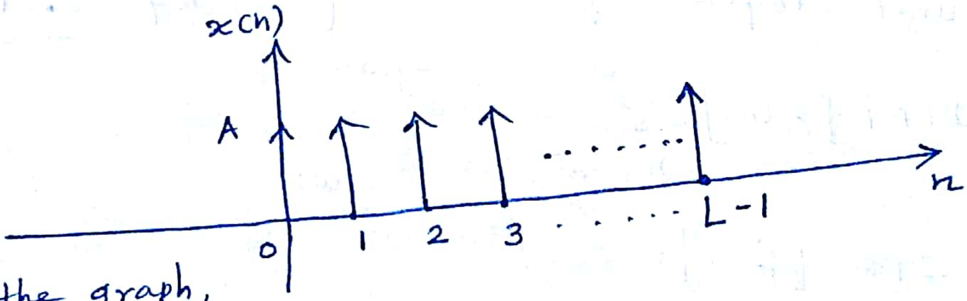
$$= \sum_{n=0}^{\infty} (\tfrac{1}{2} e^{-j\omega})^n$$

$$\boxed{GSF : \sum_{n=0}^{\infty} a^n = \frac{1}{1-a} ; |a|<1}$$

$$DTFT\left[(\tfrac{1}{2})^n u(n)\right] = \frac{1}{1 - \frac{1}{2} e^{-j\omega}}$$

4) Find DTFT of rectangular pulse with amplitude of 'A' and length 'L'.

Soln:



From the graph,

$$x(n) = \begin{cases} A & ; 0 \leq n \leq L-1 \\ 0 & ; \text{otherwise}. \end{cases}$$

$$\text{DTFT}[x(n)] = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

$$\text{DTFT}[x(n)] = \sum_{n=0}^{L-1} A\, e^{-j\omega n}$$ [∵ using the given specification]

$$= A \sum_{n=0}^{L-1} (e^{-j\omega})^{n}$$

G.S.F. $\sum_{n=N_1}^{N_2} a^{n} = \frac{a^{N_1} - a^{N_2+1}}{1-a}$

Here, $N_1 = 0$ & $N_2 = L-1$

$$= A\left[\frac{(e^{-j\omega})^{0} - (e^{-j\omega})^{L-1+1}}{1-e^{-j\omega}}\right]$$

$$= A\left[\frac{1 - e^{-j\omega L}}{1-e^{-j\omega}}\right]$$

$$= A\left[\frac{e^{\frac{j\omega L}{2}} \cdot e^{\frac{-j\omega L}{2}} - e^{\frac{-j\omega L}{2}} \cdot e^{\frac{-j\omega L}{2}}}{e^{\frac{j\omega}{2}} \cdot e^{\frac{-j\omega}{2}} - e^{\frac{-j\omega}{2}} \cdot e^{\frac{-j\omega}{2}}}\right]$$

$$= \frac{A\, e^{\frac{-j\omega L}{2}}\left[e^{\frac{j\omega L}{2}} - e^{\frac{-j\omega L}{2}}\right]}{e^{\frac{-j\omega}{2}}\left[e^{\frac{j\omega}{2}} - e^{\frac{-j\omega}{2}}\right]}$$

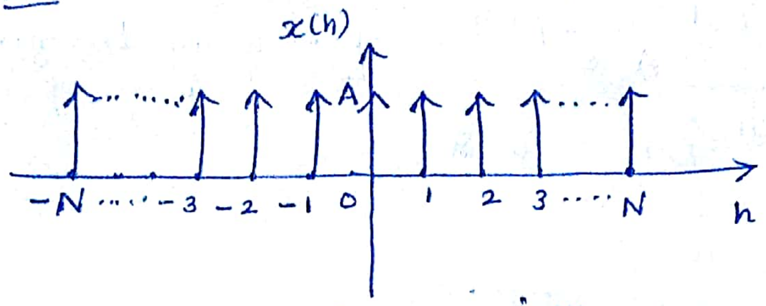
$$= A \cdot e^{\frac{-j\omega L}{2}} \cdot e^{\frac{j\omega}{2}} \frac{\left[\cancel{2j} \sin\left(\frac{\omega L}{2}\right)\right]}{\cancel{2j} \sin\left(\frac{\omega}{2}\right)}$$

$$\boxed{\text{DTFT}[x(n)] = A\, e^{-j\frac{\omega}{2}(L-1)} \cdot \left[\frac{\sin\left(\frac{\omega L}{2}\right)}{\sin\left(\frac{\omega}{2}\right)}\right]}$$

where, x(n) → Rectangular Pulse with amplitude 'A' and Length 'L'.

5) Find the DTFT of $x(n) = \begin{cases} A & ; & |n| \leq N \\ 0 & ; & |n| > N . \end{cases}$

Soln:



Discrete [Rectangular pulse].

$$DTFT[x(n)] = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

$$= \sum_{n=-N}^{N} A e^{-j\omega n} \quad [\because \text{using the given specification}]$$

$$= A \sum_{n=-N}^{N} (e^{-j\omega})^{n}$$

$$\boxed{\begin{array}{l} GSF: \displaystyle\sum_{n=N_1}^{N_2} a^n = \frac{a^{N_1} - a^{N_2+1}}{1-a} \\ \text{Here, } N_1 = -N \ \& \ N_2 = N \end{array}}$$

$$DTFT[x(n)] = A\left[\frac{e^{-j\omega(-N)} - e^{-j\omega(N+1)}}{1 - e^{-j\omega}}\right]$$

$$= A\left[\frac{e^{j\omega N} - e^{-j\omega N} \cdot e^{-j\omega}}{1 - e^{-j\omega}}\right]$$

$$= A\left[\frac{e^{j\omega N} \cdot e^{j\frac{\omega}{2}} e^{-j\frac{\omega}{2}} - e^{-j\omega N} \cdot e^{-j\frac{\omega}{2}} e^{-j\frac{\omega}{2}}}{e^{j\frac{\omega}{2}} e^{-j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}} e^{-j\frac{\omega}{2}}}\right]$$

$$= A \frac{\cancel{e^{-j\frac{\omega}{2}}}\left[e^{j\omega(N+\frac{1}{2})} - e^{-j\omega(N+\frac{1}{2})}\right]}{\cancel{e^{-j\frac{\omega}{2}}}\left[e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}\right]}$$

$$\text{DTFT}[x(n)] = X(\omega) = A \cdot \frac{\cancel{2j} \sin\omega(N+\frac{1}{2})}{\cancel{2j} \sin(\frac{\omega}{2})}$$

$$\boxed{X(\omega) = A \frac{\sin\omega(N+\frac{1}{2})}{\sin(\frac{\omega}{2})}}$$

6) Find DTFT of the following signals:

a) $x_1(n) = (n-1)^2 x(n)$

b) $x_2(n) = (1-n)\, x(n)$.

a) soln:

$$\text{DTFT}[x(n)] = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

$$\text{DTFT}[x_1(n)] = \sum_{n=-\infty}^{\infty} x_1(n)\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} [(n-1)^2 x(n)]\, e^{-j\omega n}.$$

$$= \sum_{n=-\infty}^{\infty} [(n^2-2n+1)\, x(n)]\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} n^2 x(n)\, e^{-j\omega n} - 2\sum_{n=-\infty}^{\infty} n\, x(n)\, e^{-j\omega n} + \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

Differentiation in frequency domain

$$\text{DTFT}[-jn\, x(n)] = \frac{dX(\omega)}{d\omega}$$

$$\text{DTFT}[(-j)^2 n^2 x(n) = \frac{d^2X(\omega)}{d\omega^2}$$

$$\vdots \qquad \vdots$$

$$\text{DTFT}[(-j)^n n^n x(n) = \frac{d^n X(\omega)}{d\omega^n}$$

$$X_1(\omega) = \frac{1}{(-j)^2}\frac{d^2X(\omega)}{d\omega^2} - 2\cdot\left(\frac{1}{-j}\right)\frac{dX(\omega)}{d\omega} + X(\omega)$$

$$\boxed{X_1(\omega) = -\frac{d^2X(\omega)}{d\omega^2} - 2j\frac{dX(\omega)}{d\omega} + X(\omega).}$$

$$\left[\begin{array}{l} \because -j \times -j = j^2 = -1 \\ \frac{1}{-j} = \frac{1}{-j} \times \frac{-j}{-j} = \frac{-j}{-j^2} = \frac{-j}{-(-1)} = \frac{-j}{1} = -j \end{array}\right]$$

b) soln:

$$\text{DTFT}[x(n)] = \sum_{n=-\infty}^{\infty} x(n)\cdot e^{-j\omega n}$$

$$\text{DTFT}[x_2(n)] = \sum_{n=-\infty}^{\infty} x_2(n)\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} [(1-n)x(n)]\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} - \sum_{n=-\infty}^{\infty} n x(n)\, e^{-j\omega n}$$

$$X_2(\omega) = X(\omega) - \left(\frac{1}{-j}\right)\frac{dX(\omega)}{d\omega}$$

$$\boxed{X_2(\omega) = X(\omega) - j\frac{dX(\omega)}{d\omega}} \quad \left[\because \frac{1}{j} = -j\right]$$

7) Find the DTFT of the following signals:

a) $\delta(n-n_0)$

b) $\delta(n-5)$

c) $\delta(n+7)$

Soln:

a) $DTFT[\delta(n-n_0)] = e^{-j\omega n_0} \; DTFT[\delta(n)]$

[$\because$ using Time shifting prop.]

$$= e^{-j\omega n_0} \cdot 1 = e^{-j\omega n_0}$$

b) $DTFT[\delta(n-5)] = e^{-j5\omega} \; DTFT[\delta(n)]$ [$\because$ using Time shifting prop.]

$$= e^{-j5\omega} \cdot 1 = e^{-j5\omega}$$

c) $DTFT[\delta(n+7)] = e^{j7\omega} \; DTFT[\delta(n)]$ [$\because$ using Time shifting prop.]

$$= e^{j7\omega} \cdot 1 = e^{j7\omega}$$

8) DTFT of $a^n u(n)$:

Soln:

$$DTFT[x(n)] = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

$$DTFT[a^n u(n)] = \sum_{n=-\infty}^{\infty} a^n u(n) e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} a^n \cdot e^{-j\omega n} \qquad \left[\because u(n) = \begin{cases} 1 ; & n \geq 0 \\ 0 ; & n < 0 \end{cases}\right]$$

$$= \sum_{n=0}^{\infty} (a e^{-j\omega})^n$$

$$\boxed{\begin{array}{l}\text{G.S.F:} \\ \sum_{n=0}^{\infty} A^n = \frac{1}{1-A} ; |A|<1 \\ \text{Here, } A = ae^{-j\omega}\end{array}}$$

$$\text{DTFT}\left[a^n u(n)\right] = \frac{1}{1-ae^{-j\omega}}$$

9) Compute DTFT of $(\frac{1}{2})^n u(n)$ and draw its magnitude spectrum.

Soln:

W.K.T., $\text{DTFT}[a^n u(n)] = \frac{1}{1-ae^{-j\omega}}$

So,

$$\boxed{\text{DTFT}\left[(\tfrac{1}{2})^n u(n)\right] = \frac{1}{1-\frac{1}{2}e^{-j\omega}}}$$

$$X(\omega) = \frac{1}{1-0.5e^{-j\omega}} = \frac{1}{1-0.5[\cos\omega - j\sin\omega]}$$

$$= \frac{1}{(1-0.5\cos\omega)+j0.5\sin\omega}$$

$$= \frac{1}{\underbrace{(1-0.5\cos\omega)}_{a}\underset{+}{+}j\underbrace{0.5\sin\omega}_{b}} \times \frac{(1-0.5\cos\omega)-j0.5\sin\omega}{\underbrace{(1-0.5\cos\omega)}_{a}\underset{-}{-}j\underbrace{0.5\sin\omega}_{b}}$$

$$= \frac{(1-0.5\cos\omega)-j0.5\sin\omega}{(1-0.5\cos\omega)^2-(j0.5\sin\omega)^2}$$

$$= \frac{(1-0.5\cos\omega)-j0.5\sin\omega}{[1^2-2\times1\times0.5\cos\omega+0.5^2\cos^2\omega]-(j^2 0.5^2\sin^2\omega)}$$

$$= \frac{(1-0.5\cos\omega)-j0.5\sin\omega}{1-\cos\omega+0.25\cos^2\omega+0.25\sin^2\omega} \quad [\because j^2=-1]$$

$$X(\omega) = \frac{(1-0.5\cos\omega)-j0.5\sin\omega}{1-\cos\omega+0.25\left[\cos^2\omega+\sin^2\omega\right]}$$

$$X(\omega) = \frac{(1-0.5\cos\omega)-j0.5\sin\omega}{1.25-\cos\omega} \qquad \left[\because \cos^2\omega+\sin^2\omega=1\right]$$

$$\boxed{X(\omega) = \frac{(1-0.5\cos\omega)}{(1.25-\cos\omega)} - j\,\frac{0.5\sin\omega}{(1.25-\cos\omega)}}$$

* Magnitude Spectrum:

$|X(\omega)|$ v ω

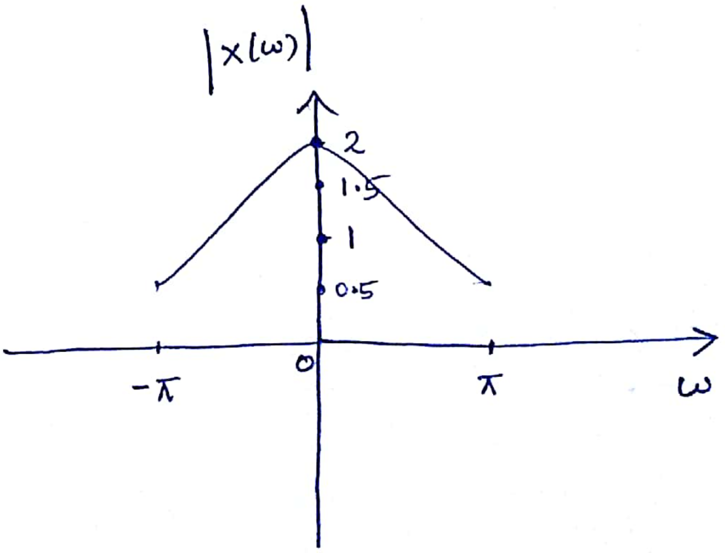
$$|X(\omega)| = \sqrt{(R.P)^2+(I.P)^2} = \sqrt{\left(\frac{1-0.5\cos\omega}{1.25-\cos\omega}\right)^2+\left(\frac{-0.5\sin\omega}{1.25-\cos\omega}\right)^2}$$

$$= \sqrt{\frac{(1-0.5\cos\omega)^2+(0.5\sin\omega)^2}{(1.25-\cos\omega)^2}} = \frac{\sqrt{1-\cos\omega+0.25\cos^2\omega+0.25\sin^2\omega}}{1.25-\cos\omega}$$

$$= \frac{\sqrt{1-\cos\omega+0.25(\cos^2\omega+\sin^2\omega)}}{1.25-\cos\omega} = \frac{\sqrt{1.25-\cos\omega}}{1.25-\cos\omega}$$

$$|X(\omega)| = \frac{1}{\sqrt{1.25-\cos\omega}}$$

ω	$\lvert X(\omega)\rvert$
$-\pi$	0.67
0	2
$+\pi$	0.67



→ Z-Transform :

*Definition :

$$Z \cdot T[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \rightarrow \text{Bilateral (or) Double sided Z-Transform}$$

$$Z \cdot T[x(n)] = X(z) = \sum_{n=0}^{\infty} x(n) z^{-n} \rightarrow \text{Unilateral (or) single sided Z-Transform.}$$

→ Inverse Z-Transform

It is computed by three methods.

i) Partial fraction method

ii) Long Division (or) Power series Expansion method

iii) contour Integration (or) Residue method

→ Relation between DTFT and Z-Transform.

$$DTFT[x(n)] = X(\omega) = X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

$$ZT[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

From the above two equations, we observed that the relation between DTFT and Z-Transform is $\underline{z = e^{j\omega}}$

→ Properties of Z-Transform :

1) Linearity :

$$Z \cdot T[a x_1(n) + b x_2(n)] = a X_1(z) + b X_2(z)$$

Proof:

$$Z \cdot T[x(n)] = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$Z \cdot T[a x_1(n) + b x_2(n)] = \sum_{n=-\infty}^{\infty} [a x_1(n) + b x_2(n)] z^{-n}$$

$$= a \sum_{n=-\infty}^{\infty} x_1(n) z^{-n} + b \sum_{n=-\infty}^{\infty} x_2(n) z^{-n}$$

$$Z \cdot T[a x_1(n) + b x_2(n)] = a X_1(z) + b X_2(z)$$

Hence proved.

2) Time shifting property:

$$z.T[x(n-n_0)] = z^{-n_0} X(z).$$

Soln:

$$z.T[x(n)] = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$z.T[x(n-n_0)] = \sum_{n=-\infty}^{\infty} x(n-n_0) z^{-n}$$

$$\boxed{\begin{array}{l}\text{Let} \\ m = n-n_0 \\ n = m+n_0\end{array}}$$

$$= \sum_{m=-\infty}^{\infty} x(m) z^{-(m+n_0)}$$

$$= z^{-n_0} \sum_{m=-\infty}^{\infty} x(m) \cdot z^{-m}$$

$$\boxed{z.T[x(n-n_0)] = z^{-n_0} X(z)}$$

Hence Proved.

3) Differentiation in frequency (or) z-domain:

$$z.T[n x(n)] = -z \frac{d X(z)}{dz}$$

Proof:

$$z.T[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$\frac{dX(z)}{dz} = \sum_{n=-\infty}^{\infty} x(n) \frac{d}{dz}(z^{-n})$$

$$\frac{dX(z)}{dz} = \sum_{n=-\infty}^{\infty} x(n) [-n z^{-n-1}]$$

$$= -\sum_{n=-\infty}^{\infty} n x(n) z^{-n} \cdot z^{-1}$$

$$\frac{dX(z)}{dz} = -\frac{1}{z} \sum_{n=-\infty}^{\infty} n\,x(n)\, z^{-n}$$

$$-z\frac{dX(z)}{dz} = \sum_{n=-\infty}^{\infty} n\,x(n)\, z^{-n}$$

$$-z\frac{dX(z)}{dz} = Z.T\left[n\,x(n)\right]$$

Hence proved.

4) Convolution Property:

$$Z.T\left[x_1(n) * x_2(n)\right] = X_1(z).X_2(z)$$

Proof:

$$Z.T\left[x(n)\right] = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

$$Z.T\left[x_1(n) * x_2(n)\right] = \sum_{n=-\infty}^{\infty} \left[x_1(n) * x_2(n)\right] z^{-n}$$

Convolution summation Formula

$$x_1(n) * x_2(n) = \sum_{k=-\infty}^{\infty} x_1(k) \cdot x_2(n-k)$$

$$= \sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} x_1(k).\, x_2(n-k)\right] z^{-n}$$

Rearranging the terms

$$= \sum_{k=-\infty}^{\infty} x_1(k) \underbrace{\sum_{n=-\infty}^{\infty} x_2(n-k)\, z^{-n}}$$

$$= \underbrace{\sum_{k=-\infty}^{\infty} x_1(k).\, z^{-k}}, X_2(Z)$$ [∵ using Time shifting Prop.]

$$\boxed{Z.T\left[x_1(n) * x_2(n)\right] = X_1(Z).\, X_2(Z)}$$

Hence proved.

5) Initial value Theorem

$$x(0) = \underset{z \to \infty}{Lt} \quad X(z)$$

6) Final value Theorem

$$x(\infty) = \underset{z \to 1}{Lt} \quad (1 - z^{-1}) X(z).$$

7) Time Reversal Property:

$$ZT[x(-n)] = X(z^{-1}) = X(\tfrac{1}{z}).$$

Proof:

$$Z.T[x(n)] = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$Z.T[x(-n)] = \sum_{n=-\infty}^{\infty} x(-n) z^{-n}$$

$$\boxed{\begin{array}{l} \text{Let} \\ \quad m = -n \\ \quad n = -m \end{array}}$$

$$= \sum_{m=-\infty}^{\infty} x(m) \, . \, z^{-[-m]}.$$

$$= \sum_{m=-\infty}^{\infty} x(m) (z^{-1})^{-m}$$

$$ZT[x(-n)] = X(z^{-1}) = X(\tfrac{1}{z})$$

Hence proved.

→ Region of convergence (Roc) and Problems based on z-Transform:

* **Region of convergence: (Roc)**

The values of 'z' for which X(z) converges is called Roc. It determines the stability and causality of the system.

* Problems:

1) Find the z-Transform of $\delta(n)$:

Soln:

$$z.T[x(n)] = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$z.T[\delta(n)] = \sum_{n=-\infty}^{\infty} \delta(n) z^{-n}$$

$$\boxed{\delta(n) = \begin{cases} 1 & ; n=0 \\ 0 & ; n \neq 0 \end{cases}}$$

$$= \delta(0) z^{0} = 1 \cdot 1 .$$

$\boxed{zT[\delta(n)] = 1}$; Roc: Entire z-plane.

2) Find the z-Transform of $\delta(n-k)$:

Soln:

$$z.T[\delta(n-k)] = z^{-k} \; z.T[\delta(n)] \quad [\because \text{using time shifting prop.}].$$

$$= z^{-k} . 1$$

$\boxed{zT[\delta(n-k)] = z^{-k}}$; Roc: Entire 'z'-plane except z=0.

Note:

$$z.T[\delta(n-8)] = z^{-8}$$

$$z.T[x(n-6)] = z^{-6} X(z).$$

3) Find the Z-Transform of $\delta(n+k)$:

Soln:

$$Z.T[\delta(n+k)] = z^{k} \; Z.T[\delta(n)] \quad [\because \text{ using time shifting prop.}]$$

$$= z^{k} \cdot 1$$

$$\boxed{Z.T[\delta(n+k)] = z^{k}}$$; Roc : Entire z-plane except $z=\infty$.

4) Find the Z-Transform of $u(n)$:

Soln:

$$Z.T[x(n)] = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

$$Z.T[u(n)] = \sum_{n=-\infty}^{\infty} u(n)\, z^{-n}$$

$$\boxed{u(n) = \begin{cases} 1 ; & n \geq 0 \\ 0 ; & n < 0 \end{cases}}$$

$$= \sum_{n=0}^{\infty} z^{-n} = \sum_{n=0}^{\infty} (z^{-1})^{n}$$

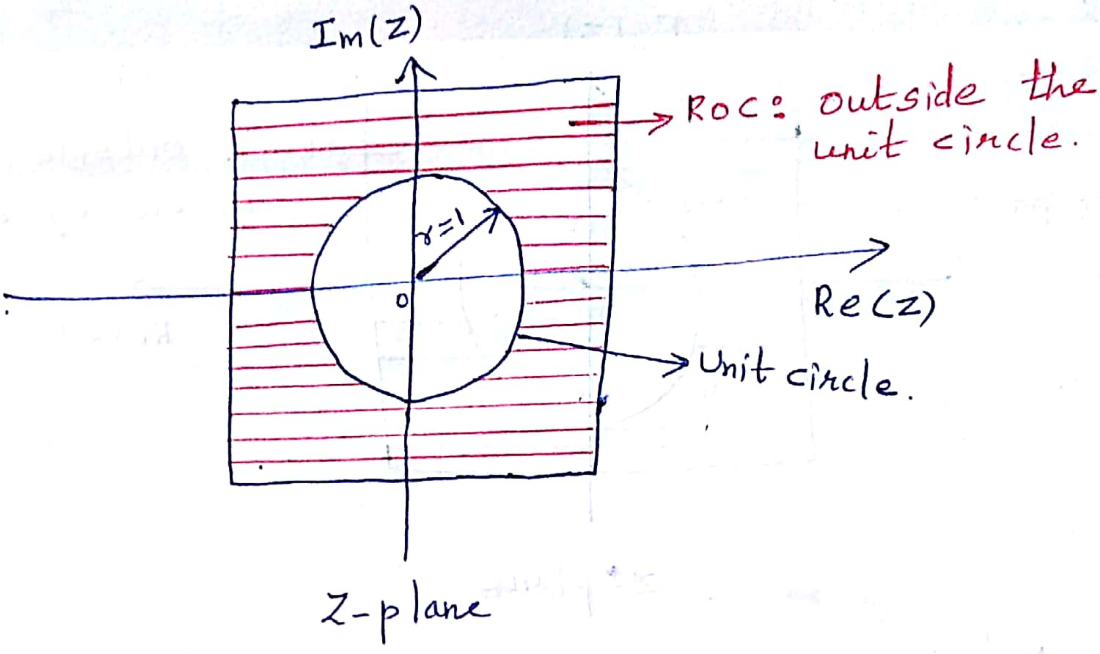
$$\boxed{\text{GSF}: \sum_{n=0}^{\infty} a^{n} = \frac{1}{1-a} ; |a|<1}$$

$$\boxed{Z.T[u(n)] = \frac{1}{1-z^{-1}} \; ; \; \text{Roc} : |z^{-1}|<1}$$

(or)

$$Z.T[u(n)] = \frac{1}{1-\frac{1}{z}} \; ; \; \text{Roc} ; \left|\frac{1}{z}\right|<1$$

$$\boxed{Z.T[u(n)] = \frac{z}{z-1} \; ; \; \text{Roc} : |z|>1}$$



5) Find the z- Transform of $a^n u(n)$:

Soln:

$$Z.T[x(n)] = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$ZT[a^n u(n)] = \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n}$$

$$\boxed{u(n) = \begin{cases} 1 & ; n \geqslant 0 \\ 0 & ; n < 0 \end{cases}}$$

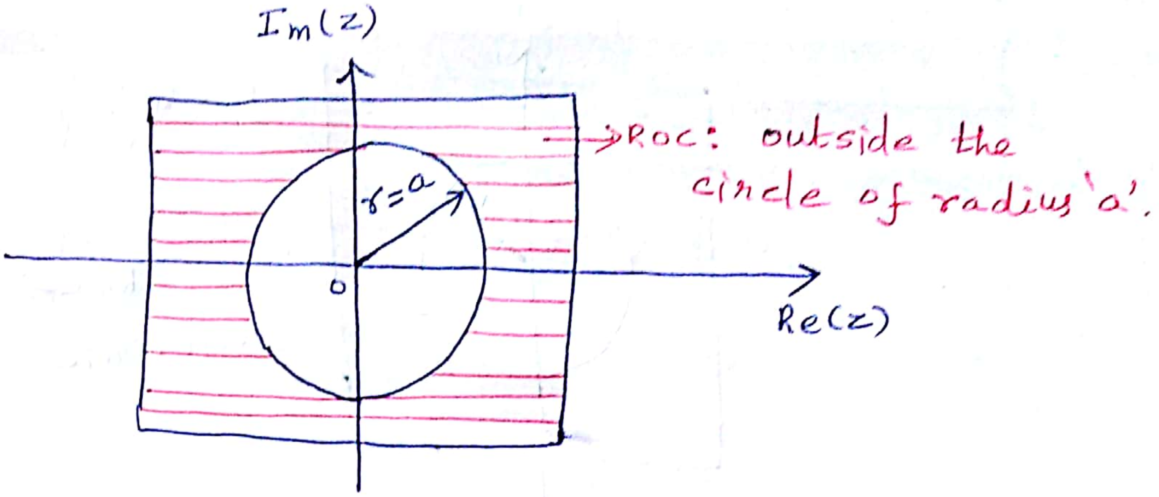
$$= \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

$$\boxed{\text{GSF: } \sum_{n=0}^{\infty} a^n = \frac{1}{1-a} \; ; \; |a|<1}$$

$$ZT[a^n u(n)] = \frac{1}{1-az^{-1}} \; ; \; \text{Roc: } |az^{-1}| < 1$$

(or)

$$Z.T[a^n u(n)] = \frac{z}{z-a} \; ; \; \text{Roc: } |z| > a$$



→ROC: outside the circle of radius 'a'.

z-plane.

Note:

$$Z.T\left[\left(\frac{1}{8}\right)^n u(n)\right] = \frac{1}{1-\frac{1}{8}z^{-1}} \text{ (or) } \frac{z}{z-\frac{1}{8}} \; ; \; ROC: |z| > \frac{1}{8}$$

6) Find the Z-Transform of $-b^n u(-n-1)$.

Soln:

$$z.T[x(n)] = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$z.T\left[-b^n u(-n-1)\right] = \sum_{n=-\infty}^{\infty} -b^n u(-n-1) z^{-n}$$

$$= -\sum_{n=-\infty}^{\infty} b^n u(-n-1) z^{-n}.$$

$u(n) = \begin{cases} 1 & ; n \geq 0 \\ 0 & ; n < 0 \end{cases}$ $u(-n-1) = \begin{cases} 1 ; & n \leq -1 \\ 0 ; & n > -1 \end{cases}$	$-n-1 = 0$ $\boxed{n = -1}$ $-n-1 = 1$ $\boxed{n = -2}$ $-n-1 = 2$ $\boxed{n = -3}$

$$= -\sum_{n=-\infty}^{-1} b^n z^{-n}$$

$$= -\sum_{n=1}^{\infty} b^{-n} z^{n}$$

$$= -\sum_{n=1}^{\infty} (b^{-1} z)^{n}$$

$$\boxed{\text{GSF : } \sum_{n=1}^{\infty} a^{n} = \frac{a}{1-a} \; ; \; |a|<1}$$

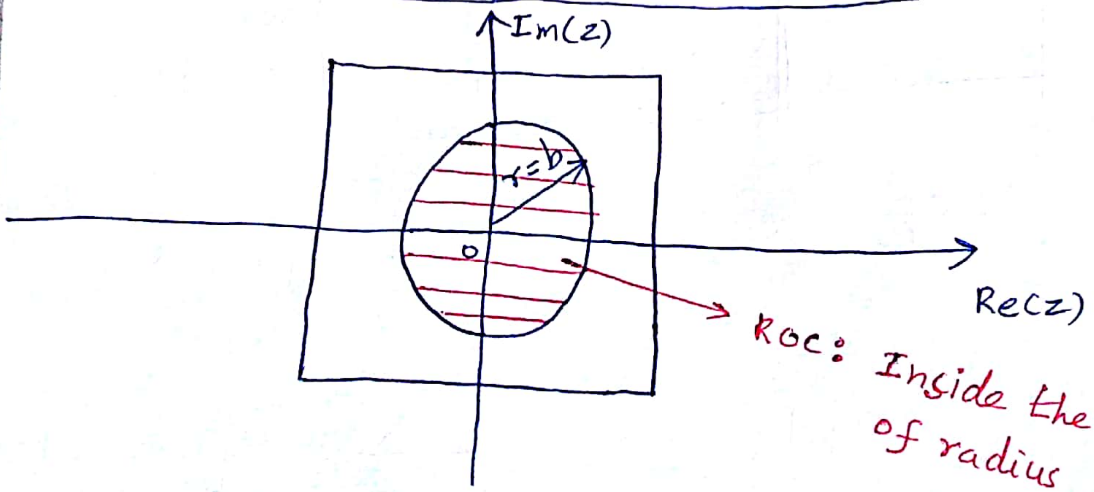
$$= -\left[\frac{b^{-1}z}{1-b^{-1}z}\right] \; ; \; |b^{-1}z|<1$$

$$= -\left[\frac{\frac{z}{b}}{1-\frac{z}{b}}\right] \; ; \; \left|\frac{z}{b}\right|<1$$

$$= -\left[\frac{\frac{z}{b}}{\frac{b-z}{b}}\right] \; ; \; |z|<b$$

$$= -\left[\frac{z}{-(z-b)}\right] \; ; \; |z|<b$$

$$\boxed{Z.T\left[-b^{n}u(-n-1)\right] = \frac{z}{z-b} \; ; \; \text{ROC: } |z|<b}$$



ROC: Inside the circle of radius 'b'.

7) Find the z-Transform of $a^n u(n-1)$:

Soln:

$$Z.T[x(n)] = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$z.T[a^n u(n-1)] = \sum_{n=-\infty}^{\infty} a^n u(n-1) z^{-n}$$

$$u(n) = \begin{cases} 1 & ; n \geq 0 \\ 0 & ; n < 0 \end{cases}$$

$$u(n-1) = \begin{cases} 1 & ; n \geq 1 \\ 0 & ; n < 1 \end{cases}$$

$n-1=0$
$\boxed{n=1}$
$n-1=1$
$\boxed{n=2}$
$n-1=2$
$\boxed{n=3}$

$$= \sum_{n=1}^{\infty} a^n z^{-n} = \sum_{n=1}^{\infty} (az^{-1})^n$$

GSF:

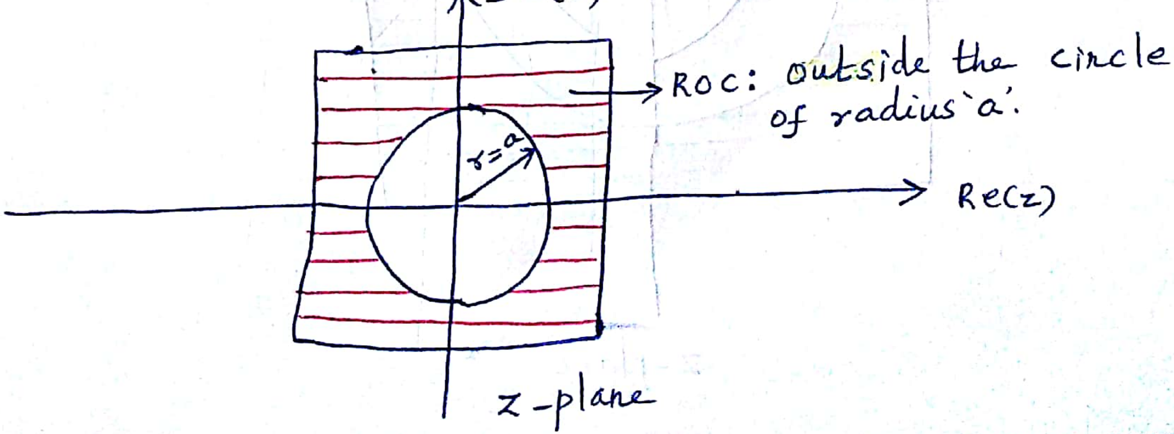
$$\sum_{n=1}^{\infty} a^n = \frac{a}{1-a} \; ; \; |a|<1$$

$$ZT[a^n u(n-1)] = \frac{az^{-1}}{1-az^{-1}} \; ; \quad ROC: |az^{-1}|<1$$

(or)

$$= \frac{\frac{a}{z}}{\frac{z-a}{z}} \; ; \quad ROC: \left|\frac{a}{z}\right|<1$$

$$\boxed{ZT[a^n u(n-1)] = \frac{a}{z-a} \; ; \quad ROC: |z|>a}$$



ROC: outside the circle of radius 'a'.

Note:

1) $Z.T[a^n u(n)] = \frac{z}{z-a}$; ROC : $|z| > a$

2) $Z.T[-b^n u(-n-1)] = \frac{z}{z-b}$; ROC : $|z| < b$

3) $Z.T[a^n u(n-1)] = \frac{a}{z-a}$; ROC : $|z| > a$.

8) Find the Z-Transform of $a^n u(n) - b^n u(-n-1)$:

soln:

$$Z.T[x(n)] = ZT[a^n u(n) - b^n u(-n-1)]$$

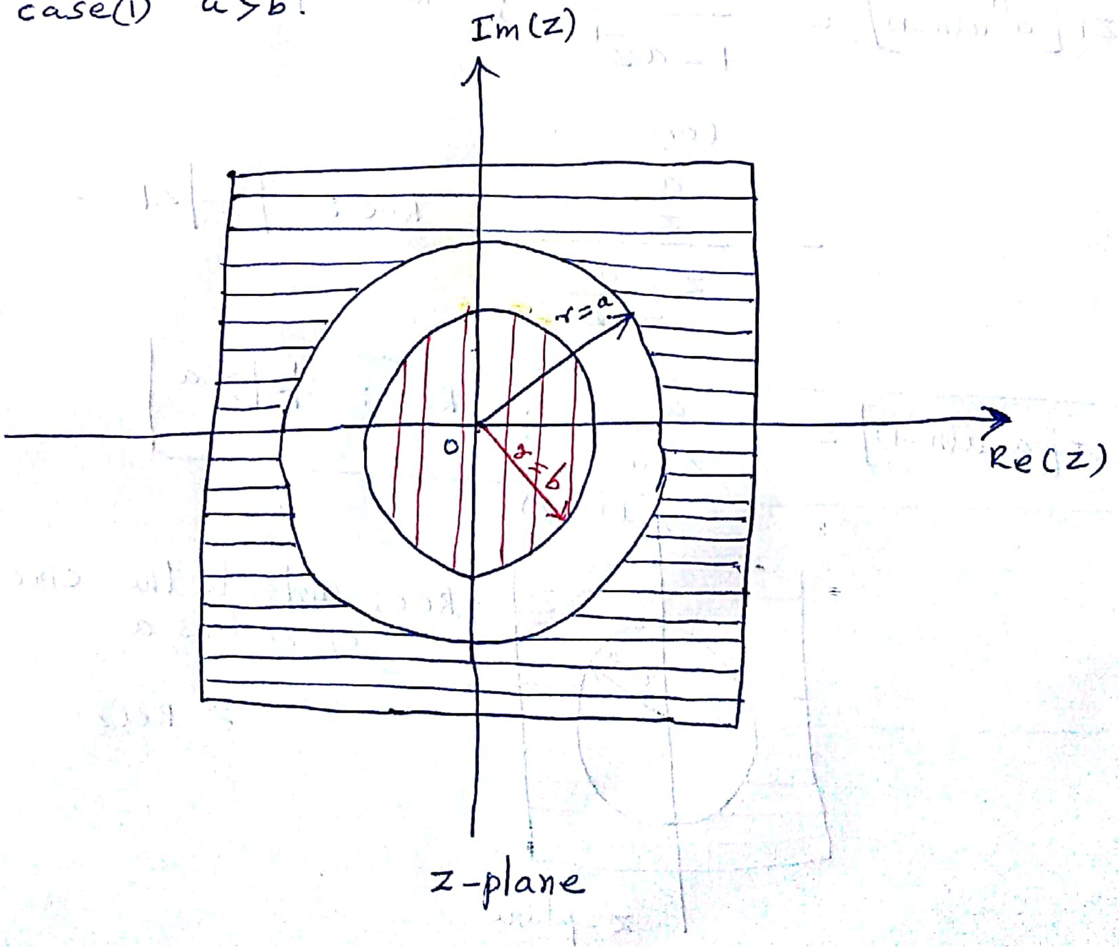
$$= ZT[a^n u(n)] + ZT[-b^n u(-n-1)]$$

$$= \frac{z}{z-a} \text{ ; ROC : } |z| > a + \frac{z}{z-b} \text{ ; ROC : } |z| < b$$

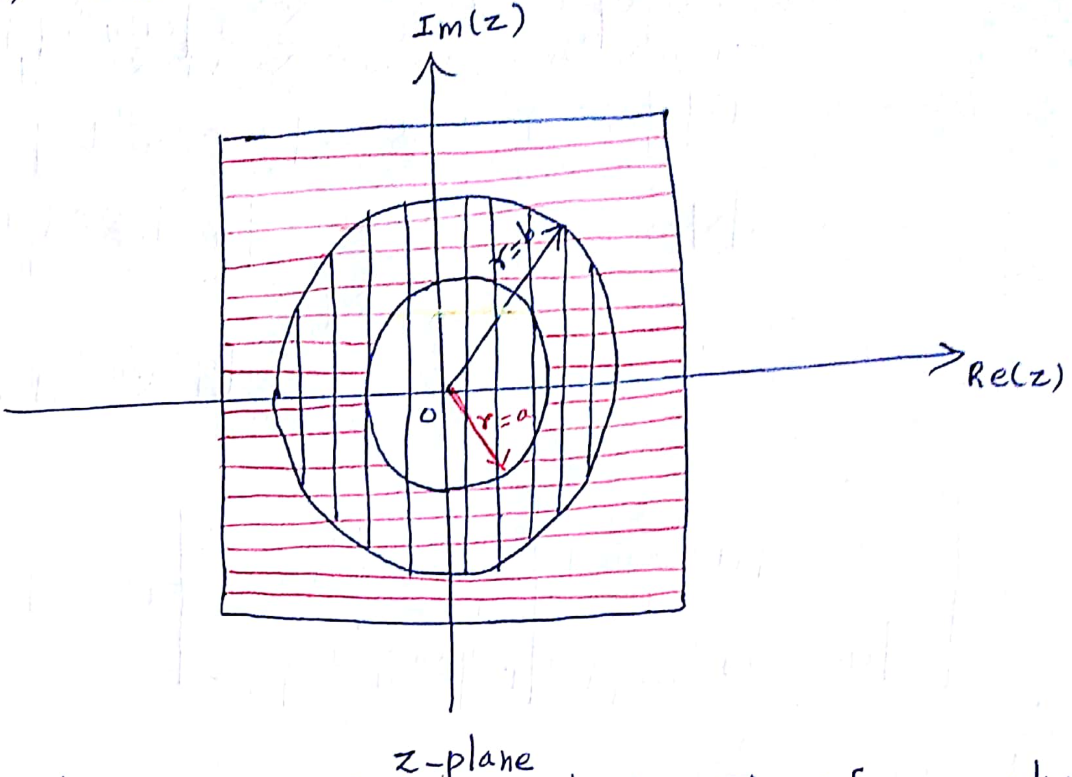
$$\boxed{X(z) = \frac{z}{z-a} + \frac{z}{z-b} \text{ ; ROC : } a < |z| < b.}$$

selection of a and b values:

case(i) a>b.



case(ii) $a<b$



Overlap occurs at $a<|z|<b$. Therefore, always select $a<b$.

→ Problems based on Z-Transform

1) Find the z-Transform of $\frac{a^n}{n!}$; $n \geq 0$.

Soln:

$$Z \cdot T[x(n)] = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$z \cdot T\left[\frac{a^n}{n!}\right] = \sum_{n=-\infty}^{\infty} \frac{a^n}{n!} z^{-n} = \sum_{n=0}^{\infty} \frac{a^n}{n!} z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{(az^{-1})^n}{n!}$$

$$\boxed{\text{G.S.F.} \quad \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x}$$

$$= e^{az^{-1}}$$

$$\boxed{Z \cdot T\left[\frac{a^n}{n!}\right] = e^{(a/z)}}$$

2) Find the z-Transform of $n a^n u(n)$.

soln:

W.k.T. the differentiation in z-domain prop. of Z.T,

$$Z \cdot T[x(n)] = X(z)$$

$$z \cdot T[n\, x(n)] = -z \frac{d X(z)}{dz}.$$

Here; $ZT[a^n u(n)] = X(z) = \frac{z}{z-a}$

$$Z \cdot T[n a^n u(n)] = -z \frac{d}{dz}\left[\frac{z}{z-a}\right]$$

$$= -z\left[\frac{(z-a).1 - z(1-0)}{(z-a)2}\right] = -z \frac{[\cancel{z}-a-\cancel{z}]}{(z-a)^2}$$

$$\boxed{Z \cdot T[n a^n u(n)] = \frac{az}{(z-a)^2}}$$

3) Find the z-Transform of $\cos\omega_0 n\, u(n)$:

Soln:

$$z\cdot T[x(n)] = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$z\cdot T[\cos\omega_0 n\, u(n)] = \sum_{n=-\infty}^{\infty} \cos\omega_0 n\, u(n)\, z^{-n}$$

$$= \sum_{n=0}^{\infty} \cos\omega_0 n\, z^{-n} \quad \left[\because u(n) = \begin{cases} 1 ; & n \geq 0 \\ 0 ; & n < 0 \end{cases}\right]$$

$$= \sum_{n=0}^{\infty} \left[\frac{e^{j\omega_0 n} + e^{-j\omega_0 n}}{2}\right] z^{-n}$$

$$= \frac{1}{2}\left[\sum_{n=0}^{\infty} (e^{j\omega_0} \cdot z^{-1})^n + \sum_{n=0} (e^{-j\omega_0} \cdot z^{-1})^n\right]$$

$$= \frac{1}{2}\left[\frac{1}{1-e^{j\omega_0} z^{-1}} + \frac{1}{1-e^{-j\omega_0} z^{-1}}\right]$$

$$= \frac{1}{2}\left[\frac{1-e^{-j\omega_0} z^{-1} + 1 - e^{j\omega_0} z^{-1}}{(1-e^{j\omega_0} z^{-1})(1-e^{-j\omega_0} z^{-1})}\right]$$

$$= \frac{1}{2}\left[\frac{2 - z^{-1}(e^{j\omega_0} + e^{-j\omega_0})}{1-e^{-j\omega_0} z^{-1} - e^{j\omega_0} z^{-1} + z^{-2}}\right]$$

$$= \frac{1}{2}\left[\frac{2 - z^{-1} \cdot 2\cos\omega_0}{1 - z^{-1}(e^{j\omega_0} + e^{-j\omega_0}) + z^{-2}}\right]$$

$$= \frac{1}{\cancel{2}}\left[\frac{\cancel{2}(1-\cos\omega_0 z^{-1})}{1 - 2\cos\omega_0 z^{-1} + z^{-2}}\right]$$

$$\boxed{z\cdot T[\cos\omega_0 n\, u(n)] = \frac{1-\cos\omega_0 z^{-1}}{1 - 2\cos\omega_0 z^{-1} + z^{-2}}}$$

4) Find the z-Transform of $a^n \cos\omega_0 n \, u(n)$:

Soln:

$$Z.T\left[a^n \cos\omega_0 n \, u(n)\right] = \underset{z \to \frac{z}{a}}{Lt} \; Z.T\left[\cos\omega_0 n \, u(n)\right]$$

$$= \underset{z \to \frac{z}{a}}{Lt} \left[\frac{1 - \cos\omega_0 z^{-1}}{1 - 2\cos\omega_0 z^{-1} + z^{-2}}\right]$$

$$\boxed{Z.T\left[a^n \cos\omega_0 n \, u(n)\right] = \frac{1 - \cos\omega_0 \left(\frac{z}{a}\right)^{-1}}{1 - 2\cos\omega_0 \left(\frac{z}{a}\right)^{-1} + \left(\frac{z}{a}\right)^{-2}}}$$

5) Find the z-Transform of $\sin\omega_0 n \, u(n)$.

Soln:

$$Z.T[x(n)] = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$Z.T\left[\sin\omega_0 n \, u(n)\right] = \sum_{n=-\infty}^{\infty} \sin\omega_0 n \, u(n) \, z^{-n}$$

$$= \sum_{n=0}^{\infty} \sin\omega_0 n \; z^{-n} \quad \left[\because u(n) = \begin{cases} 1 ; & n \geq 0 \\ 0 ; & n < 0 \end{cases}\right]$$

$$= \sum_{n=0}^{\infty} \left[\frac{e^{j\omega_0 n} - e^{-j\omega_0 n}}{2j}\right] z^{-n}$$

$$= \frac{1}{2j}\left[\sum_{n=0}^{\infty} \left(e^{j\omega_0} z^{-1}\right)^n - \sum_{n=0}^{\infty} \left(e^{-j\omega_0} - z^{-1}\right)^n\right]$$

$$= \frac{1}{2j}\left[\frac{1}{1 - e^{j\omega_0} z^{-1}} - \frac{1}{1 - e^{-j\omega_0} z^{-1}}\right]$$

$$= \frac{1}{2j}\left[\frac{\cancel{1}-e^{-j\omega_0}z^{-1}-\cancel{1}+e^{j\omega_0}z^{-1}}{(1-e^{j\omega_0}z^{-1})(1-e^{-j\omega_0}z^{-1})}\right]$$

$$= \frac{1}{2j}\left[\frac{(e^{j\omega_0}-e^{-j\omega_0})z^{-1}}{1-e^{-j\omega_0}z^{-1}-e^{j\omega_0}z^{-1}+z^{-2}}\right]$$

$$= \frac{1}{\cancel{2j}}\left[\frac{\cancel{2j}\sin\omega_0 z^{-1}}{1-(e^{j\omega_0}+e^{-j\omega_0})z^{-1}+z^{-2}}\right]$$

$$\boxed{Z.T[\sin\omega_0 n u(n)] = \frac{\sin\omega_0 z^{-1}}{1-2\cos\omega_0 z^{-1}+z^{-2}}}$$

6) Find the z-Transform of $a^n \sin\omega_0 n\, u(n)$:

Soln:

$$Z.T[a^n \sin\omega_0 n u(n)] = \underset{z\to\frac{z}{a}}{Lt}\; Z.T[\sin\omega_0 n\, u(n)]$$

$$= \underset{z\to\frac{z}{a}}{Lt}\left[\frac{\sin\omega_0 z^{-1}}{1-2\cos\omega_0 z^{-1}+z^{-2}}\right]$$

$$\boxed{Z.T[a^n \sin\omega_0 n\, u(n)] = \frac{\sin\omega_0\left(\frac{z}{a}\right)^{-1}}{1-2\cos\omega_0\left(\frac{z}{a}\right)^{-1}+\left(\frac{z}{a}\right)^{-2}}}$$

→ Problems based on Inverse z-Transform:

I) Long division (or) Power Series Expansion method:

1) Find the inverse z-Transform of $X(z) = \dfrac{1-\frac{1}{2}z^{-1}}{1+\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}}$ by long division method.

i) Roc : $|z| > \frac{1}{2}$ ii) Roc : $|z| < \frac{1}{2}$

Soln:

i) Roc : $|z| > \frac{1}{2}$

concept: For causal system, Roc lies outside the circle.

causal system : i) Polynomials → Negative powers of 'z'.

ii) Right $\overset{\text{Hand}}{\wedge}$ sided sequence

$$1 - \frac{5}{4}z^{-1} + \frac{13}{16}z^{-2} \ldots\ldots\ldots$$

$1 + \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}$ (higher to lower)

$1 - \frac{1}{2}z^{-1}$ [higher to lower power]

$$1 + \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}$$
$$(-)\ (-)\quad (-)$$

$$0 \quad -\frac{5}{4}z^{-1} - \frac{1}{8}z^{-2}$$

$$-\frac{5}{4}z^{-1} - \frac{15}{16}z^{-2} - \frac{5}{32}z^{-3}$$
$$(+)\quad (+)\quad (+)$$

$$0 \quad +\frac{13}{16}z^{-2} + \frac{5}{32}z^{-3}$$

$$+\frac{13}{16}z^{-2} + \frac{39}{64}z^{-3} + \frac{13}{128}z^{-4}$$
$$(-)\quad (-)\quad (-)$$

$$0 \quad -\frac{29}{64}z^{-3} - \frac{13}{128}z^{-4}.$$

Quotient:

$\frac{1}{1} = 1$

$\dfrac{-\frac{5}{4}z^{-1}}{1} = -\frac{5}{4}z^{-1}$

$\dfrac{\frac{13}{16}z^{-2}}{1} = \frac{13}{16}z^{-2}$

simplification:

$-\frac{1}{2} - \frac{3}{4} = \frac{-2-3}{4}$

$= -\frac{5}{4}$

$\frac{3}{4} \times -\frac{5}{4} = \frac{-15}{16}$

$-\frac{1}{8} + \frac{15}{16} = \frac{-2+15}{16}$

$= \frac{13}{16}$

$\frac{13}{16} \times \frac{3}{4} = \frac{39}{64}$

$\frac{13}{16} \times \frac{1}{8} = \frac{13}{128}$

$\frac{5}{32} - \frac{39}{64} =$

$\frac{10-39}{64}$

$= \frac{-29}{64}$

Now,

$$X(z) = \left\{1 - \frac{5}{4}z^{-1} + \frac{13}{16}z^{-2} \ldots\ldots\right.$$

Taking Inv. z.T,

$$x(n) = \left\{\underset{\substack{\uparrow \\ n=0.}}{1}, -\frac{5}{4}, +\frac{13}{16}, \ldots\right.$$

ii) Roc $|z| < \frac{1}{2}$:

concept : For noncausal system, Roc lies inside the circle.

Noncausal system : i) Polynomials → +ve powers of 'z'.

ii) Left Hand sided sequence.

$\frac{1}{8}z^{-2} + \frac{3}{4}z^{-1} + 1$

(lower to higher)

$$-4z + 32z^{2} - 160z^{3}$$

$-\frac{1}{2}z^{-1} + 1$ (lower to higher)

$$-\frac{1}{2}z^{-1} - 3 - 4z$$

$$(+) \quad (+) \quad (+)$$

$$0 \quad + 4 + 4z$$

$$+ 4 + 24z + 32z^{2}$$

$$(-) \quad (-) \quad (-)$$

$$0 - 20z - 32z^{2}$$

$$- 20z - 120z^{2} - 160z^{3}$$

$$(+) \quad (+) \quad (+)$$

$$0 \quad + 88z^{2} + 160z^{3}$$

Quotient:

$$\frac{-\frac{1}{2}z^{-1}}{+\frac{1}{8}z^{-2}} = -4z$$

$$\frac{4}{\frac{1}{8}z^{-2}} = 32 z^{2}$$

$$\frac{-20z}{\frac{1}{8}z^{-2}} = -160z^{3}$$

Simplification

$$-4 \times \frac{1}{8} = -\frac{1}{2}$$

$$-4 \times \frac{3}{4} = -3$$

$$32 \times \frac{1}{8} = 4$$

$$\overset{8}{\cancel{32}} \times \frac{3}{\cancel{4}} = 24$$

$$-160 \times \frac{1}{8} = -20$$

$$-\overset{40}{\cancel{160}} \times \frac{3}{\cancel{4}} = -120$$

Now,

$$X(z) = \{ \ldots\ldots -160z^{3} + 32z^{2} - 4z \}.$$

Taking Inv. Z.T,

$$x(n) = \{ \ldots\ldots -160, 32, \underset{\substack{\uparrow \\ n=-1}}{-4} \}.$$

(or)

$$x(n) = \{ \ldots, -160, 32, -4, \underset{\substack{\uparrow \\ n=0}}{0} \}$$

Note : i) For causal s/m,

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n} = x(0) + x(1) z^{-1} + x(2) z^{-3} + \ldots$$

ii) For non causal s/m,

$$X(Z) = \sum_{n=-\infty}^{0} x(n) z^{-n} = \ldots\ldots + x(-2) z^{+2} + x(-1) z^{+1} + x(0).$$

2). Using Long Division method, determine Z-Transform of $x(n)$

$X(z) = \frac{z+1}{z^2-3z+2}$ if a) $x(n)$ is causal b) $x(n)$ is anticausal.

Soln:

a) $x(n)$ is causal:

→ Right Hand sided sequence & -ve powers of 'z'.

$z^{-1} + 4z^{-2} + 10z^{-3} \ldots$

z^2-3z+2 [higher to lower]) $z+1$ [higher to lower]

$z - 3 + 2z^{-1}$

$(-) \quad (+) \quad (-)$

$0 + 4 - 2z^{-1}$

$+4 - 12z^{-1} + 8z^{-2}$

$(-) \quad (+) \quad (-)$

$0 \quad +10z^{-1} - 8z^{-2}$

$+10z^{-1} - 30z^{-2} + 20z^{-3}$

$(-) \quad (+) \quad (-)$

$0 \quad 22z^{-2} - 20z^{-3}$

Quotient.

$\frac{z}{z^2} = z^{-1}$

$\frac{4}{z^2} = 4z^{-2}$

$\frac{10z^{-1}}{z^2} = 10z^{-3}$

simplification:

$4 \times -3 = -12$

$4 \times 2 = 8$

$10 \times -3 = -30$

$10 \times 2 = 20$

Now,

$X(z) = \{z^{-1} + 4z^{-2} + 10z^{-3} + \ldots$

Taking Inv. Z.T,

$x(n) = \{\underset{\substack{\uparrow \\ n=1}}{1}, 4, 10, \ldots\ldots$

b) $x(n)$ is anticausal.

→ Left Hand sided sequence → +Powers of 'z'.

$$\frac{1}{2}+\frac{5}{4}z+\frac{13}{8}z^2-----$$

$2-3z+z^2$ [lower to higher]

$1+z$ [lower to higher]

$1-\frac{3}{2}z+\frac{1}{2}z^2$

(−) (+) (−)

$0+\frac{5}{2}z-\frac{1}{2}z^2$

$+\frac{5}{2}z-\frac{15}{4}z^2+\frac{5}{4}z^3$

(−) (+) (−)

$0\quad+\frac{13}{4}z^2-\frac{5}{4}z^3$

$+\frac{13}{4}z^2-\frac{39}{8}z^3+\frac{13}{8}z^4$

(−) (+) (−)

$0\quad+\frac{29}{8}z^3-\frac{13}{8}z^4$

Now,

$$X(z)=\left\{\cdots\cdots\frac{13}{8}z^2+\frac{5}{4}z+\frac{1}{2}\right\}$$

Taking Inv. Z.T,

$$x(n)=\left\{\cdots\cdots\frac{13}{8},\ \underset{\underset{n=-1}{\uparrow}}{\frac{5}{4}},\ \underset{\underset{n=0}{\uparrow}}{\frac{1}{2}}\right\}$$

Quotient:

$\frac{1}{2}$

$\frac{\frac{5}{2}z}{2}=\frac{5}{4}z$

$\frac{\frac{13}{4}z^2}{2}=\frac{13}{8}z^2$

Simplification:

$\frac{1}{2}\times-3=-\frac{3}{2}$

$\frac{3}{2}+1=\frac{3+2}{2}=\frac{5}{2}$

$\frac{5}{4}\times-3=-\frac{15}{4}$

$-\frac{1}{2}+\frac{15}{4}=\frac{-2+15}{4}=\frac{13}{4}$

$\frac{13}{8}\times 2=\frac{13}{4}$

$\frac{13}{8}\times-3=-\frac{39}{8}$

$-\frac{5}{4}+\frac{39}{8}=\frac{-10+39}{8}=\frac{29}{8}$

→ Inverse z-Transform by Partial Fraction method.

a) simple pole type

1) Find the inverse z-Transform of $X(z) = \frac{1}{1-1.5z^{-1}+0.5z^{-2}}$

if a) ROC : $|z|>1$ b) ROC: $|z|<0.5$ c) ROC: $0.5<|z|<1$.

Soln:

$$X(z) = \frac{1}{1-1.5z^{-1}+0.5z^{-2}}$$

Convert -ve powers of 'z' into +ve powers of 'z'.

$$X(z) = \frac{z^2}{z^2-1.5z+0.5}$$

If Dr. degree > Nr. degree, we can proceed partial fraction.

$$\frac{X(z)}{z} = \frac{z}{z^2-1.5z+0.5}$$

Factorization of Dr. polynomial : $a = 1,\ b=-1.5,\ c=0.5$

$$z = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-(-1.5) \pm \sqrt{(-1.5)^2-4\times1\times0.5}}{2\times1}$$

$$z = \frac{1.5 \pm \sqrt{2.25-2}}{2} = \frac{1.5 \pm \sqrt{0.25}}{2} = \frac{1.5 \pm 0.5}{2} = 1,\ 0.5$$

$z = 1$ and $z = 0.5$

$$\frac{X(z)}{z} = \frac{z}{(z-1)(z-0.5)}$$

$$\frac{X(z)}{z} = \frac{K_0}{z-1} + \frac{K_1}{z-0.5} \longrightarrow ①.$$

$$K_0 = \frac{X(z)}{z} \times (z-1) \Big|_{z=1}$$

$$= \frac{z}{\cancel{(z-1)}(z-0.5)} \times \cancel{(z-1)} \Big|_{z=1} = \frac{1}{(1-0.5)} = \frac{1}{0.5}$$

$$\boxed{K_0 = 2}$$

$$K_1 = \frac{X(z)}{z} \times (z-0.5) \Big|_{z=0.5}$$

$$K_1 = \frac{z}{(z-1)\cancel{(z-0.5)}} \times \cancel{(z-0.5)} \Big|_{z=0.5} = \frac{0.5}{(0.5-1)} = \frac{0.5}{-0.5}$$

$$\boxed{K_1 = -1}$$

Sub. K_0 and K_1 values in eqn ①,

$$\frac{X(z)}{z} = \frac{2}{z-1} - \frac{1}{z-0.5}$$

$$X(z) = 2 . \frac{z}{z-1} - \frac{z}{z-0.5} \longrightarrow ② .$$

a). ROC: $|z| > 1$: [causal system]

Taking Inv. Z.T of eqn. ②,

$$\boxed{x(n) = 2\,(1)^n\, u(n) - (0.5)^n\, u(n)}$$

b) ROC: $|z| < 0.5$ [anticausal system]

Taking Inv. Z.T of eqn ②,

$$x(n) = 2\,(1)^n\, u(n) - [-(0.5)^n\, u(-n-1)]$$

$$\boxed{x(n) = 2\,(1)^n\, u(n) + (0.5)^n\, u(-n-1)}$$

c) Roc : $0.5 < |z| < 1$

Roc : $|z| > 0.5$ & $|z| < 1$

Taking Inv. Z.T of eqn ②,

$$\boxed{x(n) = 2.\left[-(1)^n u(-n-1)\right] - (0.5)^n u(n).}$$

2) Find the Inverse Z-Transform of $X(z) = \dfrac{1 - \frac{1}{2}z^{-1}}{1 + \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}}$ if Roc : $|z| > \frac{1}{2}$.

Soln :

$$X(z) = \frac{1 - \frac{1}{2}z^{-1}}{1 + \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}}$$

Convert -ve powers of 'z' into +ve powers of 'z'.

$$X(z) = \frac{z^2 - \frac{1}{2}z}{z^2 + \frac{3}{4}z + \frac{1}{8}}$$

If Dr. degree > Nr. degree, we can proceed partial fraction.

$$\frac{X(z)}{z} = \frac{z - \frac{1}{2}}{z^2 + \frac{3}{4}z + \frac{1}{8}} = \frac{z - 0.5}{z^2 + 0.75z + 0.125}$$

Factorization of Dr. polynomial : $a = 1, b = 0.75, c = 0.125$

$$z = \frac{-0.75 \pm \sqrt{0.75^2 - 4 \times 1 \times 0.125}}{2 \times 1} = \frac{-0.75 \pm \sqrt{0.0625}}{2} = -0.25, -0.5$$

$z = -0.25$ (or) $z = -0.5$

$$\frac{X(z)}{z} = \frac{z - 0.5}{(z + 0.5)(z + 0.25)}$$

$$\frac{X(z)}{z} = \frac{k_0}{z + 0.5} + \frac{k_1}{z + 0.25} \longrightarrow ①$$

Scanned by CamScanner

$$K_0 = \frac{X(z)}{z} \times (z+0.5) \Big|_{z=-0.5}$$

$$K_0 = \frac{z-0.5}{(z+0.25)\cancel{(z+0.5)}} \times \cancel{(z+0.5)} \Big|_{z=-0.5}$$

$$K_0 = \frac{-0.25-0.5}{(-0.5+0.25)} = \frac{-1}{-0.25}$$

$$\boxed{K_0 = 4}$$

$$K_1 = \frac{X(z)}{z} \times (z+0.25) \Big|_{z=-0.25}$$

$$K_1 = \frac{z-0.5}{\cancel{(z+0.25)}(z+0.5)} \times \cancel{(z+0.25)} \Big|_{z=-0.25}$$

$$k_1 = \frac{-0.25-0.5}{(-0.25+0.5)} = \frac{-0.75}{0.25}$$

$$\boxed{K_1 = -3}$$

sub. K_0 and K_1 values in eqn. ①,

$$\frac{X(z)}{z} = \frac{4}{z+0.5} - \frac{3}{z+0.25}$$

$$X(z) = 4 \cdot \frac{z}{z+0.5} - 3 \cdot \frac{z}{z+0.25} \rightarrow ②.$$

Given: ROC : $|z| > \frac{1}{2}$ [causal system]

Taking Inv. Z.T of eqn ②,

$$\boxed{x(n) = 4\,(-0.5)^n\,u(n) - 3\,(-0.25)^n\,u(n)}$$

3) Find the Inverse Z-Transform of $X(z) = \frac{1-\frac{1}{2}z^{-1}}{1-\frac{1}{4}z^{-2}}$ if ROC: $|z| > \frac{1}{2}$.

Soln:

$$X(z) = \frac{1-\frac{1}{2}z^{-1}}{1-\frac{1}{4}z^{-2}}$$

convert -ve powers of 'z' into +ve powers of 'z'.

$$X(z) = \frac{z^2 - \frac{1}{2}z}{z^2 - \frac{1}{4}} = \frac{z^2 - 0.5z}{z^2 - 0.25}$$

If Dr. degree > Nr. degree, we can proceed partial fraction.

$$\frac{X(z)}{z} = \frac{z-0.5}{z^2-0.25} = \frac{z-0.5}{z^2-0.5^2}$$

$$\frac{X(z)}{z} = \frac{z-0.5}{(z+0.5)(z-0.5)} = \frac{1}{(z+0.5)}$$

$$X(z) = \frac{z}{z+0.5} \longrightarrow ①$$

Given: ROC : $|z| > \frac{1}{2}$. [causal system]

Taking Inv. Z.T of eqn.①,

$$\boxed{x(n) = (-0.5)^n \, u(n)}$$

Scanned by CamScanner

→ Inverse Z-Transform by Partial Fraction Method:

b) Multipoles type

1) Find $x(n)$ from the following function :

$$X(z) = \frac{1}{(1+z^{-1})(1-z^{-1})^2} \quad ; \quad \text{Roc: } |z|>1$$

Soln:

$$X(z) = \frac{1}{(1+z^{-1})(1-z^{-1})^2}$$

convert -ve powers of 'z' into +ve powers of 'z'.

$$X(z) = \frac{z^3}{(z+1)(z-1)^2}$$

If Dr. degree > Nr. degree, we can proceed partial fraction.

$$\frac{X(z)}{z} = \frac{z^2}{(z+1)(z-1)^2}$$

$$\frac{X(z)}{z} = \frac{A}{z+1} + \frac{K_0}{(z-1)^2} + \frac{K_1}{(z-1)} \rightarrow ①$$

$$A = \left.\frac{X(z)}{z} \times (z+1)\right|_{z=-1}$$

$$A = \left.\frac{z^2}{\cancel{(z+1)}(z-1)^2} \times \cancel{(z+1)}\right|_{z=-1} = \frac{(-1)^2}{(-1-1)^2} = \frac{1}{4}$$

$$\boxed{A = \frac{1}{4}}$$

Multipole residue formula:

$$K_j = \frac{1}{j!} \left.\frac{d^j}{dz^j}\left[\frac{X(z)}{z} \cdot (z-P_k)^n\right]\right|_{z=P_k} ; \; j=0,1,2,\ldots n$$

$$K_0 = \frac{1}{0!}\frac{d^0}{dz^0}\left[\frac{X(z)}{z}\cdot(z-1)^2\right]\Bigg|_{z=1}$$

$$K_0 = \frac{z^2}{(z+1)\cancel{(z-1)^2}} \times \cancel{(z-1)^2}\Bigg|_{z=1} = \frac{1^2}{(1+1)} = \frac{1}{2} \quad \left[\because 0! = 1,\ \frac{d^0}{dz^0} = 1\right]$$

$$\boxed{K_0 = \frac{1}{2}}$$

$$K_1 = \frac{1}{1!}\frac{d}{dz}\left[\frac{X(z)}{z}\cdot(z-1)^2\right]\Bigg|_{z=1}$$

$$= \frac{d}{dz}\left[\frac{z^2}{(z+1)\cancel{(z-1)^2}} \times \cancel{(z-1)^2}\right]\Bigg|_{z=1}$$

$$= \frac{d}{dz}\left[\frac{z^2}{z+1}\right]\Bigg|_{z=1} = \frac{(z+1)(2z) - z^2(1+0)}{(z+1)^2}\Bigg|_{z=1}$$

$$K_1 = \frac{(1+1)(2\cdot1) - 1^2}{(1+1)^2} = \frac{4-1}{4} = \frac{3}{4}.$$

$$\boxed{K_1 = \frac{3}{4}}$$

sub. K_0, K_1 and A values in eqn. ①,

$$\frac{X(z)}{z} = \frac{1/4}{z+1} + \frac{1/2}{(z-1)^2} + \frac{3/4}{z-1}$$

$$X(z) = \left(\frac{1}{4}\right)\cdot\frac{z}{z+1} + \left(\frac{1}{2}\right)\frac{z}{(z-1)^2} + \left(\frac{3}{4}\right)\frac{z}{z-1}. \rightarrow ②.$$

$$\frac{z}{(z-a)^k} \xrightarrow{IZT} \frac{n(n-1)(n-2)\ldots\ldots(n-[k-2])\,a^{n-(k-1)}}{(k-1)!}\cdot u(n)$$

$$\frac{z}{z-a} \xRightarrow{IZT} a^n u(n) \qquad \frac{z}{(z-a)^3} = \frac{n(n-1)a^{n-2}}{2!}\cdot u(n)$$

$$\frac{z}{(z-a)^2} \rightarrow n a^{n-1} u(n) \qquad \frac{z}{(z-a)^4} = \frac{n(n-1)(n-2)a^{n-3}}{3!}\cdot u(n)$$

Given: ROC: $|z| > 1$.

Taking Inv. Z.T of eqn ②,

$$\boxed{x(n) = \left(\frac{1}{4}\right)(-1)^n u(n) + \left(\frac{1}{2}\right)\cdot n(1)^{n-1} u(n) + \left(\frac{3}{4}\right)(1)^n u(n)}$$

2) Find the inverse z-Transform of $X(z) = \frac{z(z-1)}{(z+1)(z+2)^3}$.

Soln:

$$X(z) = \frac{z(z-1)}{(z+1)(z+2)^3}$$

$$\frac{X(z)}{z} = \frac{z-1}{(z+1)(z+2)^3}$$

$$\frac{X(z)}{z} = \frac{A}{z+1} + \frac{K_0}{(z+2)^3} + \frac{K_1}{(z+2)^2} + \frac{K_2}{z+2} \longrightarrow ①$$

$$A = \frac{X(z)}{z}\cdot(z+1)\Big|_{z=-1}$$

$$A = \frac{z-1}{\cancel{(z+1)}(z+2)^3} \times \cancel{(z+1)}\Big|_{z=-1} = \frac{-1-1}{(-1+2)^3} = \frac{-2}{(1)^3} = \frac{-2}{1}$$

$$\boxed{A = -2}$$

$$\boxed{K_j = \frac{1}{j!}\frac{d^j}{dz^j}\left[\frac{X(z)}{z}\cdot(z-P_k)^n\right]\Bigg|_{z=P_k} \quad ; j = 0, 1, 2, \ldots n}$$

$$K_0 = \frac{1}{0!}\frac{d^0}{dz^0}\left[\frac{z-1}{(z+1)\cancel{(z+2)^3}} \times \cancel{(z+2)^3}\right]\Bigg|_{z=-2} = \frac{-2-1}{-2+1} = \frac{-3}{-1}$$

$$\boxed{K_0 = 3}$$

$$k_1 = \frac{1}{1!}\frac{d}{dz}\left[\frac{z-1}{(z+1)\cancel{(z+2)^3}} \times \cancel{(z+2)^3}\right]\Bigg|_{z=-2}$$

$$k_1 = \frac{d}{dz}\left[\frac{z-1}{z+1}\right]\Bigg|_{z=-2} = \frac{(z+1)[1-0]-[z-1][1+0]}{(z+1)^2}\Bigg|_{z=-2}$$

$$k_1 = \frac{(-2+1)-(-2-1)}{(-2+1)^2} = \frac{(-1)-(-3)}{(-1)^2} = \frac{-1+3}{1} = \frac{2}{1}$$

$$\boxed{k_1 = 2}$$

$$k_2 = \frac{1}{2!}\frac{d^2}{dz^2}\left[\frac{z-1}{(z+1)\cancel{(z+2)^3}} \cdot \cancel{(z+2)^3}\right]\Bigg|_{z=-2}$$

$$k_2 = \frac{1}{2}\frac{d^2}{dz^2}\left[\frac{z-1}{z+1}\right]\Bigg|_{z=-2}$$

$$k_2 = \frac{1}{2}\frac{d}{dz}\left[\frac{(z+1)(1-0)-(z-1)(1+0)}{(z+1)^2}\right]\Bigg|_{z=-2}$$

$$k_2 = \frac{1}{2}\frac{d}{dz}\left[\frac{\cancel{z}+1-\cancel{z}+1}{(z+1)^2}\right]\Bigg|_{z=-2} = \frac{1}{\cancel{2}}\frac{d}{dz}\left[\frac{\cancel{2}}{(z+1)^2}\right]\Bigg|_{z=-2}$$

$$k_2 = \frac{d}{dz}\left[\frac{1}{(z+1)^2}\right]\Bigg|_{z=-2} = \frac{d}{dz}\left[(z+1)^{-2}\right]\Bigg|_{z=-2}$$

$$k_2 = -2(z+1)^{-3}\Big|_{z=-2} = \frac{-2}{(z+1)^3}\Bigg|_{z=-2} = \frac{-2}{(-2+1)^3} = \frac{-2}{(-1)^3}$$

$$k_2 = \frac{-2}{-1}$$

$$\boxed{k_2 = 2}$$

Sub. K_0, K_1, K_2 and A in eqn ①,

$$\frac{X(z)}{z} = \frac{-2}{z+1} + \frac{3}{(z+2)^3} + \frac{2}{(z+2)^2} + \frac{2}{z+2}$$

$$\boxed{\frac{z}{(z-a)^k} \xRightarrow{IZT} \frac{n(n-1)(n-2)\cdots\cdots(n-(k-2))\, a^{n-(k-1)}}{(k-1)!} u(n)}$$

$$X(z) = -2 \cdot \frac{z}{z+1} + 3 \frac{z}{(z+2)^3} + 2 \cdot \frac{z}{(z+2)^2} + 2 \cdot \frac{z}{z+2} \rightarrow ②.$$

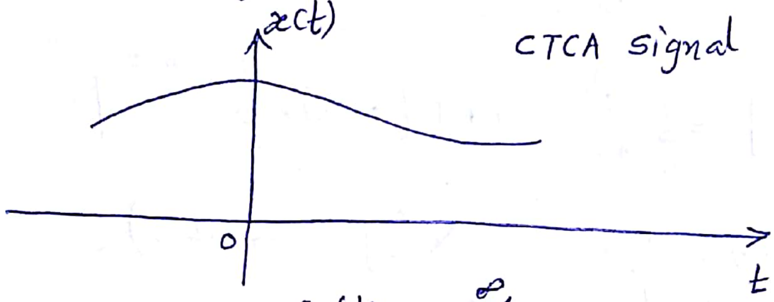
Taking Inv. Z.T of eqn.②,

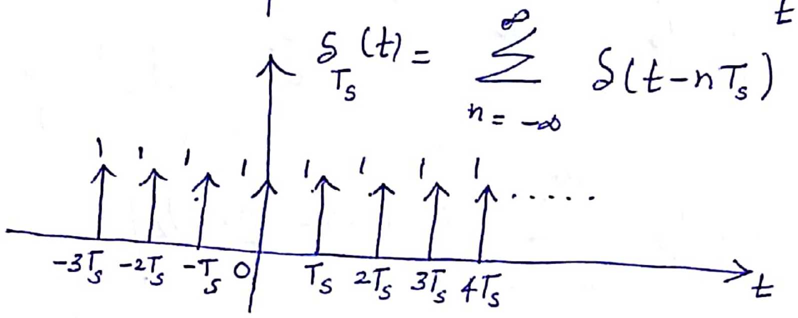
$$\boxed{x(n) = -2\,(-1)^n u(n) + 3 \cdot \frac{n(n-1)}{2!} (-2)^{n-2} u(n) + \frac{2n\,(-2)^{n-1}}{1!} u(n) + 2\,(-2)^n u(n).}$$

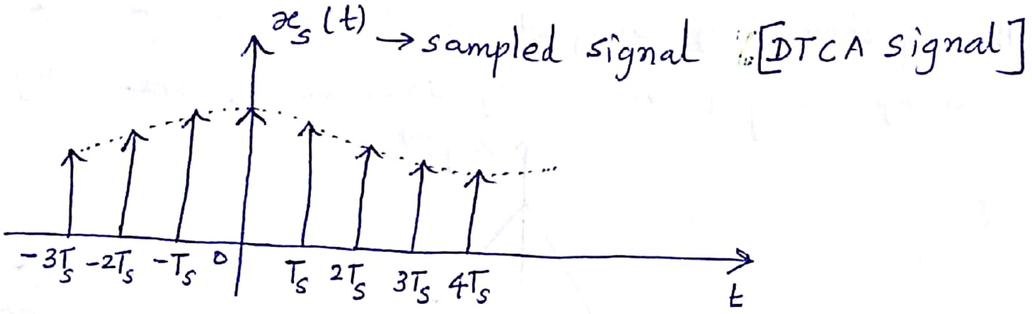
→ Baseband signal sampling :

* Sampling :

The process by which converting continuous Time Continuous Amplitude (CTCA) into Discrete Time continuous Amplitude (DTCA) is called sampling.







Let $x(t) \rightarrow$ CTCA signal.

$\delta_{T_s}(t) \rightarrow$ Pulse Train

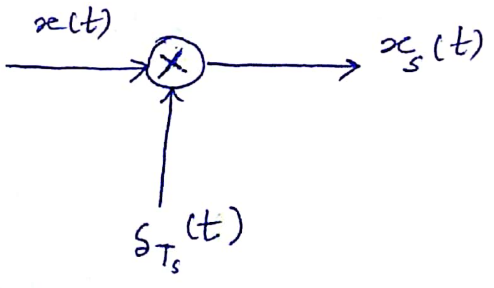
$$\delta_{T_s}(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT_s)$$

Fourier series Representation of pulse train is given by

$$\delta_{T_s}(t) = \frac{1}{T_s} \sum_{m=-\infty}^{\infty} e^{j\Omega_0 m t} \rightarrow ①$$

where,

$$\Omega_0 = \frac{2\pi}{T_s}$$



From the diagram, the sampled signal

$$x_s(t) = x(t) \cdot \delta_{T_s}(t)$$

$$x_s(t) = \frac{1}{T_s} \sum_{m=-\infty}^{\infty} x(t) \cdot e^{j\Omega_0 m t} \qquad [\because \text{using eqn ①}]$$

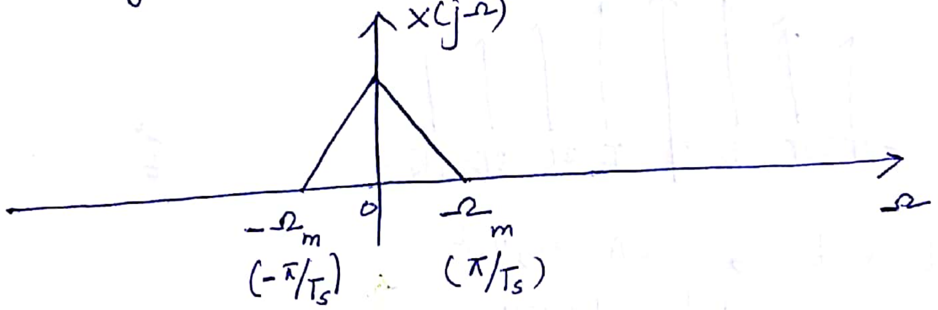
$$\text{CTFT}\left[x_s(t)\right] = \frac{1}{T_s} \sum_{m=-\infty}^{\infty} \text{CTFT}\left[x(t) \cdot e^{j\Omega_0 m t}\right]$$

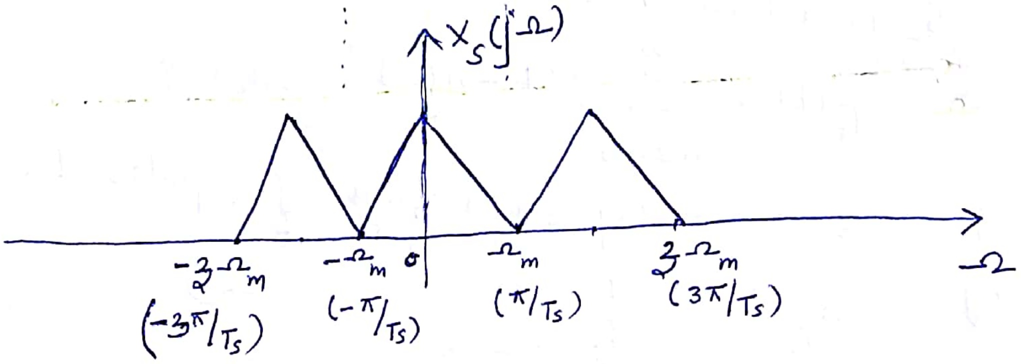
$$X_s(j\Omega) = \frac{1}{T_s} \sum_{m=-\infty}^{\infty} X(j(\Omega - m\Omega_0)).$$

[∵ using Frequency shifting property of CTFT]

$$\boxed{X_s(j\Omega) = \frac{1}{T_s} \sum_{m=-\infty}^{\infty} X\left(j\left(\Omega - \frac{2\pi}{T_s} m\right)\right)}$$

The spectrum of sampled signal is equal to the infinite sum of delayed version of spectrum of the original signal.

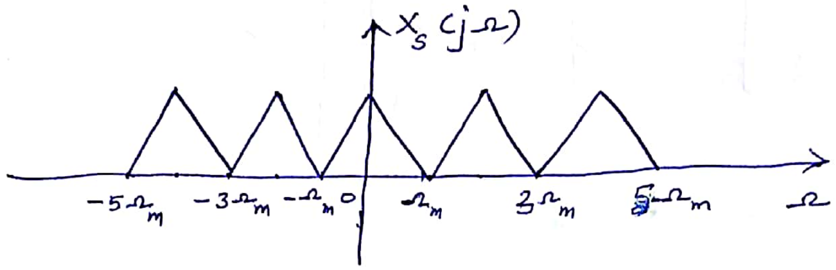




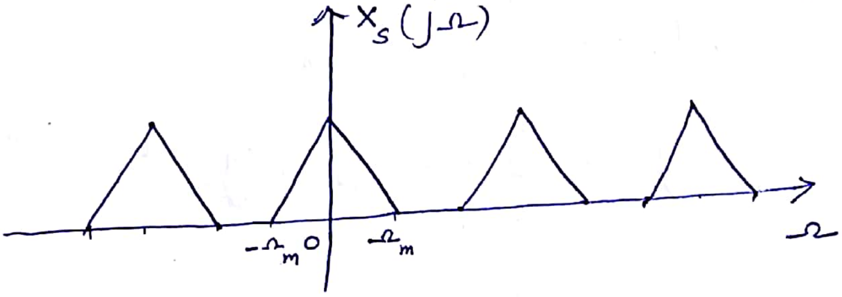
* Sampling Theorem:

The band limited signal $x(t)$ with $|X(j\Omega)| = 0$ for $|\Omega| > \Omega_m$ is uniquely determined from its samples if the sampling frequency $f_s \geq 2f_m$. where, $f_m \rightarrow$ maximum frequency component in the signal $x(t)$.

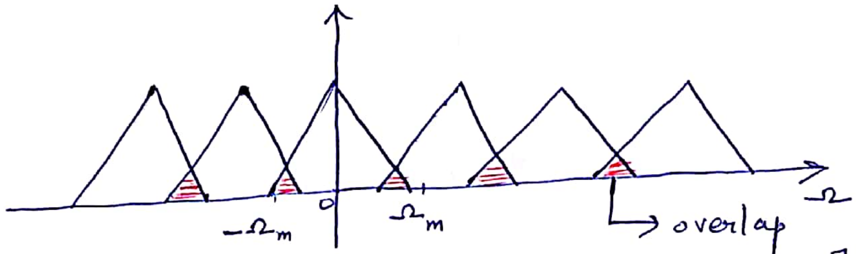
case(i) : $f_s = 2f_m$ [No aliasing]



case(ii) : $f_s > 2f_m$ No overlap [No aliasing]



case(iii) : $f_s < 2f_m$



[Aliasing].

* Aliasing

If sampling frequency $f_s < 2f_m$, overlap occurs between high frequency components and low frequency components, i.e, high frequency components appear as low frequency components, is called as aliasing.

* To avoid aliasing,

i) The signal $x(t)$ is strictly bandlimited with $|X(j\Omega)| = 0$ for $|\Omega| > \Omega_m$.

ii) 'f_s' is greater than or equal to '$2f_m$'.

iii) Using anti-aliasing filter [Low Pass Filter]

Note:

1) Nyquist rate : $f_s = 2f_m$

$f_s \rightarrow$ sampling frequency (or) sampling rate.

2) Nyquist interval : $T_s = \frac{1}{f_s} = \frac{1}{2f_m}$

3) Folding frequency $= f_s$ - frequency at which aliasing occurs.

→ Problems based on Sampling Theorem:

1) Determine the nyquist sampling rate and nyquist sampling interval for the following signals:

i) $x(t) = 25 \cos 500\pi t$

ii) $x(t) = \text{sinc}^2\, 200t$

iii) $x(t) = \text{sinc}(200\pi t) + 3\,\text{sinc}^2(120\pi t)$

iv) $x(t) = \text{sinc}(100\pi t) \cdot \text{sinc}(200\pi t)$.

Soln:

i) $x(t) = 25 \cos 500\pi t$

general form: $A \cos 2\pi f t$.

$$2\pi f_m = 500\pi$$

$$f_m = \frac{500}{2} = 250\,Hz.$$

Nyquist rate $= f_s = 2f_m = 2 \times 250 = 500\,Hz.$

Nyquist interval $= T_s = \frac{1}{f_s} = \frac{1}{500} = 2 \times 10^{-3}\,sec = 2\,msec.$

ii) $x(t) = \text{sinc}^2\, 200t$

general form: $\text{sinc}\,\theta = \frac{\sin\theta}{\theta}$

$$x(t) = \left(\frac{\sin 200t}{200t}\right)^2 = \frac{\sin^2 200t}{40000t^2}$$

$$x(t) = \frac{1 - \cos 2(200t)}{2(40000t^2)} \qquad \left[\because \sin^2\theta = \frac{1-\cos 2\theta}{2}\right]$$

$$x(t) = \underbrace{\frac{1}{80000t^2}}_{[DC]} - \underbrace{\frac{\cos 400t}{80000t^2}}_{[A.C]}.$$

For D.C signal, frequency = 0.

General form: $A \cos 2\pi f t$:

$$2\pi f_m = 400$$

$$f_m = \frac{400}{2\pi} = \frac{200}{\pi}\,HZ.$$

Nyquist rate $= f_s = 2f_m = 2 \times \frac{200}{\pi} = \frac{400}{\pi}\,HZ$

Nyquist interval $= T_s = \frac{1}{f_s} = \frac{1}{400/\pi} = \frac{\pi}{400}\,sec.$

iii) $x(t) = \text{sinc}(200\pi t) + 3\,\text{sinc}^2(120\pi t)$

$$x(t) = \frac{\sin 200\pi t}{200\pi t} + 3\left(\frac{\sin 120\pi t}{120\pi t}\right)^2 \qquad \left[\because \text{sinc}\,\theta = \frac{\sin\theta}{\theta}\right]$$

$$= \frac{\sin 200\pi t}{200\pi t} + \frac{3\sin^2 120\pi t}{14400\pi^2 t^2}$$

$$= \frac{\sin 200\pi t}{200\pi t} + \frac{3[1-\cos 2(120\pi t)]}{2(14400\pi^2 t^2)} \qquad \left[\because \sin^2\theta = \frac{1-\cos 2\theta}{2}\right]$$

$$x(t) = \underset{(AC)}{\frac{\sin 200\pi t}{200\pi t}} + \underset{(DC)}{\frac{3}{28800\pi^2 t^2}} - \underset{(AC).}{\frac{3\cos 240\pi t}{28800\pi t^2}}$$

general form: $A\cos 2\pi f t$ & $A\sin 2\pi f t$.

$2\pi f_1 = 200\pi$

$f_1 = \frac{200}{2} = 100\,HZ$

$2\pi f_2 = 240\pi$

$f_2 = \frac{240}{2} = 120\,HZ.$

so, $f_m = f_2 = 120\,HZ.$

Nyquist rate $= f_s = 2f_m = 2\times 120 = 240\,Hz.$

Nyquist interval $= T_s = \frac{1}{f_s} = \frac{1}{240} = 4.17\times 10^{-3}\,\text{sec} = 4.17\,\text{msec}.$

iv) $x(t) = \text{sinc}(100\pi t)\cdot\text{sinc}(200\pi t)$

$$x(t) = \frac{\sin 100\pi t}{100\pi t}\cdot\frac{\sin 200\pi t}{200\pi t}$$

$$x(t) = \frac{\sin 100\pi t\cdot\sin 200\pi t}{20000\pi^2 t^2} = \frac{1}{20000\pi^2 t^2}\times \sin 200\pi t\cdot\sin 100\pi t \longrightarrow ①$$

$$\sin\left(\frac{A+B}{2}\right)\cdot\sin\left(\frac{A-B}{2}\right) = \cos B - \cos A \longrightarrow ②.$$

Here, $A = f_1$ & $B = f_2$

Compare eqn. ①&②,

$2\pi\left(\frac{f_1+f_2}{2}\right) = 200\pi$

$f_1 + f_2 = 200 \longrightarrow ③$

$2\pi\left(\frac{f_1-f_2}{2}\right) = 100\pi$

$f_1 - f_2 = 100 \longrightarrow ④.$

③ $\Rightarrow f_1 + f_2 = 200$

④ $\Rightarrow \underset{(-)}{f_1} - \underset{(+)}{f_2} = \underset{(-)}{100}$

③-④ $\Rightarrow \quad 2f_2 = 100$

$$\boxed{f_2 = \frac{100}{2} = 50\,Hz}$$

Sub. $f_2 = 50\,Hz$ in eqn.①,

① $\Rightarrow f_1 + 50 = 200$

$f_1 = 200 - 50$

$$\boxed{f_1 = 150\,Hz}$$

So, $f_m = f_1 = 150\,Hz.$

Nyquist rate $= f_s = 2f_m = 2\times150 = 300\,Hz.$

Nyquist interval $= T_s = \frac{1}{f_s} = \frac{1}{300} = 3.33\times10^{-3}$ Sec $= 3.33$ mSec

---

2) An analog signal $f(t) = \sin(480\pi t) + 3\sin(720\pi t)$ is sampled at 600 times per sec. Determine

i) Nyquist rate

ii) Folding frequency

iii) What are the frequencies in radians resulting in the discrete time signal $x(n)$?

iv) If $x(n)$ is passed through an ideal digital to analog converter, what is the reconstructed signal $y(t)$?

Soln:

$f(t) = \sin(480\pi t) + 3\sin(720\pi t)$

Here,

$2\pi f_1 = 480\pi$	$2\pi f_2 = 720\pi$
$f_1 = \frac{480}{2}$	$f_2 = \frac{720}{2}$
$f_1 = 240\,Hz$	$f_2 = 360\,HZ.$

So, $f_m = f_2 = 360Hz$.

i) Nyquist rate $= f_s = 2f_m = 2 \times 360 = 720 \text{ Hz}$.

ii) Folding frequency

Given: Sampling frequency $= f_s = 600 \text{ Hz}$.

To avoid aliasing,

case (i) : $f_s \geqslant 2f_m$

$600 \geqslant 2f_1$

$600 \geqslant 2 \times 240$

$600 \geqslant 480$. This is true. In this condition, aliasing does not occur.

case (ii) :

$f_s \geqslant 2f_m$

$f_s \geqslant 2f_2$

$600 \geqslant 2 \times 360$

$600 \geqslant 720$

$600 \ngeqslant 720$. This is not true. In this condition, aliasing occurs. i.e, $f_2 = 360 \text{ HZ}$, aliasing occurs.

Folding frequency $= f_s$ − frequency at which aliasing occurs.

$= 600 - 360$

$= 240 \text{ HZ}$.

iii) Frequency in radians:

$$\omega = 2\pi \times \frac{\text{frequency at which no aliasing}}{f_s}$$

$$= 2\pi \times \frac{240}{600}$$

$= 0.8\pi$ radians.

iv) Aliasing occurs at $f_2 = f_m = 360\,Hz$. i.e, high frequency component appears as low frequency component.

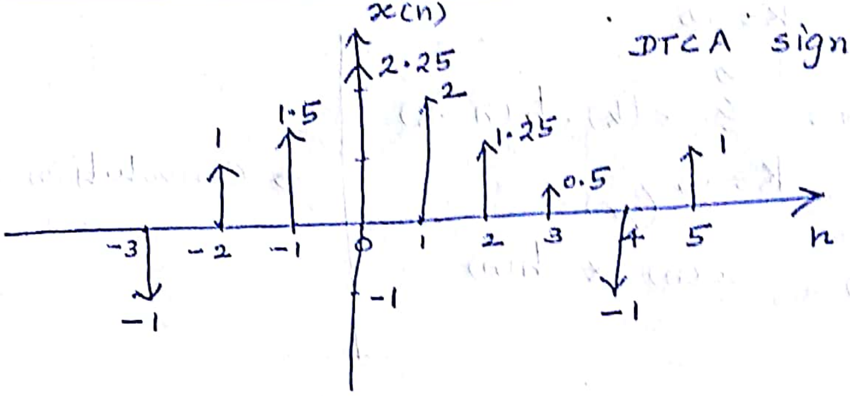
Here, the high frequency component 360 Hz appears as the low frequency component 240 Hz.

∴ The reconstructed signal $= y(t) = f(t) = \sin(480\pi)t + 3\sin(480\pi)t$

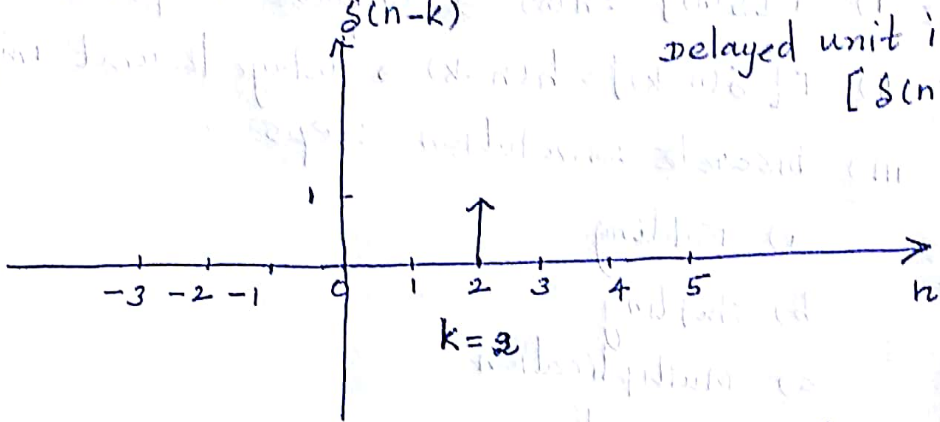
$$y(t) = f(t) = 4\sin(480\pi)t.$$

Unit V - Linear Time Invariant Discrete Time systems

→ convolution sum formula:

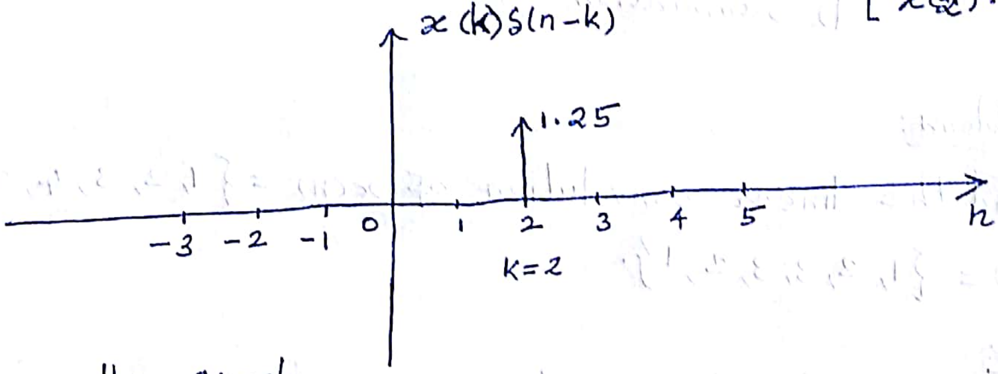


DTCA signal



Delayed unit impulse signal
$[\delta(n-2)]$

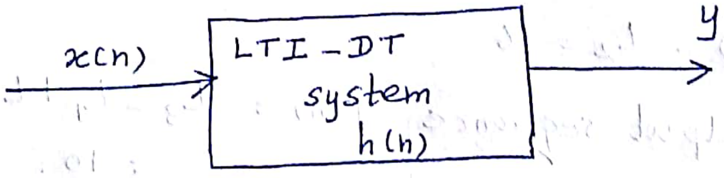
$[x(2) \cdot \delta(n-2)]$



From the graph,

$$x(n) = \sum_{k=-\infty}^{\infty} x(k) \cdot \delta(n-k)$$

$y(n) = T[x(n)]$



system output : $y(n) = T[x(n)] = T\left[\sum_{k=-\infty}^{\infty} x(k) \cdot \delta(n-k)\right]$

$$y(n) = \sum_{k=-\infty}^{\infty} x(k) \cdot T[\delta(n-k)]$$

$$\boxed{\begin{aligned} y(n) &= \sum_{k=-\infty}^{\infty} x(k) \cdot h(n-k) \\ &\text{(or)} \\ y(n) &= x(n) * h(n) \end{aligned}} \longrightarrow \text{Convolution sum Formula}$$

Note: i) $T[\delta(n)] = h(n)$ → Impulse Response

ii) $T[\delta(n-k)] = h(n-k)$ → delayed unit impulse response

iii) Discrete convolution steps

a) Folding

b) shifting

c) Multiplication

d) summation

Problems:

1) Find the linear convolution of $x(n) = \{1, 2, 3, 4, 5\}$ with $h(n) = \{1, 2, 3, 3, 2, 1\}$.

Soln:

$x(n) = \{1, 2, 3, 4, 5\}$

$h(n) = \{1, 2, 3, 3, 2, 1\}$

Length of $x(n) = L_1 = 5$

Length of $h(n) = L_2 = 6$

Length of output sequence $y(n) = L_3 = L_1 + L_2 - 1 = 5+6-1$

$= 10.$

$y(n) = x(n) * h(n).$

$h(n)$ \ $x(n)$	1	2	3	4	5
1	1	2	3	4	5
2	2	4	6	8	10
3	3	6	9	12	15
3	3	6	9	12	15
2	2	4	6	8	10
1	1	2	3	4	5

system response (or) system output : $y(n) = \{1, 4, 10, 19, 30, 36, 35, 26, 14, 5\}$

2) Find the linear convolution of $x(n) = \{1, \underset{\uparrow}{1}, -1, 2, 1\}$ with $h(n) = \{1, -2, -3, \underset{\uparrow}{4}\}$.

Soln:

$x(n) = \{1, \underset{\uparrow}{1}, -1, 2, 1\}$

Length of $x(n) = L_1 = 5$

$h(n) = \{1, -2, -3, \underset{\uparrow}{4}\}$

Length of $h(n) = L_2 = 4$

Length of output sequence $y(n) = L_3 = L_1 + L_2 - 1 = 5+4-1 = 8$.

$x(n)$ starts at '$n = -1$'

$h(n)$ starts at '$n = -3$'

$y(n)$ starts at '$n = (-1)+(-3) = -4$'

Scanned by CamScanner

$h(n)$ \ $x(n)$	1	1	-1	2	1
1	1	1	-1	2	1
-2	-2	-2	2	-4	-2
-3	-3	-3	3	-6	-3
4	4	4	-4	8	4

system response (or) system output :

$$y(n) = \{1, -1, -6, +5, \underset{\uparrow}{4}, -12, 5, 4\}$$

3) Find the linear convolution of $x(n) = \{\underset{\uparrow}{1}, 2, 3, 4, 5, 6\}$ with $h(n) = \{2, \underset{\uparrow}{-4}, -6, -8\}$.

Soln:

$x(n) = \{\underset{\uparrow}{1}, 2, 3, 4, 5, 6\}$

Length of $x(n) = L_1 = 6$

$h(n) = \{2, \underset{\uparrow}{-4}, -6, -8\}$

Length of $h(n) = L_2 = 4$

Length of output sequence $y(n) = L_3 = L_1 + L_2 - 1 = 6 + 4 - 1 = 9$

$x(n)$ starts at '$n = 0$'

$h(n)$ starts at '$n = -1$'.

$y(n)$ starts at '$n = 0 + (-1) = -1$'.

$h(n)$ \ $x(n)$	1	2	3	4	5	6
2	2	4	6	8	10	12
-4	-4	-8	-12	-16	-20	-24
-6	-6	-12	-18	-24	-30	-36
-8	-8	-16	-24	-32	-40	-48

system response = system output = $y(n) = \{2, \underset{\uparrow}{0}, -8, -24, -40, -56, -86, -76, -48\}$

→ Properties of Discrete Convolution :

a) Commutative

$y(n) = x(n) * h(n) = h(n) * x(n)$

b) Associative

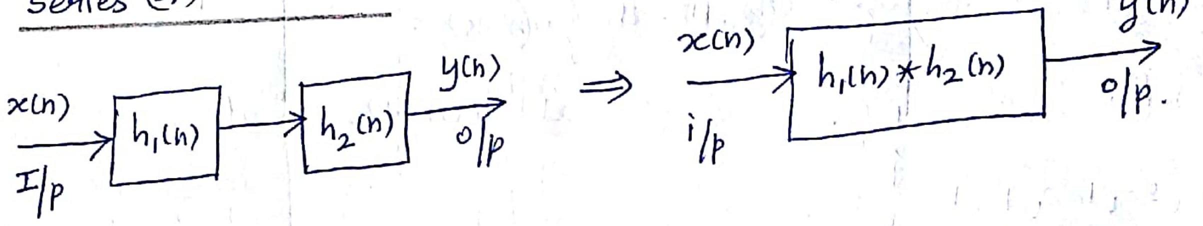
$y(n) = x(n) * [h_1(n) * h_2(n)] = [x(n) * h_1(n)] * h_2(n)$

c) Distributive

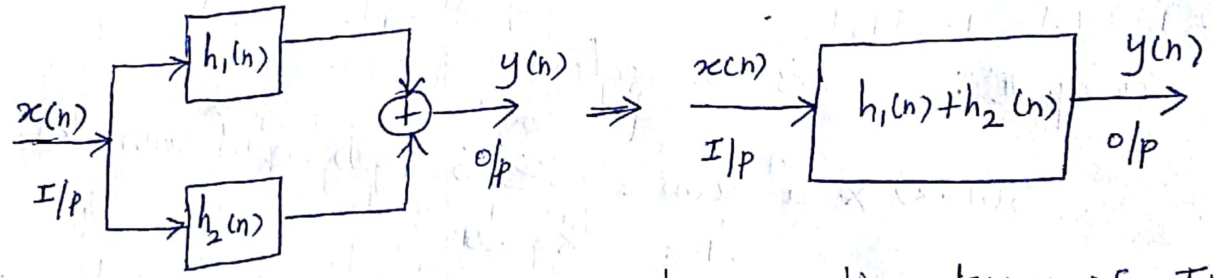
$y(n) = x(n) * [h_1(n) + h_2(n)] = x(n) * h_1(n) + x(n) * h_2(n).$

→ Impulse Response Connection Types :

i) Series (or) cascade

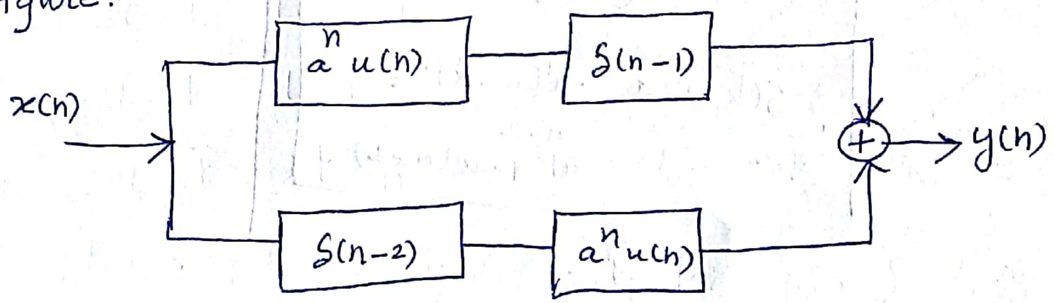


ii) Parallel



→ Problems based on properties and connection types of Impulse Response :

1) Find the overall impulse response of the system shown in figure.



Soln:

Group I : series connection :

$$a^n u(n) * \delta(n-1) = \sum_{k=-\infty}^{\infty} \delta(k-1) \cdot a^{n-k} u(n-k)$$

[∵ using convolution summation formula].

$$\boxed{\begin{array}{l} \delta(k-1) = \begin{cases} 1 & ; \; k = 1 \\ 0 & ; \; k \neq 1. \end{cases} \\ \therefore \delta(k-1) \cdot a^{n-k} u(n-k) = \\ \qquad a^{n-1} u(n-1). \end{array}}$$

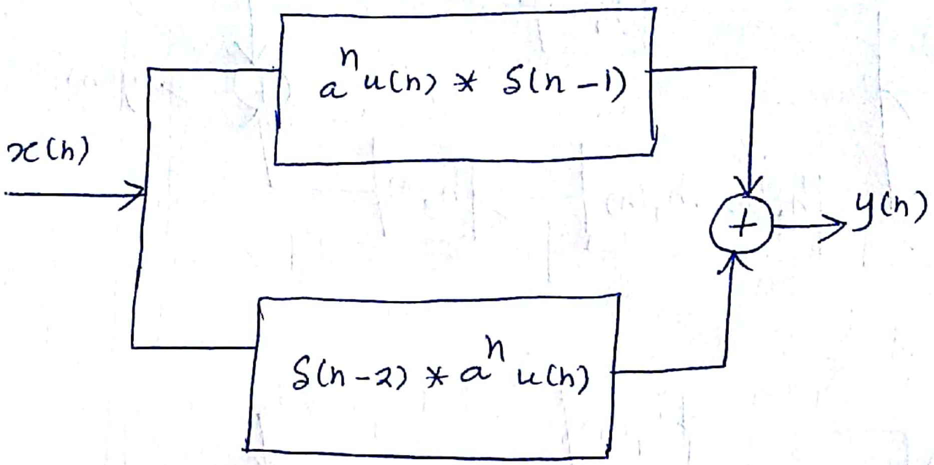
$$= a^{n-1} u(n-1).$$

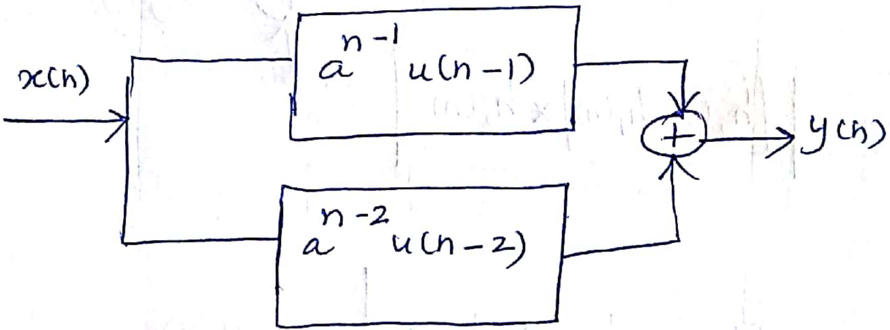
Group II : series connection

$$\delta(n-2) * a^n u(n) = \sum_{k=-\infty}^{\infty} \delta(k-2) \cdot a^{n-k} u(n-k)$$

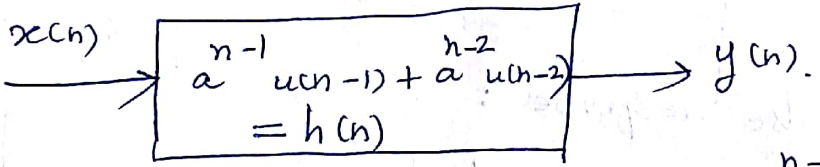
[∵ using convolution summation formula].

$$\boxed{\begin{array}{l} \delta(k-2) = \begin{cases} 1 & ; \; k = 2 \\ 0 & ; \; k \neq 2 \end{cases} \\ \therefore \delta(k-2) a^{n-k} \cdot u(n-k) = \\ \qquad a^{n-2} \cdot u(n-2) \end{array}}$$



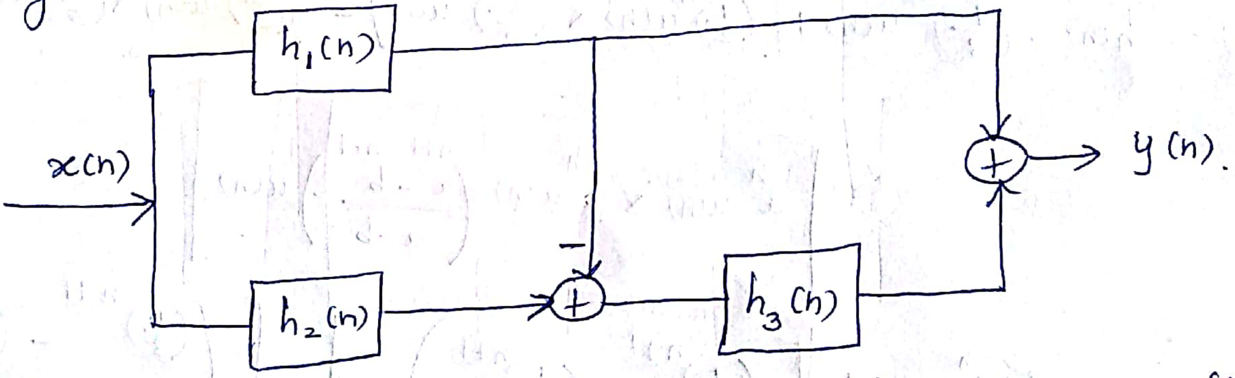


Parallel Connection:



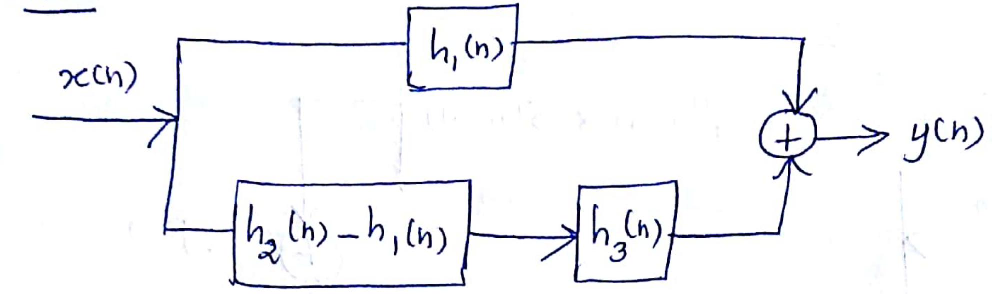
Overall Impulse Response : $h(n) = a^{n-1} u(n-1) + a^{n-2} u(n-2)$

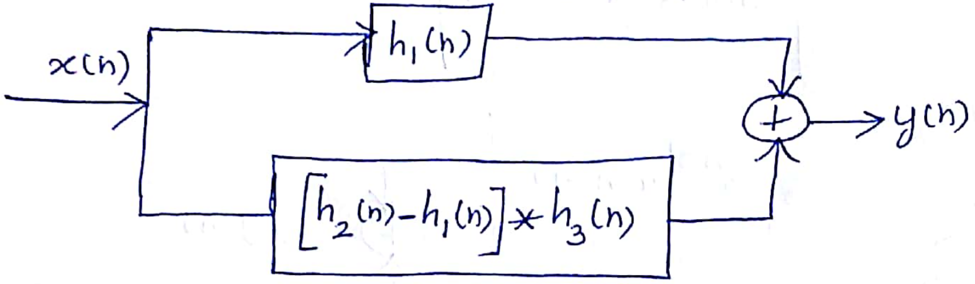
2) Find the overall response of causal system shown in figure.

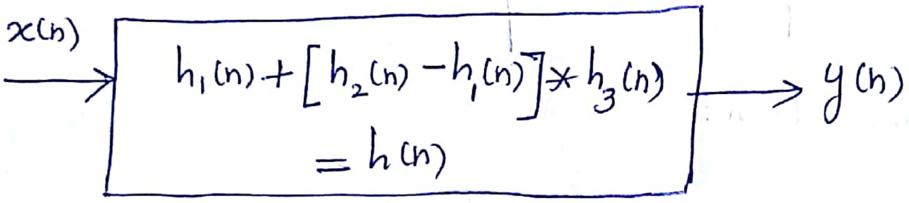


where, $h_1(n) = \left(\frac{1}{3}\right)^n u(n)$, $h_2(n) = \left(\frac{1}{2}\right)^n u(n)$, $h_3(n) = \left(\frac{1}{5}\right)^n u(n)$.

Soln:







overall impulse response :

$$h(n) = h_1(n) + [h_2(n) - h_1(n)] * h_3(n)$$

$$h(n) = h_1(n) + [h_2(n) * h_3(n)] - [h_1(n) * h_3(n)]$$

$$h(n) = \left(\frac{1}{3}\right)^n u(n) + \left[\left(\frac{1}{2}\right)^n u(n) * \left(\frac{1}{5}\right)^n u(n)\right] - \left[\left(\frac{1}{3}\right)^n u(n) * \left(\frac{1}{5}\right)^n u(n)\right]$$

$$\boxed{a^n u(n) * b^n u(n) = \left(\frac{a^{n+1} - b^{n+1}}{a-b}\right) u(n)}$$

$$h(n) = \left(\frac{1}{3}\right)^n u(n) + \left[\left(\frac{\left(\frac{1}{2}\right)^{n+1} - \left(\frac{1}{5}\right)^{n+1}}{\frac{1}{2} - \frac{1}{5}}\right) u(n)\right] - \left[\left(\frac{\left(\frac{1}{3}\right)^{n+1} - \left(\frac{1}{5}\right)^{n+1}}{\frac{1}{3} - \frac{1}{5}}\right) u(n)\right]$$

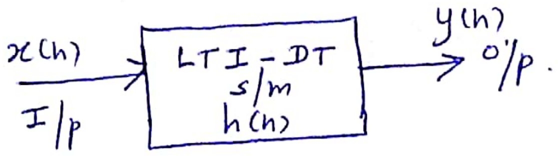
$$h(n) = \left(\frac{1}{3}\right)^n u(n) + \frac{10}{3}\left[\left(\frac{1}{2}\right)^{n+1} - \left(\frac{1}{5}\right)^{n+1}\right]u(n) - \frac{15}{2}\left[\left(\frac{1}{3}\right)^{n+1} - \left(\frac{1}{5}\right)^{n+1}\right]u(n)$$

→ Causality and stability of LTI - Discrete Time systems:

* **Condition for causality:**

Impulse Response: $h(n) = 0$; for $n < 0$.

* **Condition for stability:**



$$y(n) = x(n) * h(n) \quad (or)$$

$$= h(n) * x(n)$$

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\, x(n-k) \longrightarrow ①$$

For stable system, the input $x(n)$ is bounded. i.e, $|x(n)| < \infty$. Say, $|x(n)| = M_x$ $[<\infty]$

Taking absolute value of eqn ①,

$$|y(n)| = \sum_{k=-\infty}^{\infty} |h(k)| \cdot |x(n-k)|$$

$$|y(n)| = \sum_{k=-\infty}^{\infty} |h(k)| \cdot M_x \quad \hookrightarrow ② \qquad [\because |x(n-k)| \cdot \text{is also equal to } M_x].$$

For stable system, the output $y(n)$ is also bounded.

i.e, $|y(n)| < \infty$. say, $|y(n)| = M_y$ $[<\infty]$.

Now,

eqn. ② ⇒ $$M_y = \sum_{k=-\infty}^{\infty} |h(k)| M_x$$

$$\sum_{k=-\infty}^{\infty} |h(k)| = \frac{M_y}{M_x} \text{ (finite)}$$

$$\boxed{\sum_{k=-\infty}^{\infty} |h(k)| < \infty} \to \text{condition for stability.}$$

→ Problems based on causality and stability of LTI-Discrete Time system:

U.Q 1) Determine the range of values 'a' and 'b' for which the LTI system with impulse response is stable.

$$h(n) = \begin{cases} a^n & ; n \geq 0 \\ b^n & ; n < 0 \end{cases}$$

Soln:

condition for stability:

$$\sum_{n=-\infty}^{\infty} |h(n)| = \sum_{n=0}^{\infty} |a^n| + \sum_{n=-\infty}^{-1} |b^n|$$

$$= \sum_{n=0}^{\infty} |a^n| + \sum_{n=1}^{\infty} |b^{-n}|$$

$$\sum_{n=-\infty}^{\infty} |h(n) = \sum_{n=0}^{\infty} |a^n| + \sum_{n=1}^{\infty} |b^{-1}|^n$$

Using Geometrical summation formula, we can simplify the above expressions.

$$\sum_{n=-\infty}^{\infty} |h(n)| = \frac{1}{1-a} + \frac{b^{-1}}{1-b^{-1}} \quad ; \quad \text{ROC}: a<1 \ \& \ b^{-1}<1$$

$$= \frac{1}{1-a} + \frac{\frac{1}{b}}{1-\frac{1}{b}} \quad ; \quad \text{ROC}: a<1 \ \& \ \frac{1}{b}<1$$

$$\sum_{n=-\infty}^{\infty} |h(n)| = \frac{1}{1-a} + \frac{\frac{1}{b}}{\frac{b-1}{b}} \quad ; \quad \text{ROC}: a<1 \ \& \ 1<b$$

$$\boxed{\sum_{n=-\infty}^{\infty} |h(n)| = \frac{1}{1-a} + \frac{1}{b-1} \quad ; \quad \text{ROC}: a<1<b}$$

2) $y(n) = x(n) + \frac{1}{8} x(n-1) + \frac{1}{3} x(n-2)$. Find whether the given system is stable or not.

Soln:

$$y(n) = x(n) + \frac{1}{8} x(n-1) + \frac{1}{3} x(n-2)$$

Taking Z-T on both the sides,

$$Y(z) = X(z) + \frac{1}{8} z^{-1} X(z) + \frac{1}{3} z^{-2} X(z)$$

$$Y(z) = X(z)\left[1 + \frac{1}{8} z^{-1} + \frac{1}{3} z^{-2}\right]$$

$$H(z) = \frac{Y(z)}{X(z)} = 1 + \frac{1}{8} z^{-1} + \frac{1}{3} z^{-2} \longrightarrow ①.$$

w.k.T,

$$H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n}$$

Expand the above equation,

$$= \cdots\cdots + h(-1) z^{1} + h(0) + h(1) z^{-1} + h(2) z^{-2} + \cdots\cdots \;\; \hookrightarrow ②$$

Compare eqns. ① & ②,

we get, $h(0) = 1$, $h(1) = \frac{1}{8}$, $h(2) = \frac{1}{3}$.

condition for stability :

$$\sum_{n=-\infty}^{\infty} |h(n)| = \sum_{n=0}^{2} |h(n)| = h(0) + h(1) + h(2)$$

$$= 1 + \frac{1}{8} + \frac{1}{3}$$

$$= \frac{24+3+8}{24}$$

$$\boxed{\sum_{n=-\infty}^{\infty} |h(n)| = \frac{35}{24} \; (< \infty)}$$

∴ The given system is stable system.

→ Difference Equations:

The input and output characteristics of a LTI-Discrete Time system is described by Linear Constant Co-efficient Difference Equation [LCCDE]. It is given by

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k) \rightarrow ①$$

$a_k \rightarrow$ output co-efficients.

$b_k \rightarrow$ input co-efficients.

$y(n) \rightarrow$ present output (or) current output

$x(n) \rightarrow$ present input (or) current input.

$y(n-1), y(n-2), \ldots \rightarrow$ Past outputs.

$x(n-1), x(n-2), \ldots \rightarrow$ Past inputs.

Now, eqn ① ⇒ $a_0 y(n) + \sum_{k=1}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k)$

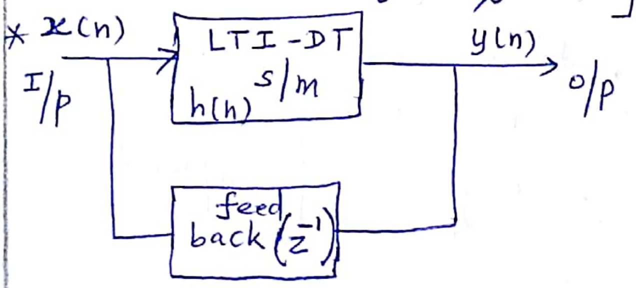
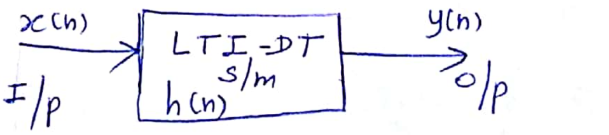
$$a_0 y(n) = \sum_{k=0}^{M} b_k x(n-k) - \sum_{k=1}^{N} a_k y(n-k)$$

$$\boxed{y(n) = \frac{1}{a_0}\left[\sum_{k=0}^{M} b_k x(n-k) - \sum_{k=1}^{N} a_k y(n-k)\right]}$$

→ Difference between FIR system and IIR system:

FIR system.	IIR system
* Finite Impulse Response system.	* Infinite Impulse Response system.
* h(n) contains finite number of samples.	* h(n) contains infinite number of samples.
* system output : $y(n) = \sum_{k=0}^{N-1} h(k).\, x(n-k)$	* system output : $y(n) = \sum_{k=-\infty}^{\infty} h(k)\, x(n-k)$
* Less memory space is required to store the samples.	* More memory space is required to store the samples.

→ Difference between Recursive system and Non-recursive system:

Recursive system	Non-Recursive system
* Present output y(n) depends on present input, past inputs and past outputs. i.e, $y(n) = F[x(n), x(n-1), x(n-2) \cdots\cdots y(n-1), y(n-2) \cdots]$	* Present output depends on present input and past inputs. y(n) doesn't depend on past outputs. i.e, $y(n) = F[x(n), x(n-1), x(n-2) \cdots]$
* 	* 
* It has feed back.	* It has no feedback.
* It is also called as system with memory.	* It is also called as system with no memory.

→ System function of LTI-Discrete Time system:

System Function (or) Transfer Function : $H(z) = \dfrac{Z.T\,[y(n)]}{Z.T\,[x(n)]}$

$$H(z) = \frac{Y(z)}{X(z)}$$

→ Poles and zero's of LTI-Discrete Time system:

$$H(z) = \frac{(z-z_1)(z-z_2)\cdots(z-z_M)}{(z-P_1)(z-P_2)\cdots(z-P_N)}$$

* Poles: The values of 'z' for which the system function H(z) is infinite (∞). ∴ The denominator roots are called poles.

* zeros: The values of 'z' for which the system function H(z) is zero (0). ∴ The numerator roots are called zero's.

→ Problems:

1) Determine the pole-zero plot of the following function: $X(z) = \dfrac{1}{1-az^{-1}}$ and find out the time domain sequence for ROC : $|z| > a$.

Soln:

$$X(z) = \frac{1}{1-az^{-1}}$$

$$X(z) = \frac{1}{1-\frac{a}{z}} = \frac{1}{\frac{z-a}{z}}$$

$$\boxed{X(z) = \frac{z}{z-a}} \longrightarrow ①.$$

Time domain sequence x(n):

Taking Inv. Z.T of eqn ①,

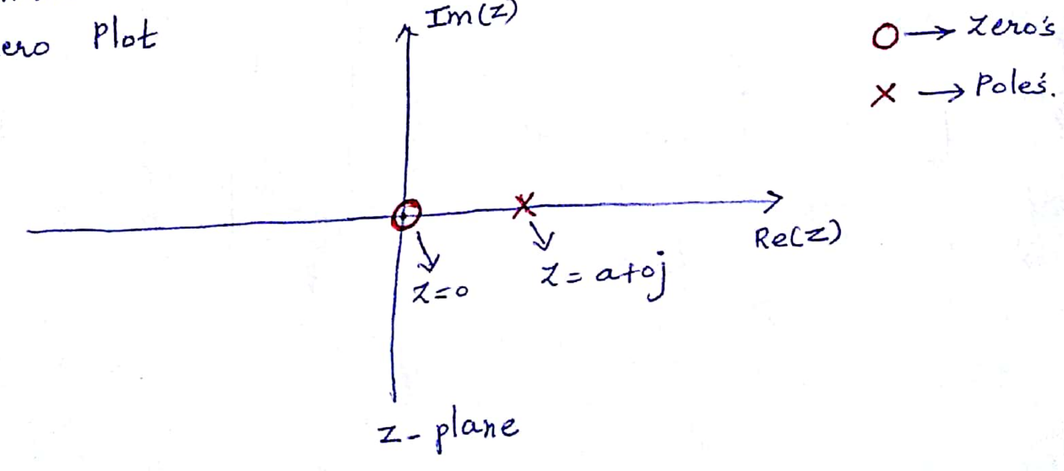
$$\boxed{x(n) = a^n u(n)}$$

[∵ ROC $|z| > a$, causal s/m]

Numerator Roots: $z = 0$ → zero's.

Denominator Roots: $z = a$ → Pole's.

Pole-zero Plot



2) What is the transfer function of a system whose poles are at $-4 \pm j0.5$ and zero at -0.6?

Soln:

Transfer Function : $H(z) = \frac{(z-z_1)(z-z_2)\ldots\ldots(z-z_M)}{(z-P_1)(z-P_2)\ldots\ldots(z-z_N)}$

$$H(z) = \frac{z+0.6}{(\underbrace{z+4}_{a}-\underbrace{j0.5}_{b})(\underbrace{z+4}_{a}+\underbrace{j0.5}_{b})}$$

$$H(z) = \frac{z+0.6}{(z+4)^2-(j0.5)^2}$$

$$H(z) = \frac{z+0.6}{z^2+8z+16+0.25} \qquad [\because j^2=-1]$$

$$\boxed{H(z) = \frac{z+0.6}{z^2+8z+16.25}}$$

→ Solutions to the difference equations:

1) Note: with initial conditions

$$Z.T[y(n-2)] = z^{-2}Y(z) + z^{-1}y(-1) + y(-2)$$

$$Z.T[y(n-1)] = z^{-1}Y(z) + y(-1)$$

2) Note: without initial conditions.

$$Z.T[y(n-2)] = z^{-2}Y(z)$$

$$Z.T[y(n-1)] = z^{-1}Y(z).$$

→ Problems based on solutions to the difference equation:

1) Given that $y(-1)=5$, $y(-2)=0$; Solve the difference equation $y(n) = 3y(n-1) - 4y(n-2)$ for $n \geq 0$.

Soln:

$$y(n) = 3y(n-1) + 4y(n-2) \quad ; \quad n \geq 0 \quad [\because \text{causal } s/m].$$

Taking Z.T on both the sides

$$Y(z) = 3\left[z^{-1}Y(z) + y(-1)\right] + 4\left[z^{-2}Y(z) + z^{-1}y(-1) + y(-2)\right]$$

$$Y(z) = 3z^{-1}Y(z) + 3(5) + 4z^{-2}Y(z) + 20z^{-1} + 0.$$

$$Y(z)\left[1 - 3z^{-1} - 4z^{-2}\right] = 15 + 20z^{-1}$$

$$Y(z) = \frac{15 + 20z^{-1}}{1 - 3z^{-1} - 4z^{-2}} \longrightarrow ①$$

To find $y(n)$:

Convert -ve power of 'z' into +ve power of 'z' in eqn ①.

$$Y(z) = \frac{z^2(15 + 20z^{-1})}{z^2(1 - 3z^{-1} - 4z^{-2})}.$$

$$Y(z) = \frac{15z^2 + 20z}{z^2 - 3z - 4}$$

If Dr. degree > Nr. degree, we can proceed partial fraction.

$$\frac{Y(z)}{z} = \frac{15z + 20}{z^2 - 3z - 4}.$$

$$\frac{Y(z)}{z} = \frac{15z + 20}{(z-4)(z+1)}$$

$$\frac{Y(z)}{z} = \frac{K_0}{z-4} + \frac{K_1}{z+1} \longrightarrow ②.$$

Factorization of Dr. poly.

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$z = \frac{3 \pm \sqrt{(-3)^2 - 4 \times 1 \times (-4)}}{2 \times 1} = \frac{3 \pm \sqrt{9+16}}{2}$$

$$= \frac{3 \pm \sqrt{25}}{2} = \frac{3 \pm 5}{2} = \frac{3+5}{2}, \frac{3-5}{2}$$

$$= \frac{8}{2}, \frac{-2}{2} = 4, -1$$

$$K_0 = \frac{Y(z)}{z} \times (z-4) \Big|_{z=4}$$

$$K_0 = \frac{15z+20}{\cancel{(z-4)}(z+1)} \times \cancel{(z-4)} \Big|_{z=4}$$

$$K_0 = \frac{15(4)+20}{(4+1)} = \frac{60+20}{5} = \frac{80}{5} = 16$$

$$\boxed{K_0 = 16}$$

$$K_1 = \frac{Y(z)}{z} \times (z+1) \Big|_{z=-1}$$

$$K_1 = \frac{15z+20}{(z-4)\cancel{(z+1)}} \times \cancel{(z+1)} \Big|_{z=-1}$$

$$K_1 = \frac{15(-1)+20}{(-1-4)} = \frac{-15+20}{-5} = \frac{5}{-5} = -1$$

$$\boxed{K_1 = -1}$$

Sub. K_0 and K_1 values in eqn ②,

$$② \Rightarrow \frac{Y(z)}{z} = \frac{16}{z-4} - \frac{1}{z+1}$$

$$Y(z) = (16) \cdot \frac{z}{z-4} - \frac{z}{z+1}$$

Taking Inverse z-Transform of the above equation,

$$\boxed{y(n) = 16 \cdot (4)^n u(n) - (-1)^n u(n)}$$

[∵ given the s/m is causal]

→ Solutions to the difference equations:

1) Solve the difference equation $y(n) - \frac{5}{6} y(n-1) + \frac{1}{6} y(n-2) = x(n)$ with $x(n) = \delta(n) - \frac{1}{3} \delta(n-1)$. Assume zero initial condition.

Soln:

$$y(n) - \frac{5}{6} y(n-1) + \frac{1}{6} y(n-2) = x(n).$$

Taking z.T,

$$Y(z) - \frac{5}{6} z^{-1} Y(z) + \frac{1}{6} z^{-2} Y(z) = X(z)$$

$$Y(z) \left[1 - \frac{5}{6} z^{-1} + \frac{1}{6} z^{-2}\right] = X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - \frac{5}{6} z^{-1} + \frac{1}{6} z^{-2}}$$

Convert −ve powers of 'z' into +ve powers of 'z'

$$H(z) = \frac{z^2}{z^2 \left[1 - \frac{5}{6} z^{-1} + \frac{1}{6} z^{-2}\right]}$$

$$\boxed{H(z) = \frac{z^2}{z^2 - \frac{5}{6} z + \frac{1}{6}}}$$

Given: $x(n) = \delta(n) - \frac{1}{3} \delta(n-1)$

Taking z.T,

$$X(z) = 1 - \frac{1}{3} z^{-1}$$

Convert −ve powers of 'z' into +ve powers of 'z'

$$X(z) = \frac{z\left(1 - \frac{1}{3} . z^{-1}\right)}{z} = \frac{z - \frac{1}{3}}{z}$$

$$\boxed{X(z) = \frac{z - \frac{1}{3}}{z}}$$

* To find y(n):

$$H(z) = \frac{Y(z)}{X(z)}$$

$$Y(z) = H(z) \cdot X(z)$$

$$Y(z) = \frac{z^{\cancel{2}}}{\left(z^2 - \frac{5}{6}z + \frac{1}{6}\right)} \times \frac{\left(z - \frac{1}{3}\right)}{\cancel{z}}$$

$$Y(z) = \frac{z\left(z - \frac{1}{3}\right)}{z^2 - \frac{5}{6}z + \frac{1}{6}}.$$

If Dr. degree > Nr. degree, we can proceed partial fraction.

$$\therefore \frac{Y(z)}{z} = \frac{z - \frac{1}{3}}{z^2 - \frac{5}{6}z + \frac{1}{6}} \longrightarrow ①$$

Factorization of Dr. Polynomial.

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$z = \frac{+\frac{5}{6} \pm \sqrt{\left(\frac{-5}{6}\right)^2 - 4 \times 1 \times \frac{1}{6}}}{2} = \frac{\frac{5}{6} \pm \sqrt{\frac{25}{36} - \frac{4}{6}}}{2}$$

$$= \frac{\frac{5}{6} \pm \sqrt{\frac{25-24}{36}}}{2} = \frac{\frac{5}{6} \pm \sqrt{\frac{1}{36}}}{2} = \frac{\frac{5}{6} \pm \frac{1}{6}}{2}.$$

$$= \frac{5 \pm 1}{12} = \frac{5+1}{12}, \frac{5-1}{12} = \frac{6}{12}, \frac{4}{12}$$

$$z = \frac{1}{2}, \frac{1}{3}.$$

$$① \Rightarrow \frac{Y(z)}{z} = \frac{\cancel{z-\frac{1}{3}}}{(z-\frac{1}{2})\cancel{(z-\frac{1}{3})}}$$

$$\frac{Y(z)}{z} = \frac{1}{z-\frac{1}{2}}$$

$$Y(z) = \frac{z}{z-\frac{1}{2}} \rightarrow ②$$

Taking Inv. Z.T of $Y(z)$,

$$\boxed{y(n) = (\tfrac{1}{2})^n u(n)}$$

2) Find the output for the equation $y(n) = 0.7y(n-1) - 0.1y(n-2) + 2x(n) - x(n-2)$ when the input sequence $x(n) = u(n)$.

Soln:

$$y(n) = 0.7y(n-1) - 0.1y(n-2) + 2x(n) - x(n-2)$$

Taking Z.T,

$$Y(z) = 0.7z^{-1}Y(z) - 0.1z^{-2}Y(z) + 2X(z) - z^{-2}X(z)$$

$$Y(z)[1 - 0.7z^{-1} + 0.1z^{-2}] = X(z)[2 - z^{-2}]$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{2 - z^{-2}}{1 - 0.7z^{-1} + 0.1z^{-2}}$$

Convert -ve powers of 'z' into +ve powers of 'z'.

$$H(z) = \frac{z^2[2 - z^{-2}]}{z^2[1 - 0.7z^{-1} + 0.1z^{-2}]}$$

$$\boxed{H(z) = \frac{2z^2 - 1}{z^2 - 0.7z + 0.1}}$$

Given:

$$x(n) = u(n)$$

Taking Z·T, $\quad Z \cdot T[x(n)] = Z \cdot T[u(n)]$

$$\boxed{X(z) = \frac{z}{z-1}}$$

W.k.T,

$$H(z) = \frac{Y(z)}{X(z)}$$

$$Y(z) = H(z) \cdot X(z)$$

$$Y(z) = \frac{2z^2-1}{(z^2-0.7z+0.1)} \times \frac{z}{(z-1)}$$

If Dr. degree > Nr. degree, we can proceed partial fraction.

So,

$$\frac{Y(z)}{z} = \frac{2z^2-1}{(z-1)(z^2-0.7z+0.1)}$$

Factorization of Dr. Polynomial

$$z = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{0.7 \pm \sqrt{(-0.7)^2-4\times1\times0.1}}{2\times1}$$

$$z = \frac{0.7 \pm \sqrt{0.49-0.4}}{2} = \frac{0.7 \pm \sqrt{0.09}}{2} = \frac{0.7 \pm 0.3}{2}$$

$$z = \frac{0.7+0.3}{2}, \frac{0.7-0.3}{2} = \frac{1}{2}, \frac{0.4}{2}$$

$$\boxed{z = 0.5, 0.2}$$

Now,

$$\frac{Y(z)}{z} = \frac{2z^2-1}{(z-1)(z-0.5)(z-0.2)} \longrightarrow ①$$

$$\frac{Y(z)}{z} = \frac{K_0}{z-1} + \frac{K_1}{z-0.5} + \frac{K_2}{z-0.2} \longrightarrow ②$$

$$K_0 = \frac{Y(z)}{z} \times (z-1) \Big|_{z=1}$$

$$K_0 = \frac{2z^2-1}{\cancel{(z-1)}(z-0.5)(z-0.2)} \times \cancel{(z-1)} \Big|_{z=1}$$

$$K_0 = \frac{2(1)^2-1}{(1-0.5)(1-0.2)} = \frac{2-1}{(0.5)(0.8)} = 2.5$$

$$\boxed{K_0 = 2.5}$$

$$K_1 = \frac{Y(z)}{z} \times (z-0.5) \Big|_{z=0.5}$$

$$K_1 = \frac{2z^2-1}{(z-1)\cancel{(z-0.5)}(z-0.2)} \times \cancel{(z-0.5)} \Big|_{z=0.5}$$

$$K_1 = \frac{2(0.5)^2-1}{(0.5-1)(0.5-0.2)} = \frac{-0.5}{(-0.5)(0.3)} = 3.33$$

$$\boxed{K_1 = 3.33}$$

$$K_2 = \frac{Y(z)}{z} \times (z-0.2) \Big|_{z=0.2}$$

$$K_2 = \frac{2z^2-1}{(z-1)(z-0.5)\cancel{(z-0.2)}} \times \cancel{(z-0.2)} \Big|_{z=0.2}$$

$$K_2 = \frac{2(0.2)^2-1}{(0.2-1)(0.2-0.5)} = \frac{-0.92}{(-0.8)(-0.3)} = -3.833$$

$$\boxed{K_2 = -3.833}$$

Now,

$$② \Rightarrow \frac{Y(z)}{z} = \frac{2.5}{z-1} + \frac{3.33}{z-0.5} - \frac{3.833}{z-0.2}$$

$$Y(z) = 2.5\left(\frac{z}{z-1}\right) + 3.33\left(\frac{z}{z-0.5}\right) - 3.833\left(\frac{z}{z-0.2}\right)$$

Taking Inv. z.T,

$$\boxed{y(n) = 2.5\,(1)^n u(n) + 3.33\,(0.5)^n u(n) - 3.833\,(0.2)^n u(n)}$$

Scanned by CamScanner

→ Problems based on Analysis of Discrete Time system using Z-Transform :

1) System function of LTI system is given as $H(z) = \frac{3-4z^{-1}}{1-3.5z^{-1}+1.5z^{-2}}$. Specify the ROC of H(z) and determine the unit sample response for the following condition. i) causal system ii) anticausal system iii) stable system.

Soln:

$$H(z) = \frac{3-4z^{-1}}{1-3.5z^{-1}+1.5z^{-2}}$$

Convert -ve power of 'z' into +ve power of 'z':

$$H(z) = \frac{z^2(3-4z^{-1})}{z^2(1-3.5z^{-1}+1.5z^{-2})}$$

$$H(z) = \frac{3z^2-4z}{z^2-3.5z+1.5}$$

If Dr. degree > Nr. degree, we can proceed partial fraction.

$$H(z) = \frac{z(3z-4)}{z^2-3.5z+1.5}$$

$$\frac{H(z)}{z} = \frac{3z-4}{z^2-3.5z+1.5}$$

Factorization of Dr. polynomial

$$z = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{+3.5 \pm \sqrt{(-3.5)^2-4\times1\times1.5}}{2}$$

$$z = \frac{3.5 \pm \sqrt{12.25-6}}{2} = \frac{3.5 \pm 2.5}{2} = \frac{3.5+2.5}{2}, \frac{3.5-2.5}{2}$$

$z = 3, 0.5$

Now,

$$\frac{H(z)}{z} = \frac{3z-4}{(z-3)(z-0.5)}$$

$$\frac{H(z)}{z} = \frac{K_0}{z-3} + \frac{K_1}{z-0.5} \longrightarrow ①$$

$$K_0 = \frac{H(z)}{z} \times (z-3) \Big|_{z=3}$$

$$= \frac{3z-4}{\cancel{(z-3)}(z-0.5)} \times \cancel{(z-3)} \Big|_{z=3}$$

$$= \frac{3(3)-4}{(3-0.5)} = \frac{9-4}{2.5} = \frac{5}{2.5}$$

$$\boxed{K_0 = 2}$$

$$K_1 = \frac{H(z)}{z} \times (z-0.5) \Big|_{z=0.5}$$

$$= \frac{3z-4}{(z-3)\cancel{(z-0.5)}} \times \cancel{(z-0.5)} \Big|_{z=0.5}$$

$$= \frac{3(0.5)-4}{(0.5-3)} = \frac{1.5-4}{-2.5} = \frac{-2.5}{-2.5}$$

$$\boxed{K_1 = 1}$$

Now,

$$① \Rightarrow \frac{H(z)}{z} = \frac{2}{z-3} + \frac{1}{z-0.5}$$

$$H(z) = (2) \cdot \frac{z}{z-3} + \frac{z}{z-0.5} \longrightarrow ②$$

(i) causal system:

For causal system, Roc: $|z| > 3$ & $|z| > 0.5$.
Now, taking Inv. Z.T of eqn ②,

$$\boxed{h(n) = 2.(3)^n u(n) + (0.5)^n u(n)}$$

(ii) Anticausal system:

For anticausal system, Roc: $|z| < 3$ and $|z| < 0.5$.

Now, taking Inv. Z.T of eqn ②,

$$\boxed{h(n) = -(2)(3)^n u(-n-1) - (0.5)^n u(-n-1)}$$

(iii) Stable system:

For stable system, Roc must include the unit circle.
So, Roc: $|z| < 3$ & $|z| > 0.5$.

Now, taking Inv. Z.T of eqn ②,

$$\boxed{h(n) = -(2)(3)^n u(-n-1) + (0.5)^n u(n)}$$

2) The difference equation of a system is given below $y(n) = 0.5 y(n-1) + x(n)$. Determine
i) system function
ii) Pole zero plot of the system function
iii) Unit sample response of system.

Soln:

$$y(n) = 0.5 y(n-1) + x(n)$$

Taking Z.T,

$$Y(z) = 0.5 z^{-1} Y(z) + X(z)$$

$$Y(z)[1-0.5z^{-1}] = X(z)$$

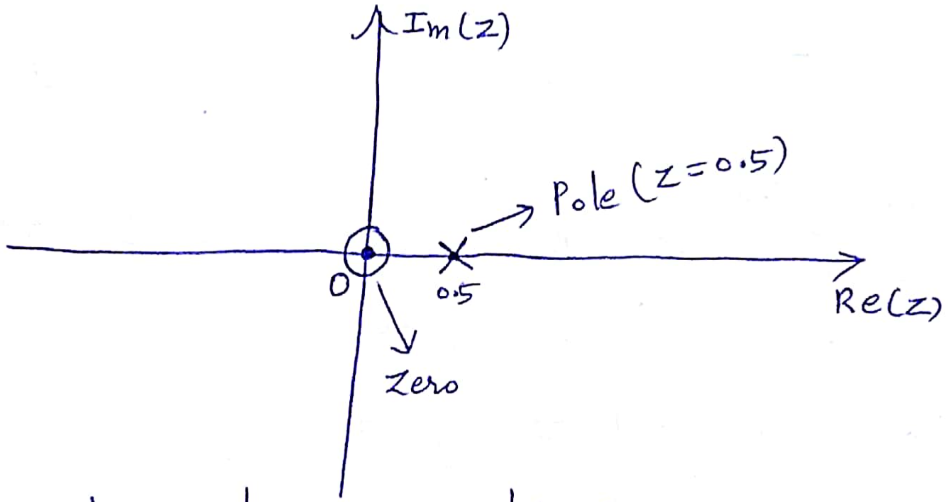
$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1-0.5z^{-1}} = \frac{1}{1-\frac{0.5}{z}} = \frac{1}{\frac{z-0.5}{z}}$$

$$\boxed{H(z) = \frac{z}{z-0.5}} \rightarrow \text{system function} \rightarrow ①$$

* Pole zero plot :

From eqn ①, Zero's : $z=0$

Pole's : $z = 0.5$



* Unit sample response h(n) :

Taking Inv. Z.T of H(z) [∵ using eqn.①]

$$\text{Inv. Z.T}[H(z)] = \text{Inv. Z.T}\left[\frac{z}{z-0.5}\right]$$

$$\boxed{h(n) = (0.5)^n u(n)}$$

3) The input to a causal LTI system is $x(n) = u(-n-1) + (\frac{1}{2})^n u(n)$.

The Z-Transform of the system is : $Y(z) = \frac{-\frac{1}{2}z^{-1}}{(1-\frac{1}{2}z^{-1})(1+z^{-1})}$.

Determine $H(z)$ and also determine output $y(n)$.

Soln:

$$x(n) = u(-n-1) + (\tfrac{1}{2})^n u(n).$$

$Z.T[-b^n u(-n-1)] = \frac{z}{z-b}$;	$ZT[a^n u(n)] = \frac{z}{z-a}$;
Roc: $\lvert z\rvert < b$	Roc: $\lvert z\rvert > a$

Now.

$$Z.T[x(n)] = Z.T[u(-n-1)] + Z.T[(\tfrac{1}{2})^n u(n)]$$

$$X(z) = \frac{-z}{z-1} + \frac{z}{z-\frac{1}{2}} = \frac{-z(z-\frac{1}{2})+z(z-1)}{(z-1)(z-\frac{1}{2})} = \frac{-\cancel{z^2}+0.5z+\cancel{z^2}-z}{(z-1)(z-0.5)}$$

$$\boxed{X(z) = \frac{-0.5z}{(z-1)(z-0.5)}} \rightarrow ①$$

$$Y(z) = \frac{-\frac{1}{2}z^{-1}}{(1-\frac{1}{2}z^{-1})(1+z^{-1})}.$$

Convert -ve power of 'z' into +ve power of 'z'.

$$Y(z) = \frac{z^2(-\frac{1}{2}z^{-1})}{z(1-\frac{1}{2}z^{-1})z(1+z^{-1})}$$

$$Y(z) = \frac{-0.5z}{(z-0.5)(z+1)} \longrightarrow ② .$$

Scanned by CamScanner

$$H(z) = \frac{Y(z)}{X(z)}$$

$$H(z) = \frac{-0.5\cancel{z}}{(z-\cancel{0.5})(z+1)} \cdot \frac{(z-1)(z-\cancel{0.5})}{-\cancel{0.5}z}$$

$$\boxed{H(z) = \frac{z-1}{z+1}}$$

* To find y(n):

$$② \Rightarrow Y(z) = \frac{-0.5z}{(z-0.5)(z+1)}$$

$$\frac{Y(z)}{z} = \frac{-0.5}{(z-0.5)(z+1)}$$

$$\frac{Y(z)}{z} = \frac{K_0}{z-0.5} + \frac{K_1}{z+1} \longrightarrow ③$$

$$K_0 = \frac{Y(z)}{z} \times (z-0.5)\Big|_{z=0.5}$$

$$K_0 = \frac{-0.5}{(z-\cancel{0.5})(z+1)} \times (z-\cancel{0.5})\Big|_{z=0.5}$$

$$K_0 = \frac{-0.5}{(0.5+1)} = \frac{-0.5}{1.5}$$

$$\boxed{K_0 = -0.333}$$

$$k_1 = \frac{Y(z)}{z} \times (z+1) \Big|_{z=-1}$$

$$k_1 = \frac{-0.5}{(z-0.5)\cancel{(z+1)}} \times \cancel{(z+1)} \Big|_{z=-1}$$

$$k_1 = \frac{-0.5}{-1-0.5} = \frac{-0.5}{-1.5}$$

$$\boxed{k_1 = 0.333}$$

Now, eqn. ③ $\Rightarrow$ $\frac{Y(z)}{z} = -\frac{0.333}{z-0.5} + \frac{0.333}{z+1}.$

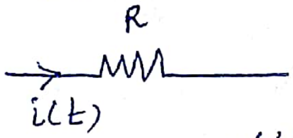
$$Y(z) = (-0.333)\frac{z}{z-0.5} + (0.333)\frac{z}{z+1}$$

Taking Inv. z.T,

$$\boxed{y(n) = (-0.333)(0.5)^n u(n) + (0.333)(-1)^n u(n)}$$

→ Analysis of Electrical Networks using Laplace Transform:

* Resistor

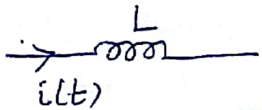


$$v(t) = i(t) \cdot R$$

$$LT[v(t)] = LT[i(t) \cdot R]$$

$$V(s) = I(s) \cdot R$$

*

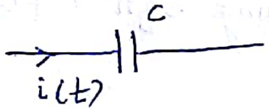


$$v(t) = L \frac{di(t)}{dt}$$

$$L \cdot T[v(t)] = L \cdot T\left[L \frac{di(t)}{dt}\right]$$

$$V(s) = L \cdot sI(s)$$

*



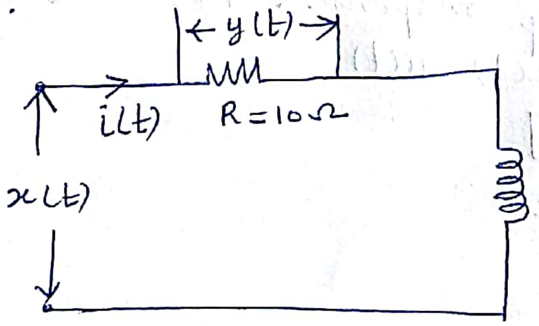
$$v(t) = \frac{1}{c} \int i(t)\, dt$$

$$L \cdot T[v(t)] = L \cdot T\left[\frac{1}{c} \int i(t)\, dt\right]$$

$$V(s) = \frac{1}{c} \cdot \frac{I(s)}{s}$$

→ Problems:

1) Find the unit step response of the circuit shown in figure.



Soln:

From the circuit,

$$x(t) = i(t) \cdot R + L \frac{di(t)}{dt}$$

Taking L.T,

$$X(s) = I(s).R + LsI(s)$$

$$X(s) = I(s)[R + Ls]$$

$$X(s) = I(s)[10 + 5s] \rightarrow ①$$

From the circuit,

$$y(t) = i(t).R$$

Taking L.T,

$$y(s) = I(s).R$$

$$y(s) = 10I(s) \rightarrow ②$$

W.K.T,

$$H(s) = \frac{y(s)}{X(s)} = \frac{10I(s)}{I(s)[10+5s]}$$ [∵ using eqn. ① & ②]

$$H(s) = \frac{10}{10+5s}$$

$$H(s) = \frac{10}{5[2+s]}$$

$$\boxed{H(s) = \frac{2}{s+2}} \rightarrow ③.$$

* To find unit step response: i.e, to find the output y(t) when the input x(t) = u(t).

$$L.T[x(t)] = L.T[u(t)]$$

$$X(s) = \frac{1}{s} \rightarrow ④.$$

W.k.T,

$$H(s) = \frac{y(s)}{X(s)}$$

$$y(s) = H(s).X(s)$$

$$y(s) = \frac{2}{(s+2)} \cdot \frac{1}{s}$$ [∵ using eqn. ③ & ④]

$$Y(s) = \frac{2}{s(s+2)} \longrightarrow ⑤$$

$$Y(s) = \frac{K_0}{s} + \frac{K_1}{s+2} \longrightarrow ⑥.$$

$$K_0 = Y(s) \cdot s \Big|_{s=0}$$

$$K_0 = \frac{2}{\cancel{s}(s+2)} \times \cancel{s} \Big|_{s=0}$$

$$K_0 = \frac{2}{0+2} = \frac{2}{2}$$

$$\boxed{K_0 = 1}$$

$$K_1 = Y(s).(s+2)\Big|_{s=-2}$$

$$K_1 = \frac{2}{s\cancel{(s+2)}} \cdot \cancel{(s+2)}\Big|_{s=-2}$$

$$K_1 = \frac{2}{-2}$$

$$\boxed{K_1 = -1}$$

Now, ⑥ $\Rightarrow$ $Y(s) = \frac{1}{s} - \frac{1}{s+2}$

Taking Inv. L.T,

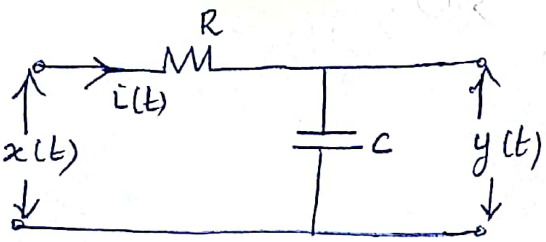
$$LT^{-1}[Y(s)] = LT^{-1}\left[\frac{1}{s}\right] - LT^{-1}\left[\frac{1}{s+2}\right]$$

$$\boxed{y(t) = u(t) - e^{-2t}u(t)}$$

→ Impulse response of ideal Low Pass Filter.

Soln:

Low Pass Filter → Integrator



x(t) → input

y(t) → output

From the circuit,

$$x(t) = i(t)R + \frac{1}{c}\int i(t)\,dt$$

Taking L.T,

$$X(s) = I(s).R + \frac{1}{c}.\frac{I(s)}{s}$$

$$X(s) = I(s)\left[R + \frac{1}{sc}\right] \longrightarrow ①$$

From the circuit,

$$y(t) = \frac{1}{c}\int i(t)\,dt$$

Taking L.T,

$$Y(s) = \frac{1}{c}.\frac{I(s)}{s.} \longrightarrow ②$$

system function (or) Transfer function : $H(s) = \dfrac{Y(s)}{X(s)}$

$$H(s) = \frac{\dfrac{\cancel{I(s)}}{cs}}{\cancel{I(s)}\left[R + \frac{1}{sc}\right]} = \frac{1}{cs\left[R + \frac{1}{sc}\right]} = \frac{1}{csR + 1}$$

$$H(s) = \frac{1}{Rc\left[s + \frac{1}{Rc}\right]} = \frac{1/Rc}{\left(s + \frac{1}{Rc}\right)} \longrightarrow ③$$

Taking Inv. L.T,

$$LT^{-1}[H(s)] = LT^{-1}\left[\frac{1/Rc}{s + \frac{1}{Rc}}\right]$$

$$\boxed{h(t) = \frac{1}{Rc}.\,e^{-(1/Rc)t}}$$ → Impulse response of Low Pass Filter.

* Unit step response:

To find y(t), when the input $x(t) = u(t)$.

$$L.T[x(t)] = L.T[u(t)]$$

$$X(s) = \frac{1}{s} \rightarrow ④$$

W.K.T,

$$H(s) = \frac{Y(s)}{X(s)}$$

$$Y(s) = H(s) \cdot X(s)$$

$$Y(s) = \frac{1/RC}{\left(s+\frac{1}{RC}\right)} \cdot \frac{1}{s} \quad [\because \text{using eqn.} ③ \& ④]$$

$$Y(s) = \frac{k_0}{s+\frac{1}{RC}} + \frac{k_1}{s} \rightarrow ⑤$$

$$k_0 = Y(s) \cdot \left(s+\frac{1}{RC}\right)\Big|_{s=-\frac{1}{RC}}$$

$$= \frac{1/RC}{\cancel{\left(s+\frac{1}{RC}\right)}} \cdot \frac{1}{s} \times \cancel{\left(s+\frac{1}{RC}\right)}\Big|_{s=-\frac{1}{RC}}$$

$$= \frac{1}{RC} \times \frac{1}{-1/RC} = \frac{RC}{-RC}$$

$$\boxed{k_0 = -1}$$

$$k_1 = Y(s) \cdot s\Big|_{s=0}$$

$$k_1 = \frac{\frac{1}{RC}}{\left(s+\frac{1}{RC}\right)} \cdot \frac{1}{\cancel{s}} \times \cancel{s}\Big|_{s=0} = \frac{1/RC}{0+1/RC} = \frac{1/RC}{1/RC}$$

$$\boxed{k_1 = 1}$$

Now, eqn.⑤ ⟹ $Y(s) = \frac{-1}{s+\frac{1}{RC}} + \frac{1}{s}$

Taking Inv. L.T,

$$LT^{-1}[Y(s)] = LT^{-1}\left[\frac{-1}{s+\frac{1}{RC}}\right] + LT^{-1}\left[\frac{1}{s}\right]$$

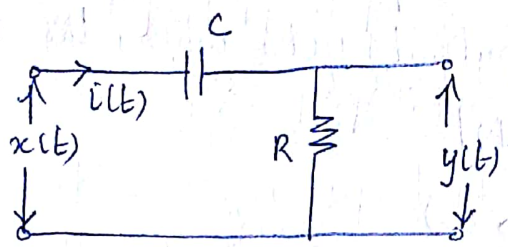
$$y(t) = -e^{\left(-\frac{1}{RC}\right)t} u(t) + u(t).$$

$$\boxed{y(t) = u(t) - e^{\left(-\frac{1}{RC}\right)t} u(t)}$$

→ Impulse Response of ideal high pass filter:

Soln:

ideal high pass filter → differentiator



x(t) → Input
y(t) → output

From the circuit,

$$x(t) = \frac{1}{c}\int i(t)\,dt + i(t)\,R$$

Taking L.T,

$$X(s) = \frac{1}{c}\,\frac{I(s)}{s} + I(s).R$$

$$X(s) = I(s)\left[\frac{1}{sc} + R\right] \rightarrow ①$$

From the circuit,

$$y(t) = i(t)\,R$$

Taking L.T,

$$Y(s) = I(s).R \rightarrow ②.$$

*System function (or) Transfer function : $H(s) = \frac{Y(s)}{X(s)}$

$$H(s) = \frac{\cancel{I(s)}.R}{\cancel{I(s)}\left[\frac{1}{sc}+R\right]}$$ [∵ using eqn. ①&②]

$$H(s) = \frac{R}{\frac{1+RSC}{Sc}} = \frac{SCR}{1+SCR}$$

$$H(s) = \frac{S\cancel{CR}}{\cancel{CR}\left[\frac{1}{CR}+s\right]} = \frac{s}{s+\frac{1}{RC}} \rightarrow ③$$

Taking Inv. L.T. of eqn. ③,

$$LT^{-1}[H(s)] = LT^{-1}\left[\frac{s}{s+\frac{1}{RC}}\right] = LT^{-1}\left[\frac{s+\frac{1}{RC}-\frac{1}{RC}}{s+\frac{1}{RC}}\right]$$

$$= LT^{-1}\left[\frac{s+\frac{1}{RC}}{s+\frac{1}{RC}} - \frac{\frac{1}{RC}}{s+\frac{1}{RC}}\right] = LT^{-1}\left[1-\left(\frac{1}{RC}\right)\cdot\frac{1}{\left(s+\frac{1}{RC}\right)}\right]$$

$$\boxed{h(t)=1-\left(\frac{1}{RC}\right)e^{\left(\frac{-1}{RC}\right)t}} \rightarrow \text{Impulse response.}$$

* To find unit step response:

i.e, to find the output y(t) when the input x(t)=u(t)

$$L.T[x(t)] = L.T[u(t)]$$

$$\boxed{X(s) = \frac{1}{s}} \rightarrow ④$$

W.k.T,

$$H(s) = \frac{Y(s)}{X(s)}$$

$$Y(s) = H(s) \cdot X(s)$$

$$Y(s) = \frac{\cancel{s}}{\left(s+\frac{1}{RC}\right)} \cdot \frac{1}{\cancel{s}} \quad [\because \text{using eqn ③ \& ④}]$$

$$Y(s) = \frac{1}{s+\frac{1}{RC}}$$

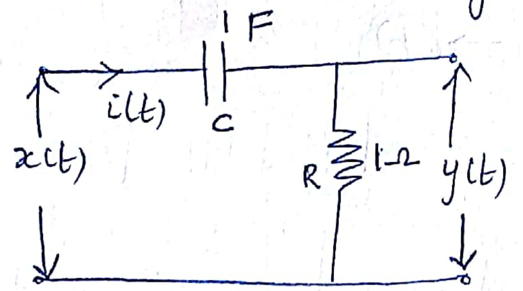
Taking Inv. L.T,

$$L.T^{-1}[Y(s)] = LT^{-1}\left[\frac{1}{s+\frac{1}{RC}}\right]$$

$$\boxed{y(t) = e^{\left(\frac{-1}{RC}\right)t} u(t)}$$

→ Problems:

1) Find the output of the system shown in figure for the input $x(t) = e^{-2t} u(t)$ using Laplace transform.



Soln:

From the circuit,

$$x(t) = \frac{1}{C}\int i(t)\,dt + i(t).R$$

Taking L.T,

$$X(s) = \frac{1}{C}.\frac{I(s)}{s} + I(s).R$$

$$X(s) = I(s)\left[\frac{1}{sC} + R\right]$$

$$X(s) = I(s)\left[\frac{1}{s} + 1\right]$$

$$X(s) = I(s)\left[\frac{1+s}{s}\right] \longrightarrow ①$$

From the circuit,

$$y(t) = i(t).R$$

$$y(t) = i(t).$$

Taking L-T,

$$Y(s) = I(s). \longrightarrow ②$$

W.K.T,

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\cancel{I(s)}}{\cancel{I(s)}\left[\frac{1+s}{s}\right]}$$

$$\boxed{H(s) = \frac{s}{s+1}} \longrightarrow ③.$$

* To find the output $y(t)$ when the input $x(t)=e^{-2t}u(t)$.

$L.T[x(t)] = L.T[e^{-2t}u(t)]$

$$X(s) = \frac{1}{s+2} \rightarrow ④$$

W.K.T,

$$H(s) = \frac{Y(s)}{X(s)}$$

$$Y(s) = H(s).X(s)$$

$$Y(s) = \frac{s}{(s+1)}.\frac{1}{(s+2)} \quad [\because \text{using eqn } ③ \,\&\, ④]$$

$$Y(s) = \frac{k_0}{s+1} + \frac{k_1}{s+2} \rightarrow ⑤$$

$$k_0 = Y(s).(s+1)\Big|_{s=-1}$$

$$= \frac{s}{\cancel{(s+1)}(s+2)} \times \cancel{(s+1)}\Big|_{s=-1} = \frac{-1}{(-1+2)} = \frac{-1}{1}$$

$$\boxed{k_0 = -1}$$

$$k_1 = Y(s).(s+2)\Big|_{s=-2}$$

$$= \frac{s}{(s+1)\cancel{(s+2)}}.\cancel{(s+2)}\Big|_{s=-2} = \frac{-2}{(-2+1)} = \frac{-2}{-1}$$

$$\boxed{k_1 = 2}$$

Now, eqn⑤ $\Rightarrow$ $Y(s) = \frac{-1}{s+1} + \frac{2}{s+2}$

Taking Inv. L.T,

$$LT^{-1}[Y(s)] = LT^{-1}\left[\frac{-1}{s+1} + \frac{2}{s+2}\right]$$

$$\boxed{y(t) = -e^{-t}u(t) + 2e^{-2t}u(t)}$$