

Unit - IPart - A :-

1. Prove that $(P \rightarrow (P \vee Q))$ is a tautology.

Proof :-

Let us prepare the truth tables for $P \rightarrow (P \vee Q)$. For any Propositions 'P' and 'Q'.

P	Q	$P \vee Q$	$P \rightarrow (P \vee Q)$
0	0	0	1
0	1	1	1
1	0	1	1
1	1	1	1

From the above table, for all possible truth values of P and Q, the compound Proposition $P \rightarrow (P \vee Q)$ is true.
 $\therefore P \rightarrow (P \vee Q)$ is a tautology.

2. Construct the truth table for $(\neg P) \vee (\neg Q)$.

Solution :-

For any proposition 'P' and 'Q', Let us form the truth table

P	Q	$\neg P$	$\neg Q$	$(\neg P) \vee (\neg Q)$
0	0	1	1	1
0	1	1	0	1
1	0	0	1	1
1	1	0	0	0

3. Symbolize the statement "All men are mortal".

Solution :-

Let S denotes the set of all men, $P(x)$ denote the statement "x is mortal". Then the given argument reads.
 $\forall x \in S, P(x)$.

$$\frac{a \in S}{\therefore p(a)}$$

Since the statement $\forall x, \exists S, p(x)$ is true and $a \in S$, it follows by the rule of universal specification that $p(a)$ is true. Thus, the given (statement)_{argument} is valid.

4. Symbolise the following statement, "All the world loves a lover".

Solution:

Let S denote the world, $p(x)$ denote the statement " x loves a lover". Then, the given argument reads,

$$\forall x \in S, p(x).$$

Since the statement $\forall x \in S, p(x)$ is true and it follows by the Rule of Universal Generalisation, that $p(x)$ is true. Thus, the given argument is valid.

5. Define proposition with examples.

A proposition is a (declarative) statement that is either true (1) or false (0), but not both.

For example,

(i) Japan is a country

(ii) Rose is red

(iii) $5+5=10$

(iv) What is the meaning of tautology?

(v) $5+x=7$

(vi) Hyderabad is a pink city.

→ (i), (ii) and (vi) are statements, [∵ (i) and (ii) are always true, (vi) are always false]

→ (iv) is not a declarative sentence, it is not a statement.

→ (v) is not take any particular value, ∴ it may be true or false depending on the situation. It is not considered as statement.

6. Define compound proposition with an example.

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"The new proposition formed by combining one or more propositions by using the connectives, called compound propositions."

Example:-

P ! It is raining today

Q ! $2+3=5$

$P \vee Q$! It is raining today or $2+3=5$.

7. Define Well Formed Formulas (WFFs) or (WFF).

Definition:-

A statement formula is an expression which is a string consisting of variables (capital letters with or without subscripts), Parentheses and connective symbols ($\neg, \vee, \rightarrow, \leftrightarrow, \perp$) which produces a statement when the variables are replaced by statements. But, every string of these symbols is not a formula we now give a recursive definition of a statement formula, called a well formed formulas (WFFs or WFF).

8. Write down the properties of well-formed formulas (WFFs).

Solution:-

→ A statement variable standing alone is a well formed formula (primitive statements)

→ If P is a well formed formula then $\neg P$ is also a formulae.

→ If P and Q are well formed formulae then $P \vee Q, P \wedge Q, P \rightarrow Q$ and $P \leftrightarrow Q$ are also well formed formulae.

→ A string of symbols containing the statement variables, connectives and Parentheses is a well-formed formulae, if it can be obtained by finitely many applications of rules 1, 2, and 3.

9. Is $(\neg p \wedge (p \vee q)) \rightarrow q$ a tautology.

Solution:

p	q	$\neg p$	$p \vee q$	$\neg p \wedge (p \vee q)$	$(\neg p \wedge (p \vee q)) \rightarrow q$
0	0	1	0	0	1
0	1	1	1	1	1
1	0	0	1	0	1
1	1	0	1	0	1

Since all the entries in the resulting column is true, hence the given proposition is a tautology.

10. Define Rules of Inference.

Consider a set of Proposition, $P_1, P_2, \dots, P_n$ and a Proposition C . Then a compound proposition of the form $(P_1 \wedge P_2 \wedge \dots \wedge P_n) \rightarrow C$, is called an 'argument'. Here, $P_1, P_2, \dots, P_n$ are called the 'premises' of the argument and 'C' is called the 'conclusion' of the argument.

11. Define Quantifiers with examples.

Consider the following propositions:

→ All Square are rectangles.

→ For every integer x , x^2 is a non-negative integer

→ Some determinants are equal to zero.

→ There exists a real number whose square is equal to itself.

In these propositions, the words 'all', 'every', 'some', 'there exists' are associated with the idea of a quantity. Such words are called quantifiers.

12. Prove that there exists a real number 'x' such that

$$x^3 + 2x^2 - 5x - 6 = 0.$$

Solution:-

It is sufficient to exhibit one real number x such that $x^3 + 2x^2 - 5x - 6 = 0$.

We check that $x = -1$, is one such real number.

13. Prove the following:

$$(i) \forall x, p(x) \Rightarrow \exists x, p(x) \quad (ii) \forall x, [p(x) \vee q(x)] \Rightarrow \forall x, p(x) \vee \exists x, q(x)$$

Solution:-

Let S denote the universe. we find that.

$$(i) \forall x, p(x) \Rightarrow p(x) \text{ is true for every } x \in S.$$

$$\Rightarrow p(a) \text{ is true for } x = a \in S.$$

$$\Rightarrow p(x) \text{ is true for some } x \in S$$

$$\Rightarrow \exists x, p(x).$$

$$(ii) \forall x, [p(x) \vee q(x)] \Rightarrow p(x) \vee q(x) \text{ is true for every } x \in S.$$

$$\Rightarrow \{p(x) \text{ is true for every } x \in S\} \vee \{q(x) \text{ is true for every } x \in S\}$$

$$\Rightarrow \forall x, p(x) \vee q(a) \text{ is true for } a \in S$$

$$\Rightarrow \forall x, p(x) \vee \exists x, p(x).$$

This proves the required result.

14. Define method of proof.

Solution:-

The proposition that commonly appear in mathematical discussions are conditionals of the form $p \rightarrow q$, where p and q are simple or compound propositions which may involve quantifiers as well. Given such a conditional, the process of establishing that the conditional is true by using the rules/laws of logic and known facts constitutes a proof of the conditional.

15. Let m and n be integers. Prove that $n^2 = m^2$ if and only if $m = n$ or $m = -n$.

Solution:

Consider the propositions

$p: n^2 = m^2$, $q: m = n$, $r: m = -n$.

We have to prove that $p \leftrightarrow (q \vee r)$ is true. First, assume that $q \vee r$ is true. Then $m = n$ or $m = -n$, so that $m^2 = n^2$, i.e., p is true.

Thus, $(q \vee r) \rightarrow p$ is true.

Next, assume that $\sim(q \vee r)$ is true, i.e., $(\sim q) \wedge (\sim r)$ is true.

Then q is false and r is false, i.e., $m \neq n$ and $m \neq -n$.

Then $m^2 \neq n^2$, i.e., $\sim p$ is true. This proves that $\sim(q \vee r) \rightarrow \sim p$

is true. Accordingly, the statement $p \rightarrow (q \vee r)$ is true.

Thus, we have proved that both of $(q \vee r) \rightarrow p$ and $p \rightarrow (q \vee r)$ are true. Hence $p \leftrightarrow (q \vee r)$ is true.

The process of establishing that a proposition is false constitutes a disproof.

Contrapositive Law: $p \rightarrow q \Leftrightarrow \neg q \rightarrow \neg p$

Conditional of Disjunction: $p \rightarrow q \Leftrightarrow \neg p \vee q \rightarrow (\cdot \cdot)$

Biconditional as conditional: $p \rightarrow q \Leftrightarrow (p \rightarrow q) \wedge (q \rightarrow p)$.

Exportation Law: $p \rightarrow (q \rightarrow r) \Leftrightarrow (p \wedge q) \rightarrow r$

Theory of Inference :-

Using rule of Inference:-

S.I. No.	Tautological Form	Rules of Inference	Name
1.	$p \Rightarrow (p \vee q)$	$\frac{p}{p \vee q}$	Addition
2.	$q \Rightarrow (p \vee q)$	$\frac{q}{p \vee q}$	Addition
3.	$p \wedge q \Rightarrow p$	$\frac{p \wedge q}{p}$	Simplification
4.	$p \wedge q \Rightarrow q$	$\frac{p \wedge q}{q}$	Simplification
5.	$[p \wedge (p \rightarrow q)] \Rightarrow q$	$\frac{p \quad p \rightarrow q}{q}$	Modus ponens
6.	$[\neg q \wedge (p \rightarrow q)] \Rightarrow \neg p$	$\frac{\neg q \quad p \rightarrow q}{\neg p}$	Modus Tollens
7.	$(p \vee q) \wedge (\neg p) \Rightarrow q$	$\frac{p \vee q \quad \neg p}{q}$	Disjunctive Syllogism
8.	$(p \rightarrow q) \wedge (q \rightarrow r) \Rightarrow p \rightarrow r$	$\frac{p \rightarrow q \quad q \rightarrow r}{p \rightarrow r}$	Hypothetical Syllogism (or) Translative rule.
9.	$[(p \vee q) \wedge (p \rightarrow r) \wedge (q \rightarrow r)] \Rightarrow r$	$\frac{p \vee q \quad p \rightarrow r \quad q \rightarrow r}{r}$	Dilemma
10.	$[(p \vee q) \wedge (\neg p \vee r)] \Rightarrow (q \vee r)$	$\frac{p \vee q \quad \neg p \vee r}{q \vee r}$	Resolution.

1) Rule - P :-

A ^{pre}misses may be introduced at any point in the derivation.

2) Rule - T :-

A formula S may be introduced in a given derivation. If S is tautology implied in any one or more of the proceeding formula. in the derivation.

3) Rule - CP :- If we can ^Pdrive S from R and a set of premisses then we can derive $R \rightarrow S$ from that set of premisses alone.

Quantifiers :-

SI. No.	Rule	Inference.
1	US	$\frac{\forall x P(x)}{P(c)}$ for some c .
2	CS	$\frac{\exists x P(x)}{P(c)}$ for a particular of c .
3	UG	$\frac{P(c) \text{ for an arbitrary } c}{\forall P(x)}$
4	EG	$\frac{P(c) \text{ for some } c}{\exists P(x)}$

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Part-B:-

1. Show that $(\neg P \wedge (\neg Q \wedge R)) \vee (Q \wedge R) \vee (P \wedge R) \Leftrightarrow R$.

Proof:-

we have,

$$[\neg P \wedge (\neg Q \wedge R)] \Leftrightarrow (\neg P \wedge \neg Q) \wedge R \quad [\because \text{By Associative Law}]$$

$$\Leftrightarrow [\neg(P \vee Q)] \wedge R \quad [\because \text{By Demorgan Law}]$$

$$\Leftrightarrow R \wedge [\neg(P \vee Q)] \longrightarrow \textcircled{1}$$

and,

$$(Q \wedge R) \vee (P \wedge R) \Leftrightarrow (R \wedge Q) \vee (R \wedge P) \quad [\because \text{By Commutative Law}]$$

$$\Leftrightarrow R \wedge (Q \vee P) \quad [\because \text{By Distributive Law}]$$

$$\Leftrightarrow R \wedge (P \vee Q) \longrightarrow \textcircled{2} \quad [\because \text{By Commutative Law}]$$

 $\therefore$ From $\textcircled{1}$ & $\textcircled{2}$, we have

$$[\neg P \wedge (\neg Q \wedge R)] \vee (Q \wedge R) \vee (P \wedge R) \Leftrightarrow \{R \wedge [\neg(P \vee Q)]\} \vee \{R \wedge (P \vee Q)\}$$

$$\Leftrightarrow R \wedge \{[\neg(P \vee Q)] \vee (P \vee Q)\}$$

$$\Leftrightarrow R \wedge T_0 \quad [\because \text{By Inverse Law}]$$

$$[\neg P \wedge (\neg Q \wedge R)] \vee (Q \wedge R) \vee (P \wedge R) \Leftrightarrow R \quad [\because \text{By Identity Law}]$$

Hence Proved.

2. Show that $\neg(P \wedge (\neg P \wedge Q))$ and $(\neg P \wedge \neg Q)$ are logically equivalent without using truth table.~~Proof~~ Solution:-

$$\text{Given, } \neg(P \wedge (\neg P \wedge Q)) \Leftrightarrow \neg((P \wedge \neg P) \wedge Q) \quad [\because \text{By Associative Law}]$$

$$\Leftrightarrow \neg(F_0 \wedge Q) \quad [\because \text{By Inverse Law}]$$

$$\Leftrightarrow \neg F_0 \vee \neg Q \quad [\because \text{Demorgan's Law}]$$

$$\Leftrightarrow T_0 \vee \neg Q$$

$$\Leftrightarrow T_0 \quad [\because \text{Domination Law}]$$

3. Using the indirect method, show that:

$$(x)(p(x) \vee q(x)) \Rightarrow x p(x) \vee (\exists x) q(x).$$

Solution:-

We shall use the indirect method of proof. Assume that $\sim [(x) p(x) \vee (\exists x) q(x)]$ as an additional premisses.

Sr. No	Statement	Reason.
1.	$\sim [(x) p(x) \vee (\exists x) q(x)]$	Assumed premisses.
2.	$(\exists x) \sim p(x) \wedge (x) \sim q(x)$	Rule T
3.	$(\exists x) \sim p(x)$	Rule T
4.	$(\exists x) \sim q(x)$	Rule T
5.	$\sim p(y)$	Rule ES
6.	$\sim q(y)$	Rule US
7.	$\sim p(y) \wedge \sim q(y)$	Rule T
8.	$\sim (p(y) \vee q(y))$	Rule T
9.	$x (p(x) \vee q(x))$	Rule P
10.	$p(y) \vee q(y)$	Rule US
11.	$[p(y) \vee q(y)] \wedge \sim [p(y) \vee q(y)]$	Rule T

which is nothing but false value.

$\therefore$ By method for contradiction, we have

$$x [p(x) \vee q(x)] \Rightarrow (x) p(x) \vee (\exists x) q(x).$$

4. Find the Principle Disjunctive Normal Form and Principle Conjunctive Normal Form of the formula $P \vee (\neg P \rightarrow (q \vee (\neg q \rightarrow r)))$ (6)

Solution:-

Let
Given, $P \vee (\neg P \rightarrow (q \vee (\neg q \rightarrow r))) = A$

$$A \Rightarrow P \vee (\neg P \vee (q \vee (\neg q \rightarrow r)))$$

[$\therefore$ Conversion Law]

$$\Rightarrow P \vee (P \vee (q \vee (\neg q \vee r)))$$

[$\therefore$ Conversion Law]

$$\Rightarrow P \vee (P \vee (q \vee (\neg q \vee r)))$$

[$\therefore$ Negation Law
Conversion Law]

$$\Rightarrow P \vee (P \vee (q \vee r))$$

[$\therefore$ Associative Law]

$$\Rightarrow P \vee (P \vee q \vee r)$$

[$\therefore$ Idempotent Law]

$$\Rightarrow (P \vee P) \vee (q \vee r)$$

[$\therefore$ Associative Law]

$$A \Rightarrow P \vee q \vee r$$

[$\therefore$ Idempotent Law]

This is the Principle Conjunctive Normal form of 'A' for P, q, r.

To find the Principle Disjunctive Normal form,

$$\therefore \neg A \equiv (\neg P \vee q \vee r) \wedge (P \vee \neg q \vee r) \wedge (P \vee q \vee \neg r) \wedge (\neg P \vee \neg q \vee r) \wedge (\neg P \vee q \vee \neg r) \wedge (P \vee \neg q \vee \neg r) \wedge (\neg P \vee \neg q \vee \neg r)$$

$$\text{Now, } A \equiv \neg(\neg A)$$

$$\equiv \neg [(\neg P \vee q \vee r) \wedge (P \vee \neg q \vee r) \wedge (P \vee q \vee \neg r) \wedge (\neg P \vee \neg q \vee r) \wedge (\neg P \vee q \vee \neg r) \wedge (P \vee \neg q \vee \neg r) \wedge (\neg P \vee \neg q \vee \neg r)]$$

$$\equiv \neg(\neg P \vee q \vee r) \vee \neg(P \vee \neg q \vee r) \vee \neg(P \vee q \vee \neg r) \vee \neg(\neg P \vee \neg q \vee r)$$

$$\vee \neg(\neg P \vee q \vee \neg r) \vee \neg(P \vee \neg q \vee \neg r) \vee \neg(\neg P \vee \neg q \vee \neg r)$$

$$A \equiv (P \wedge \neg q \wedge \neg r) \vee (\neg P \wedge q \wedge \neg r) \vee (\neg P \wedge \neg q \wedge r) \vee (P \wedge q \wedge r)$$

$$\vee (P \wedge \neg q \wedge r) \vee (\neg P \wedge q \wedge r) \vee (P \wedge q \wedge \neg r)$$

$\therefore$ This is the principle Disjunctive Normal Form.

5. Find the conjunctive Normal Form of $P \rightarrow [(P \rightarrow Q) \wedge \neg (\neg Q \vee \neg P)]$

Solution :-

$$\text{Given, } P \rightarrow [(P \rightarrow Q) \wedge \neg (\neg Q \vee \neg P)]$$

$$\Leftrightarrow P \rightarrow [(P \rightarrow Q) \wedge (Q \wedge P)]$$

[$\therefore$ Negation Law]

$$\Leftrightarrow \neg P \vee [\neg(P \vee Q) \wedge (Q \wedge P)]$$

[$\therefore$ Conversion Law]

$$\Leftrightarrow \neg P \vee [\neg(P \wedge (Q \wedge P)) \vee (Q \wedge (Q \wedge P))]$$

[$\therefore$ Distributive Law]

$$\Leftrightarrow \neg P \vee [(\neg P \wedge P) \wedge Q \vee ((Q \wedge Q) \wedge P)]$$

[$\therefore$ Associative Law]

$$\Leftrightarrow \neg P \vee [(F_0 \wedge Q) \vee (Q \wedge P)]$$

[$\therefore$ Inverse Law]

$$\Leftrightarrow \neg P \vee [F_0 \vee (Q \wedge P)]$$

[$\therefore$ Domination Law]

$$\Leftrightarrow \neg P \vee (Q \wedge P)$$

[$\therefore$ Identity Law]

$$\Leftrightarrow (\neg P \vee Q) \wedge (\neg P \vee P)$$

[$\therefore$ Distributive Law]

$$\Leftrightarrow (\neg P \vee Q) \wedge T_0$$

[$\therefore$ Inverse Law]

$$\Leftrightarrow (\neg P \vee Q) \wedge (P \vee T_0)$$

[$\therefore$ Domination Law]

which is the required conjunctive Normal Form.

Normal Forms :-

A formula which is equivalent to a given formula and which consists of sum of elementary products is called a disjunctive normal form of the given formula.

PDNF $\rightarrow$ Principle Disjunctive Normal Form	PCNF $\rightarrow$ Principle Conjunctive Normal Form.
A formula which is equivalent to a given formula which consists of sum of its miniterms is called PDNF	A formula which is equivalent to a given formula which consists of product of maxterms is called PCNF.

Unit - I :

1. Show that $(\sim P \wedge (\sim Q \wedge R)) \vee (Q \wedge R) \vee (P \wedge R) \Leftrightarrow R$
2. Show that $[(P \vee Q) \wedge \sim (\sim P \wedge (\sim Q \vee \sim R)) \vee (\sim P \wedge \sim Q) \vee (\sim P \wedge \sim R)]$ is a tautology
3. Show that $(P \rightarrow Q) \wedge (R \rightarrow Q) \Leftrightarrow (P \vee R) \rightarrow Q$
4. Show that $P \rightarrow (Q \rightarrow P) \Leftrightarrow \sim P \rightarrow (P \rightarrow Q)$
5. Show that $\sim (P \wedge Q) \rightarrow (\sim P \vee (\sim P \vee Q)) \Leftrightarrow (\sim P \vee Q)$
6. Find the principle disjunctive normal form (PDNF) of the statement $(Q \vee (P \wedge R) \wedge \sim ((P \vee R) \wedge Q))$
7. Find the distinctive normal form of the following
 - (i) $P \rightarrow \{ (P \rightarrow Q) \wedge \sim (\sim Q \vee \sim P) \}$
 - (ii) $\sim (P \vee Q) \leftrightarrow (P \wedge Q)$
8. Find the distinctive normal form of the compound proposition $\{ Q \vee (P \wedge R) \} \wedge \sim \{ (P \vee R) \wedge Q \}$.
9. Find the conjunctive normal forms of the compound propositions.
 - (i) $P \wedge (P \rightarrow Q)$, (ii) $\sim (P \vee Q) \leftrightarrow (P \wedge Q)$.
10. Using a conjunctive normal form, show that $Q \vee (P \wedge Q) \vee (\sim P \wedge \sim Q)$ is a tautology.
11. Prove that $\sim [\{ (P \vee Q) \wedge R \} \rightarrow Q] \Leftrightarrow \sim [\sim [(P \vee Q) \wedge R] \vee \sim Q] \Leftrightarrow Q \wedge R$.
12. Using truth table, for any three propositions P, Q, R .
 - (i) $[P \rightarrow (Q \wedge R)] \Leftrightarrow [P \rightarrow Q] \wedge [P \rightarrow R]$.
 - (ii) $[P \rightarrow (Q \vee R)] \Leftrightarrow [P \rightarrow Q] \vee [P \rightarrow R]$.
13. Using truth table, prove that $(P \rightarrow Q) \Leftrightarrow \sim P \vee Q$
14. Using truth table for any proposition's P, Q, R the compound Proposition. $\{ P \rightarrow (Q \rightarrow R) \} \rightarrow \{ (P \rightarrow Q) \rightarrow (P \rightarrow R) \}$ is a tautology

15. Find the possible situations in which the following are contradictions.

(i) $[(P \wedge Q) \wedge r] \rightarrow (S \vee t).$

(ii) $[P \wedge (Q \wedge r)] \rightarrow (S \vee t).$

16. Determine whether the following are tautologies.

(i) $P \vee [\neg(P \wedge Q)]$

(ii) $(P \vee Q) \vee \neg P$

(iii) $P \rightarrow (P \wedge Q)$

(iv) $P \rightarrow (P \vee Q)$

17. Prove the statement by using contradiction. "If $3n+2$ is odd, then n is odd".

Solution:-

~~Given~~

Let $P: 3n+2$ is odd

$Q: n$ is odd.

Let us assume that $P \rightarrow Q$ is false.

ie, Assume that P is true and Q is false.

$\neg$ n is ^{not} odd
 $\wedge$

$\Rightarrow n$ is even.

If n is even, then $n = 2k$ for some integer k .

$$3n+2 = 3(2k) + 2$$

$$= 6k + 2$$

$$= 2(3k+1)$$

We observe that $3n+2 = 2(3k+1)$ is divisible by 2.

$\therefore 3n+2$ is even.

This is a contradiction to the assumption for P is true.

$\therefore$ Due to this contradiction, we infer that the given conditional statement $P \rightarrow Q$ is true.

Part - A :-

1. State the Pigeonhole principle.

Solution:-

If 'm' pigeons occupy 'n' pigeonholes and if $m > n$, then two or more pigeons occupy the same pigeonhole.

This is often restated as follows...

If 'm' pigeons occupy 'n' pigeonholes and if $m > n$, then at least one pigeonhole must contain two or more pigeons in it.

2. Define Mathematical Induction.

Solution:- (Principle of Mathematical Induction).

Let 'P' be a proposition defined on the integers $n \geq 1$ such that

(1). $P(1)$ is true

(2). $P(n+1)$ is true whenever $P(n)$ is true.

Then 'P' is true for every integer $n \geq 1$.

3. Define well-ordering principle.

Solution:-

Let 'S' be a non-empty set of positive integers. Then 'S' contains a least element, that is, s . Every subset of S contains a first element.

4. Define strong Induction:

Solution:- (Principle of strong Induction):

The following are the three steps followed for proving a statement $P(n)$ by strong mathematical induction.

(i) Basic step: show that $P(1), P(2), P(3), \dots, P(q)$ are true where $q \geq 1$.

(ii) Strong Inductive Hypothesis: make an assumption that $P(k)$ is true for all $k \in \mathbb{Z}^+$.

(iii) Show that $P(k+1)$ is true based on the assumptions.

5. Difference between Mathematical Induction and strong Induction is that, in former $P(k)$ is assumed to be true, in latter $P(k), P(k-1), P(k-2), \dots, P(1)$ are assumed to be true.

6. Define product Rule:-

Solution:-

If $S_1, S_2, \dots, S_n$ are non-empty sets, then the number of elements in the cartesian product $S_1 \times S_2 \times \dots \times S_n$ is the product $n_1 n_2 \dots n_n$ for any two set 'A' and 'B'. If 'A' has 'm' elements and 'B' has 'n' elements then the cartesian product $A \times B$ will have ~~(m+n)~~ 'mn' elements.

7. Define one-to-one correspondence:-

Solution:-

A one-to-one correspondence between 2 sets A and B is just a one-to-one function from A onto B. i.e., there are exactly the same number of objects of the first type as there of the second type.

8. Define the Inclusion-Exclusive Principle:

Let A and B be any finite sets. Then

$$n(A \cup B) = n(A) + n(B) - n(A \cap B).$$

"To find the number $n(A \cup B)$ of elements in the union $A \cup B$, $n(A) + n(B)$ and then we subtract $n(A \cap B)$. i.e., we include, $n(A)$ and $n(B)$ and we exclude $A \cap B$."

The sum rule which is applied to the non-disjoint set is called "principles of Inclusion or exclusive" also called as "Sieve method".

9. Find the number of permutations of the letter of the word SUCCESS.

Solution:-

The given word has 7 letters, of which 3 are S, 2 are C and 1 each are U and E.

∴ the required number of permutations is

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$$\frac{7!}{3! \cdot 2! \cdot 1! \cdot 1!} = \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{(3 \times 2) \times (2 \times 1) \times 1 \times 1} = \frac{5040}{12} = 420$$

10. Define combinations.

Suppose we are interested in selecting (choosing) a set of r objects from a set of $n \geq r$ objects without regard to order. The set of r objects being selected is traditionally called a combination of r objects.

Since there are C combinations, the total number of permutations is $(C \cdot r!)$. But this is equal to $P(n, r)$. Thus,

$$C \cdot r! = P(n, r), \text{ (or), } C = \frac{P(n, r)}{r!}$$

11. Define Sequence.

A sequence of real numbers is a function f from $N = \{1, 2, 3, \dots\}$ to the set of real numbers R . The image of $n \in N$ is denoted by f_n (instead of $f(n)$). f_n is called the n^{th} term of the sequence.

The sequence is written as $f_1, f_2, \dots, f_n, \dots$ or briefly as $\{f_n\}$.

→ The number of terms in a sequence is finite, then the sequence is called a finite sequence. Otherwise it is called an infinite sequence.

12. Define Recursion:-

Let $\{a_n\}$ be a sequence of real number with a_n as the n^{th} term.

A recurrence relation of the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the earlier terms.

$a_1, a_2, \dots, a_{n-1}$ for all integers n with $n \geq n_0$.

A recurrence relation of a sequence is also known as a difference equation.

13. Find the first five terms of the sequence defined by $a_n = a_{n-1}^2$, $a_1 = 2$.

Solution:-

The given recurrence relation is $a_n = a_{n-1}^2$

$$a_1 = 2,$$

$$a_2 = a_1^2 = 2^2 = 4$$

$$a_3 = a_2^2 = 4^2 = 16$$

$$a_4 = a_3^2 = (16)^2 = 256$$

$$a_5 = a_4^2 = (256)^2 = 65,536.$$

Hence, the first five terms of the sequence are 2, 4, 16, 256, 65,536...

14. Among 200 people, how many of them were born on the same month.

Solution:-

$$\text{No. of Pigeon} = m = \text{no. of people} = 200$$

$$\text{No. of Holes} = n = \text{No. of months} = 12$$

$\therefore$ By generalised Pigeon hole principle,

$$\left\lceil \frac{m-1}{n} \right\rceil + 1 = \left\lceil \frac{200-1}{12} \right\rceil + 1$$

$$= \left\lceil \frac{199}{12} \right\rceil + 1$$

$$= 16 + 1$$

$$= 17, \text{ were born on the same month.}$$

Part - B.

(10)

1. Use mathematical Induction to show that $1^2 + 2^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$, whenever 'n' is a positive Integer.

Solution:

Let $P(n)$ be $1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$

To prove: $P(1)$ is true

for $n=1$; $1^2 = \frac{1(1+1)(2(1)+1)}{6} = \frac{6}{6} = 1$

$\Rightarrow 1=1$

$\Rightarrow P(1)$ is true.

Assume that $P(k)$ is true. To prove $P(k+1)$ is true. $\Rightarrow n=k+1$

$\Rightarrow P(k+1) = \frac{(k+1)[(k+1)+1][2(k+1)+1]}{6}$

$\therefore P(k+1) = \frac{(k+1)(k+2)(2k+3)}{6}$

$$\begin{aligned}(1^2 + 2^2 + 3^2 + \dots + k^2) + (k+1)^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\&= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \\&= \frac{(k+1)[k(2k+1) + 6(k+1)]}{6} \\&= \frac{(k+1)[2k^2 + k + 6k + 6]}{6} \\&= \frac{(k+1)(k+2)(2k+3)}{6}\end{aligned}$$

$= P(k+1)$

i.e., $P(k+1)$ is true, whenever $P(k)$ is true.

By the ~~principle~~ ^{Principle} of Mathematical Induction $P(n)$ is true

2) all ~~positive~~ ^{positive} integers n !

2. Use mathematical Induction to prove that

$$1^2 + 3^2 + 5^2 + \dots + (2n+1)^2 = \frac{(n+1)(2n+1)(2n+3)}{3}$$

Solution:-

$$\text{Let } S(n) : 1^2 + 3^2 + 5^2 + \dots + (2n-1)^2 = \frac{1}{3} n(2n-1)(2n+1)$$

when $n=1$,

$$S(1) : (2(1)-1)^2 = \frac{1}{3} (1)(2-1)(2+1)$$

$$1^2 = \frac{1}{3} (1)(1)(3)$$

$$\therefore \boxed{1=1}$$

So, $S(1)$ is true, the basic step is valid.

Let $S(n)$ be true for $n=k$,

$$\text{i.e., } 1^2 + 3^2 + 5^2 + \dots + (2k-1)^2 = \frac{1}{3} k(2k-1)(2k+1)$$

$$\begin{aligned} \text{now, } 1^2 + 3^2 + 5^2 + \dots + (2k-1)^2 + (2k+1)^2 &= \left[\frac{1}{3} k(2k-1)(2k+1) \right] + (2k+1)^2 \\ &= \frac{1}{3} [k(2k-1)(2k+1) + 3(2k+1)^2] \\ &= \frac{1}{3} (2k+1) [k(2k-1) + 3(2k+1)] \\ &= \frac{1}{3} (2k+1) [2k^2 - k + 6k + 3] \\ &= \frac{1}{3} (2k+1) [2k^2 + 5k + 3] \\ &= \frac{1}{3} (2k+1) [2k^2 + 4k + 3k + 3] \\ &= \frac{1}{3} (2k+1) [2k(k+1) + 3(k+1)] \\ &= \frac{1}{3} (2k+1) (2k+3)(k+1) \end{aligned}$$

∴ $S(k+1)$ is valid

Thus the inductive step is also true. $S(n)$ is true for all $n \in \mathbb{Z}^+$.

3. In a survey of 100 students it was found that 30 studied Mathematics, 54 studied statistics, 25 studied operations research, 1 studied all 3 subjects, 20 studied maths and statistics, 3 studied maths and operations research, 15 studied statistics and operations research.

- (i) How many students studied none of these subjects?
 (ii) How many studied only Mathematics?

Solution:-

Let A denotes the students who studied mathematics.

B denotes the students who studied statistics.

C denotes the students who ~~only~~ studied operations research.

$$|A|=30; |B|=54; |C|=25; |A \cap B|=20; |A \cap C|=3; |B \cap C|=15;$$

$$|A \cap B \cap C|=1.$$

By the principle of Inclusion & Exclusion.

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

$$= 30 + 54 + 25 - 20 - 3 - 15 + 1$$

$$= 110 - 38$$

$$\therefore |A \cup B \cup C| = 72.$$

$$\begin{aligned} \text{(i) The students who studied } \left. \begin{array}{l} \text{none of the subjects} \end{array} \right\} &= \overset{\text{no. of}}{\text{Total students}} - |A \cup B \cup C| \\ &= 100 - 72 \\ &= 28. \end{aligned}$$

$$\begin{aligned} \text{(ii) The students only studied } \left. \begin{array}{l} \text{mathematics} \end{array} \right\} &= n(A \cap B) - n(A \cap B \cap C) \\ &= 20 - 1 \\ &= 19. \end{aligned}$$

4. Prove by induction that a finite set with n elements has exactly 2^n subsets. (or) Prove that the no. of subsets of set having n elements is 2^n .

Solution:-

Let A be a set with n elements.

Let $p(n)$ denote the proposition 'the no. of subsets of a set A is 2^n '.

We have to prove that $p(n)$ is true for $n \geq 0$.

Let $n = 0$

$\therefore A = \phi$; so A has exactly $2^0 = 1$ subset, which is true, since ϕ is the only subset, implies $p(0)$ is true.

Assume $p(k)$ is true, $k \geq 0$.

A set with k elements has 2^k subsets is true.

To prove $p(k+1)$ is true.

' A ' set with $(k+1)$ elements has 2^{k+1} subsets is true.

Let $a \in A$, then $B = A - \{a\}$ is the set with k elements.

$\therefore$ The no. of subsets of B is 2^k by induction hypothesis.

Since every subset of A either contains a or does not contain a , the total no. of subsets of A is,

$$2^k + 2^k = 2 \cdot 2^k = 2^{k+1}$$

$\therefore p(k+1)$ is true.

$\therefore$ By principle of induction $p(n)$ is true for $n \geq 0$.

$\Rightarrow$ The no. of subsets of a set with n elements is 2^n .

5. Find no. of integers between 1 to 100 that are divisible by (i) 2, 3, 5 or 7, (ii) 2, 3, 5 but not by 7.

Solution:-

Solution:-

(12)

i) Let A, B, C and D denotes the number of the integers between 1 to 100 which are divisible by 2, 3, 5, 7.

$$|A| = \left\lfloor \frac{100}{2} \right\rfloor = 50 ; |A \cap B| = \left\lfloor \frac{100}{2 \times 3} \right\rfloor = 16 ; |B \cap D| = \left\lfloor \frac{100}{3 \times 7} \right\rfloor = 4$$

$$|B| = \left\lfloor \frac{100}{3} \right\rfloor = 33 ; |A \cap C| = \left\lfloor \frac{100}{2 \times 5} \right\rfloor = 10 ; |C \cap D| = \left\lfloor \frac{100}{5 \times 7} \right\rfloor = 2$$

$$|C| = \left\lfloor \frac{100}{5} \right\rfloor = 20 ; |A \cap D| = \left\lfloor \frac{100}{2 \times 7} \right\rfloor = 7 ; |A \cap B \cap C| = \left\lfloor \frac{100}{2 \times 3 \times 5} \right\rfloor = 3$$

$$|D| = \left\lfloor \frac{100}{7} \right\rfloor = 14 ; |B \cap C| = \left\lfloor \frac{100}{3 \times 5} \right\rfloor = 6 ; |A \cap B \cap D| = \left\lfloor \frac{100}{2 \times 3 \times 7} \right\rfloor = 2$$

$$|B \cap C \cap D| = \left\lfloor \frac{100}{3 \times 5 \times 7} \right\rfloor = 0 ; |A \cap B \cap C \cap D| = \left\lfloor \frac{100}{2 \times 3 \times 5 \times 7} \right\rfloor = 0 ; |A \cap C \cap D| = \left\lfloor \frac{100}{2 \times 5 \times 7} \right\rfloor = 1$$

By the principle of Exclusion-Inclusion,

$$\begin{aligned} |A \cup B \cup C \cup D| &= |A| + |B| + |C| + |D| - |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| \\ &\quad - |C \cap D| + |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D| - |A \cap B \cap C \cap D| \\ &= 50 + 33 + 20 + 14 - 16 - 10 - 7 - 6 - 4 - 2 + 3 + 2 + 1 + 0 - 0 \\ &= 123 - 45 \\ &= 78 \end{aligned}$$

$$|A \cup B \cup C \cup D| = 78$$

ii) The number of integers between 1 to 100 that are divisible by 2, 3, 5 but not by 7.

$$\begin{aligned} &= |A \cap B \cap C| - |A \cap B \cap C \cap D| \\ &= 3 - 0 \\ &= 3. \end{aligned}$$

6. Use mathematical induction to prove that

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

Solution:-

$$\text{Let } P(n) = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

We have to show that, $P(n)$ is true.

Basis step:-

$$n=1 \Rightarrow P(1) = \frac{1}{1+1} = \frac{1}{2} = \frac{1}{1 \cdot 2} \Rightarrow \frac{1}{2} = \frac{1}{2}.$$

$\therefore P(1)$ is true.

Inductive step: Assume that $P(k)$ is true,

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \dots + \frac{1}{k(k-1)} = \frac{k}{k-1}$$

Under this assumption, we have to show that $P(k+1)$ is true

$$\therefore P(k+1) = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \dots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)}$$

$$= \frac{k}{k-1} + \frac{1}{(k+1)(k+2)}$$

$$= \frac{k(k+1)(k+2) + (k-1)}{(k-1)(k+1)(k+2)}$$

$$= \frac{(k-1)[k(k+2) + 1]}{(k-1)(k+1)(k+2)}$$

$$= \frac{k^2 + 2k + 1}{(k+1)(k+2)}$$

$$= \frac{(k+1)(k+1)}{(k+1)(k+2)}$$

$$P(k+1) = \frac{k+1}{k+2}$$

$\therefore P(k+1)$ is true under the assumption that $P(k)$ is true

Hence by mathematical induction, $P(n)$ is true.

7. Solve $a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$ with initial conditions $a_0 = 2$, $a_1 = 5$, $a_2 = 15$.

Solution:-

Given, $a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$, $a_0 = 2$, $a_1 = 5$, $a_2 = 15$.

$$\Rightarrow a_n - 6a_{n-1} + 11a_{n-2} - 6a_{n-3} = 0$$

$\therefore n - (n-3) = 3$, it is of order 3

The characteristic equation is,

$$r^3 - 6r^2 + 11r - 6 = 0$$

By synthetic division method,

$$\begin{array}{r|rrrr} 1 & 1 & -6 & 11 & -6 \\ & & 6 & 1 & -5 \\ \hline & 1 & 0 & 12 & -11 \end{array}$$

$$r-1=0 \quad ; \quad r^2-5r+6=0$$

$$\boxed{r=1} \quad ; \quad (r-2)(r-3)=0$$

$$r-2=0$$

$$\boxed{r=2}$$

$$r-3=0$$

$$\boxed{r=3}$$

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∴ The roots are $r=1, 2, 3$.

The general solution is, $a_n = A(1)^n + B(2)^n + C(3)^n$, $n \geq 0$

$$\Rightarrow a_n = A + B(2)^n + C(3)^n, \quad a_0=2, a_1=5, a_2=15$$

→ (1)

Put $n=0$ in equ. (1), we have

$$a_0 = A + B(2)^0 + C(3)^0$$

$$2 = A + B(1) + C(1)$$

[∵ $a_0=2$ by given]

$$\Rightarrow 2 = A + B + C \longrightarrow (2)$$

Put $n=1$ in equ. (1), we have

$$a_1 = A + B(2)^1 + C(3)^1$$

$$5 = A + B(2) + C(3)$$

[∵ $a_1=5$ by given]

$$\Rightarrow 5 = A + 2B + 3C \longrightarrow (3)$$

Put $n=2$ in equ. (1), we have

$$a_2 = A + B(2)^2 + C(3)^2$$

$$15 = A + B(4) + C(9)$$

[∵ $a_2=15$ by given]

$$\Rightarrow 15 = A + 4B + 9C \longrightarrow (4)$$

$$\text{equ. (3) - (2), } A + 2B + 3C = 5$$

$$\begin{array}{r} A + B + C = 2 \quad (2) \\ \underline{A + 2B + 3C = 5 \quad (3)} \\ \hline \end{array}$$

$$B + 2C = 3 \longrightarrow (5)$$

$$\text{equ. (4) - (3), } A + 4B + 9C = 15$$

$$\begin{array}{r} A + 2B + 3C = 5 \quad (3) \\ \underline{A + 4B + 9C = 15 \quad (4)} \\ \hline \end{array}$$

$$2B + 6C = 10 \longrightarrow (6)$$

$$\text{equ. (6) } \times 2 \Rightarrow$$

$$\begin{array}{r} 2B + 6C = 10 \quad (6) \\ \underline{2B + 4C = 6 \quad (6 \times 2)} \\ \hline \end{array}$$

$$2C = 4$$

$$C = \frac{4}{2} \Rightarrow \boxed{C=2}$$

Put $c=2$ in equ. (5), we have

$$B + 2(2) = 3$$

$$B + 4 = 3$$

$$B = 3 - 4$$

$$\boxed{B = -1}$$

Put $c=2, B=-1$ in equ. (2), we have

$$A + (-1) + (2) = 2$$

$$A - 1 + 2 = 2$$

$$A + 1 = 2$$

$$A = 2 - 1$$

$$\boxed{A = 1}$$

∴ The general solution, equ. (1) becomes,

$$a_n = 1 - (2)^n + 2(3)^n, \quad n \geq 0.$$

8. Using generating function, solve the recurrence relation $a_n = 3a_{n-1} + 2, \quad n \geq 1$ with initial conditions $a_0 = 1$.

Solution:-

Given, $a_n = 3a_{n-1} + 2, \quad a_0 = 1, \quad n \geq 1$.

$$\Rightarrow a_n - 3a_{n-1} - 2 = 0$$

Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$, where $G(x)$ is the generating function

for the sequence $\{a_n\}$

$$\therefore \sum_{n=1}^{\infty} a_n x^n - 3 \sum_{n=1}^{\infty} a_{n-1} x^n - 2 \sum_{n=1}^{\infty} x^n = 0$$

We know that, $G(x) = \sum_{n=0}^{\infty} a_n x^n$

$$= a_0 x^0 + a_1 x^1 + a_2 x^2 + \dots + a_n x^n + a_{n+1} x^{n+1} + \dots$$

$$\Rightarrow \sum_{n=1}^{\infty} a_n x^n - 3 \sum_{n=1}^{\infty} a_{n-1} x^n \left(\frac{x}{x}\right) - 2 \sum_{n=1}^{\infty} x^n = 0$$

$$\sum_{n=1}^{\infty} a_n x^n - 3x \sum_{n=1}^{\infty} a_{n-1} x^{n-1} - 2 \sum_{n=1}^{\infty} x^n = 0$$

$$G(x) - a_0 - 3x[G(x)] - 2\left[\frac{1}{1-x} - 1\right] = 0 \quad (14)$$

$$G(x) - a_0 - 3x[G(x)] - 2\left[\frac{1 - (1-x)}{1-x}\right] = 0$$

$$G(x)[1-3x] - a_0 - 2\left[\frac{x - \cancel{1} + 1}{1-x}\right] = 0$$

$$G(x)[1-3x] - a_0 - \left(\frac{2x}{1-x}\right) = 0$$

$$G(x)(1-3x) - 1 - \left(\frac{2x}{1-x}\right) = 0 \quad [\because a_0 = 1 \text{ by given}]$$

$$G(x)(1-3x) = \left[1 + \frac{2x}{1-x}\right]$$

$$G(x) = \left[\frac{1-x+2x}{1-x}\right] \cdot \left(\frac{1}{1-3x}\right)$$

$$G(x) = \frac{1+x}{(1-x)(1-3x)} \rightarrow (1)$$

Let us use Partial Fraction,

$$\frac{1+x}{(1-x)(1-3x)} = \frac{A}{(1-x)} + \frac{B}{(1-3x)} \rightarrow (2)$$

$$\frac{1+x}{\cancel{(1-x)}\cancel{(1-3x)}} = \frac{A\cancel{(1-3x)} + B\cancel{(1-x)}}{\cancel{(1-x)}\cancel{(1-3x)}}$$

$$\Rightarrow 1+x = A(1-3x) + B(1-x) \rightarrow (3)$$

Put $x=1$ in equ. (3), we have

$$1+1 = A(1-3(1)) + B(1-1)$$

$$2 = A(1-3) + B(0)$$

$$2 = A(-2) + 0$$

$$\Rightarrow x = -2A$$

$$\boxed{A = -1}$$

Put $x = \frac{1}{3}$ in equ. (3), we have

$$1 + \frac{1}{3} = A(1 - \cancel{1}(\frac{1}{3})) + B(1 - \frac{1}{3})$$

$$\frac{4}{3} = A(0) + B(\frac{2}{3})$$

$$\frac{4}{3} = B \cdot \frac{2}{3}$$

$$\Rightarrow B = \frac{4}{3} \cdot \frac{3}{2}$$

$$\boxed{B=2}$$

$\therefore$ equ. (2) becomes,

$$\frac{1+x}{(1-x)(1-3x)} = \frac{-1}{(1-x)} + \frac{2}{(1-3x)}$$

$$= -(1-x)^{-1} + 2(1-3x)^{-1}$$

$$= -1(x)^n + 2(3x)^n$$

$$= (-1)x^n + 2(3)^n \cdot x^n$$

$\therefore a_n$ is coefficient of x^n in $g(x)$.

$\therefore a_n = (-1) + 2(3)^n$ is the required solution.

9. Solve the recurrence relation $a_n - a_{n-1} = 2$, $n \geq 2$ and $a_1 = 2$ by iteration method.

Solution:-

Given, $a_n - a_{n-1} = 2$

$$\Rightarrow a_{n-1} - a_{n-2} = 2$$

$$a_{n-2} - a_{n-3} = 2$$

$\vdots$

$$a_3 - a_2 = 2$$

$$a_2 - a_1 = 2$$

Adding, we get $a_n - a_1 = 2 + 2 + \dots + 2(n-1)$ terms.

$$a_n - a_1 = 2(n-1)$$

$$a_n - \cancel{2} = 2n - \cancel{2}$$

[$\because a_1 = 2$ by given]

$$\boxed{a_n = 2n}$$

This is the implicit form of a_n , we prove by induction, (15)

Let $P(n)$ denote $a_n = 2^n$, we have to prove $P(n)$ is true for $n \geq 1$.

Base step:- Here $n=1$,

$\therefore P(1)$ is $a_1 = 2$, which is true by data.

Inductive step:- Assume $P(k)$ is true,

$a_k = 2^k$ is true.

To prove $P(k+1)$ is true,

Now from the recurrence relation,

$$a_{k+1} - a_k = 2$$

$$a_{k+1} = a_k + 2$$

$$= 2^k + 2$$

$$[\because a_k = 2^k]$$

$$a_{k+1} = 2(k+1)$$

$\therefore P(k+1)$ is true.

Thus, $P(k)$ is true $\Rightarrow P(k+1)$ is true

Hence, by induction $P(n)$ is true $\forall n \geq 1$.

$$\Rightarrow a_n = 2^n \quad \forall n \geq 1.$$

10. Using the generating function, solve the difference equation,

$$y_{n+2} - y_{n+1} - 6y_n = 0, \quad y_1 = 1, y_0 = 2.$$

Solution:-

Let $G(y, z)$ be the generating function of the sequence $\{y_n\}$.

$$\text{Then, } G(y, z) = \sum_{n=0}^{\infty} y_n z^n$$

$$\text{As, } y_{n+2} - y_{n+1} - 6y_n = 0, \quad n \geq 0$$

Multiply by z^n and taking summation on both sides, we have

$$\sum_{n=0}^{\infty} [y_{n+2} - y_{n+1} - 6y_n] z^n = 0$$

$$\sum_{n=0}^{\infty} y_{n+2} z^n - \sum_{n=0}^{\infty} y_{n+1} z^n - 6 \sum_{n=0}^{\infty} y_n z^n = 0$$

$$\frac{1}{z^2} \sum_{n=0}^{\infty} y_{n+2} z^{n+2} - \frac{1}{z} \sum_{n=0}^{\infty} y_{n+1} z^{n+1} - 6 G(y, z) = 0$$

$$\frac{1}{z^2} [G(y, z) - y_0 z^0 - y_1 z^1] - \frac{1}{z} [G(y, z) - y_0 z^0] - 6 G(y, z) = 0$$

$$h(y, z) - 2 - z - 2 \cdot h(y, z) + 2z - 6z^2 \cdot h(y, z) = 0$$

$$h(y, z) - 2 \cdot h(y, z) - 6z^2 \cdot h(y, z) - 2 + z = 0$$

$$h(y, z) [1 - 2 - 6z^2] - 2 + z = 0$$

$$h(y, z) [1 - 2 - 6z^2] = 2 - z$$

$$h(y, z) = \frac{2 - z}{(1 - 2 - 6z^2)}$$

To solve the difference equation resolve $h(y, z)$ into Partial fractions.

$$1 - 2 - 6z^2 = 1 - 3z + 2z - 6z^2$$

$$= (1 - 3z) + 2z(1 - 3z)$$

$$1 - 2 - 6z^2 = (1 - 3z)(1 + 2z)$$

$$\therefore h(y, z) = \frac{2 - z}{(1 - 3z)(1 + 2z)}$$

$$\text{Let, } \frac{2 - z}{(1 - 3z)(1 + 2z)} = \frac{A}{(1 - 3z)} + \frac{B}{(1 + 2z)}$$

$$\frac{2 - z}{(1 - 3z)(1 + 2z)} = \frac{A(1 + 2z) + B(1 - 3z)}{(1 - 3z)(1 + 2z)}$$

$$\therefore 2 - z = A(1 + 2z) + B(1 - 3z)$$

Put, $z = -\frac{1}{2}$, we have

$$2 - (-\frac{1}{2}) = A(1 + 2(-\frac{1}{2})) + B(1 - 3(-\frac{1}{2}))$$

$$2 + \frac{1}{2} = A(1 - 1) + B(1 + \frac{3}{2})$$

$$\frac{5}{2} = A(0) + B(\frac{5}{2})$$

$$\frac{5}{2} = B(\frac{5}{2})$$

$$\boxed{B = 1}$$

Put, $z = \frac{1}{3}$, we have

$$2 - (\frac{1}{3}) = A(1 + 2(\frac{1}{3})) + B(1 - 3(\frac{1}{3}))$$

$$\frac{5}{3} = A(\frac{5}{3}) + B(0)$$

$$\frac{5}{3} = A\left(\frac{5}{3}\right)$$

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$$\therefore [A = 1]$$

$$G(y, z) = \frac{1}{(1-3z)} + \frac{1}{(1+2z)}$$

y_n is coefficient of z^n ,

$$= (1-3z)^{-1} + (1+2z)^{-1}$$

$$= (1 + 3z + (3z)^2 + \dots + 3^n z^n) + (1 - (2z) + (2z)^2 - \dots - 2^n z^n)$$

$$\therefore y_n = 3^n + (-1)^n 2^n$$

which is the required solution.

Note:

Generating function, (G.G.F.).

General term of the sequence (a_k)

$$(1) (1+x)^n = \sum_{k=0}^n C(n, k) x^k$$

$$= 1 + C(n, 1)x + C(n, 2)x^2 + \dots + x^n$$

$$C(n, k)$$

$$(2) (1+ax)^n = \sum_{k=0}^n C(n, k) a^k x^k$$

$$= 1 + C(n, 1)ax + C(n, 2)a^2 x^2 + \dots + a^n x^n$$

$$C(n, k) a^k$$

$$(3) (1+x^r)^n = \sum_{k=0}^n C(n, k) x^{rk}$$

$$= 1 + C(n, 1)x^r + C(n, 2)x^{2r} + \dots + x^{rn}$$

$$C(n, k/r) : \text{if } r|k$$

$$0 : \text{otherwise}$$

$$(4) \frac{1-x^{n+1}}{1-x} = \sum_{k=0}^n x^k$$

$$= 1 + x + x^2 + \dots + x^n$$

$$1, \text{ if } k \leq n$$

$$0, \text{ otherwise}$$

$$(5) \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$$

$$= 1 + x + x^2 + \dots$$

$$1$$

$$(6) \frac{1}{1-ax} = \sum_{k=0}^{\infty} a^k x^k$$

$$= 1 + ax + a^2 x^2 + \dots$$

$$a^k$$

$$(7) \frac{1}{1-x^r} = \sum_{k=0}^{\infty} x^{rk} \\ = 1 + x^r + x^{2r} + \dots$$

1, if r/k
0, otherwise

$$(8) \frac{1}{(1-x)^2} = \sum_{k=0}^{\infty} (k+1) x^k \\ = 1 + 2x + 3x^2 + 4x^3 + \dots$$

$k+1$

$$(9) \frac{1}{(1-x)^n} = \sum_{k=0}^{\infty} C(n+k-1, k) x^k \\ = 1 + C(n, 1)x + C(n+1, 2)x^2 + \dots$$

$$C(n+k-1, k) = C(n+k-1, n-1)$$

$$(10) \frac{1}{(1+x)^n} = \sum_{k=0}^{\infty} C(n+k-1, k) (-1)^k x^k \\ = 1 - C(n, 1)x + C(n+1, 2)x^2 - \dots$$

$$(-1)^k C(n+k-1, k) \\ = (-1)^k C(n+k-1, n-1)$$

$$(11) \frac{1}{(1-ax)^n} = \sum_{k=0}^{\infty} C(n+k-1, k) a^k x^k \\ = 1 + C(n, 1)ax + C(n+1, 2)a^2x^2 + \dots$$

$$C(n+k-1, a^k) a^k \\ = C(n+k-1, n-1) a^k$$

$$(12) e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \\ = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\left(\frac{1}{k}\right)!$$

$$(13) \log(1+x) = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{k} x^k \\ = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

$$(-1)^{k+1} / k$$

Part-A:-1. Define Graph:-

A graph G is a pair of sets (V, E) , where, V is a set of vertices and E is a set of edges (undirected or directed edges).

→ Vertices (or) nodes are represented as point (or) small circles.

→ Edges are represented as line segments (or) curve joining of its end vertices.

2. Define order and size:-Order:-

If G is finite, then $|V(G)|$ denotes the number of vertices in G , and is called the order of G .

Size:-

If G is finite, then $|E(G)|$ denotes the number of edges in G , and is called size of G .

3. Define Path, Simple Path and Elementary path.Path:-

A path is a sequence of vertices of a graph in which each successive pair of vertices is an edge of the graph.

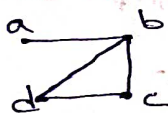
→ In graph G , abc is a path because $\{ \{a, b\}, \{b, c\} \}$ are the sequence of edges of G . In this $|V(G)| = 3$, $|E(G)| = 2$.

→ In path, vertices and edges may be repeated.

Simple path:-

A simple path is one in which no edges is repeated in the path.

Example:-

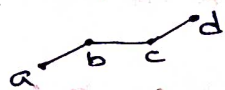


, simple path = $\{ \{a, b\}, \{b, c\}, \{c, d\}, \{d, b\} \}$

Elementary Path:-

Elementary path is one which no vertex is repeated.

Example:-



, Elementary path = $\{ \{a, b\}, \{b, c\}, \{c, d\} \}$.

4. Find the total number of edges of a complete graph with 40 vertices.

Solution:-

$$nC_2 = 40C_2 = \frac{40 \times 39}{2} = 780.$$

5. Definitions:-

Circuit:- A circuit is a path which starts and ends at the same vertex.

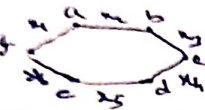
example $\rightarrow$  , Path = $\{\{a,b\}, \{b,c\}, \{c,d\}, \{d,a\}\}$

Null-graph:- A graph in which number of edges is zero is called as null graph.

example $\rightarrow$  , Null graph with four vertices and zero edges.

Simple Graph:-

A graph which contains neither loop nor parallel edges is called a simple graph. And a graph which does not contain loops is called a loop-free graph.

example $\rightarrow$ 

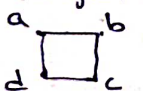
Multi-Graph:-

If more than one edge is allowed to join a pair of vertices, it is called multi-graph.

example $\rightarrow$ 

Cycle Graph:-

A cycle is a circuit in which initial vertex occurs twice and no other vertex is repeated. It is denoted by C_n . Cycle is a simple circuit. A loop is a cycle of length 1.

example $\Rightarrow$  , Path = $\{\{a,b\}, \{b,c\}, \{c,d\}, \{d,a\}\}$

$\rightarrow$ Note $\Rightarrow$ Smallest cycle is the triangle.  , Path = $\{\{a,b\}, \{b,c\}, \{c,a\}\}$

Complete Graph:-

A simple graph in which there is exactly one edge between each pair of distinct vertices, is called a complete graph denoted by K_n .

6. Define Directed and Non-directed graph:-

Directed Graph:-

A graph $G = (V, E)$ is said to be directed graph (digraph) if the elements of E are ordered pairs of vertices, i.e., edges (u, v) means edges is only from u to v . $[(a, b) \Rightarrow a \rightarrow b \text{ but not } b \rightarrow a]$

Non-directed Graph:-

A graph G is said to be non-directed graph if the elements of E are unordered pairs (sets) of vertices. $[(a \rightarrow b) \text{ or } (b \rightarrow a)]$

7. Define Isomorphism of graphs.

Let $G_1 = (V(G_1), E(G_1))$ and $G_2 = (V(G_2), E(G_2))$ be two graphs. A function $f: G_1 \rightarrow G_2$ is called an isomorphism, if

(i) f is one to one,

(ii) f is onto.

(iii) $(x, y) \in E(G_1)$ if and only if $f(x), f(y) \in E(G_2)$.

i.e., two vertices x and y are adjacent in G_1 if $f(x)$ and $f(y)$ are adjacent in G_2 . If the graph G_1 is isomorphic to G_2 , then we write $G_1 \cong G_2$.

8. Define Adjacency matrix.

Let $G(V, E)$ be a simple graph with n vertices ordered from V_1 to V_n . then the adjacency matrix $A = [a_{ij}]_{n \times n}$ of G is an $n \times n$ symmetric matrix defined by the elements.

$$a_{ij} = \begin{cases} 1, & \text{when } V_i \text{ is adjacent to } V_j \\ 0, & \text{otherwise} \end{cases}$$

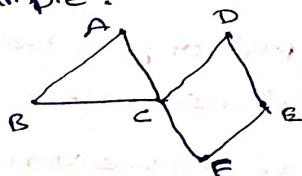
It is denoted by $A(G)$ or A_G .

9. Define Connected graph with an example.

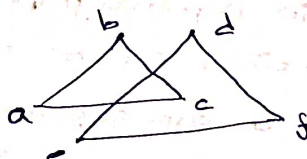
A graph G is connected if there is a path between any two of its vertices, otherwise it is disconnected.

An undirected graph is connected if there is a path between every pair of distinct vertices of the graph.

example:-

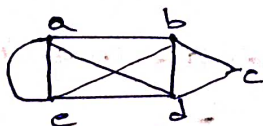


Connected graph.



Disconnected graph.

10. Consider the following graph, Find the Euler's circuit.



Solution:-

The Euler's circuit is,

$$\{ (a, b), (b, c), (c, d), (d, e), (e, a), (a, d), (d, b), (b, e), (e, c) \}.$$

11. Define Hamilton Path.

A path $v_0, v_1, v_2, \dots, v_{n-1}, v_n$ in the graph $G=(V, E)$ is called a Hamilton path if $V = \{v_0, v_1, v_2, \dots, v_{n-1}, v_n\}$ and $x_i \neq x_j$ for $0 \leq i < j \leq n$.
i.e., a Hamilton path is defined as a simple path that contains all vertices of G but the end points may be different.

12. Define Hamilton Circuit and Hamilton graph.

Hamilton circuit :-

A cycle in a graph G that contains each vertex in G exactly once, except for the starting and ending vertex that appears twice is known as Hamiltonian cycle or Hamiltonian circuit.

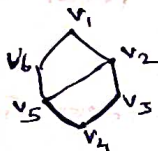
Hamilton Graph :-

A Hamiltonian graph is a graph containing a Hamiltonian cycle. A Hamiltonian path is a simple path that contains all vertices of G but where the end points may be distinct.

13. Find a graph which is Hamilton but not Eulerian?

Solution:-

This has a Hamiltonian cycle $\{v_1, v_2, v_3, v_4, v_5, v_6, v_1\}$, But it is not Eulerian because the vertices v_5, v_2 have odd degrees.



14. Define Euler's Path:-

An Euler's path in a multigraph is a path that travels all edges exactly once and vertices can be travelled at least once.

In any multigraph, we can get an Euler's Path only if there are two vertices of odd degree and remaining vertices degree is even.

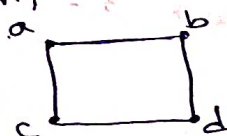
Note - Euler's path will always start at odd degree vertex and end at other odd degree vertex.

Example $\rightarrow$

Euler's Path is multigraph $\{(a, b), (b, c), (c, d), (d, e), (e, a), (b, d), (c, e)\}$

15. Draw a graph with the adjacency matrix

Solution:-



	a	b	c	d
a	0	1	1	0
b	1	0	0	1
c	1	0	0	1
d	0	1	1	0

Part-B:-

(19)

1. Show that the maximum number of edges in a simple graph with 'n' vertices is $\frac{n(n-1)}{2}$.

Proof:-

Given a simple graph G in which $|V| = n$.

We know that in a simple graph of n vertices, the maximum degree of any vertex is $(n-1)$.

We know that, $\sum \deg(V_i) = 2|E| \rightarrow (1)$.

But, $\max \{ \sum \deg(V_i) \} = (n-1) + (n-1) + \dots + n \text{ times}.$
 $= (n-1)n$

$\therefore \max \{ \sum \deg(V_i) \} = n(n-1) \rightarrow (2)$

From equ. (1) and (2), we have,

$$\sum \deg(V_i) \leq \max \{ \sum \deg(V_i) \}$$

$$\Rightarrow 2|E| \leq n(n-1)$$

$$\Rightarrow |E| \leq \frac{n(n-1)}{2}$$

Hence proved.

(or)

2. Show that the maximum number of edges in a simple graph with 'n' vertices is $\frac{n(n-1)}{2}$.

Proof:-

We prove this theorem by Mathematical Induction for $n=1$, a graph with one vertex has no edges.

$\therefore$ The result is true for $n=1$.

For $n=2$, a graph with two vertices may have at most one edge.

$$\therefore \frac{2(2-1)}{2} = 1.$$

$\therefore$ the result is true for $n=2$.

Assume now, the result is true for $n=k$.

∴ a graph with ' k ' vertices has at most $\frac{k(k-1)}{2}$ edges.

when $k=k+1$,

Let G be a graph having ' n ' vertices and G' be a graph obtained from G by deleting one vertex.

∴ $v \in V(G)$

∴ G' has k vertices, then by the hypothesis G' has at most $\frac{k(k-1)}{2}$ edges.

Now add the vertex v to G , ∴ v may be adjacent to all the k vertices of G' .

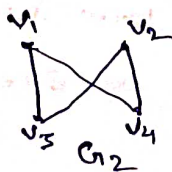
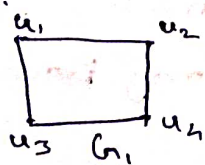
∴ The total number of edges in G are

$$\begin{aligned} \frac{k(k-1)}{2} + k &= \frac{k^2 - k}{2} + k \\ &= \frac{k^2 - k + 2k}{2} \\ &= \frac{k^2 + k}{2} \\ &= \frac{k(k+1)}{2} \\ &= \frac{(k+1)(k+1-1)}{2} \end{aligned}$$

∴ The result is true for $n=k+1$.

Hence the maximum number of edges in a simple graph with ' n ' vertices is $\frac{n(n-1)}{2}$.

3. Determine whether the following pair of graphs G_1 and G_2 are isomorphic.



Solution :-

$$G_1 : |V| = 4$$

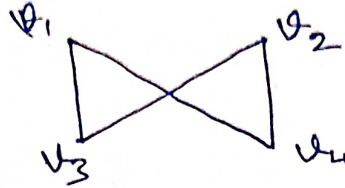
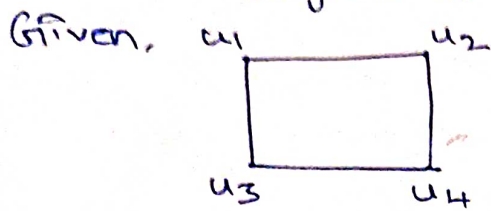
$$G_2 : |V| = 4$$

$$G_1 : |E| = 4$$

$$G_2 : |E| = 5$$

Degree of Sequence (2,2,2,2)

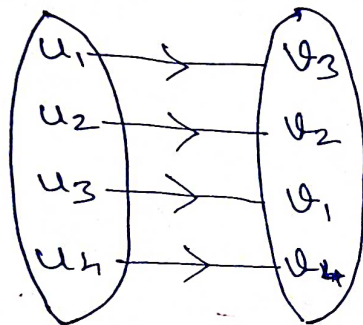
Degree of Sequence (2,2,2,2)



$$V(G_1) = \{u_1, u_2, u_3, u_4\}$$

$$V(G_2) = \{v_1, v_2, v_3, v_4\}$$

Let $f: V(G_1) \rightarrow V(G_2)$ is defined by
 $f(u_1) = v_3, f(u_2) = v_2, f(u_3) = v_1, f(u_4) = v_4$.



Representation of $f: V(G_1) \rightarrow V(G_2)$

clearly, f is one to one and onto.

Let A_1 be the adjacency matrix for the vertex ordering u_1, u_2, u_3, u_4 of the vertices of G_1 .

$$A_1 = \begin{matrix} & \begin{matrix} u_1 & u_2 & u_3 & u_4 \end{matrix} \\ \begin{matrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{matrix} & \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{matrix}$$

Let A_2 be the adjacency matrix for the vertex ordering v_1, v_2, v_3, v_4 of the vertices of G_2 .

$$A_2 = \begin{matrix} & \begin{matrix} v_3 & v_2 & v_1 & v_4 \end{matrix} \\ \begin{matrix} v_3 \\ v_2 \\ v_1 \\ v_4 \end{matrix} & \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{matrix}$$

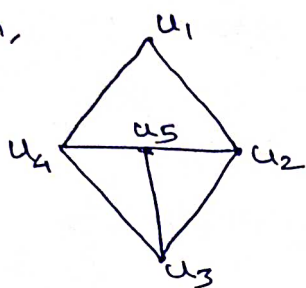
The adjacency matrices A_1 and A_2 are ~~non~~ identical. So, f is ~~non~~ isomorphism.

Hence G_1 and G_2 are isomorphic.

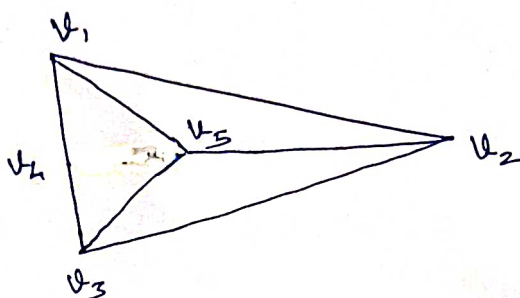
4. Establish the isomorphism of the following pairs of graphs by considering their adjacency matrices.

Solution:-

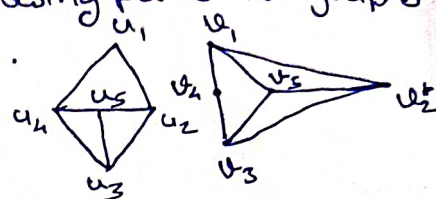
Given,



G_1



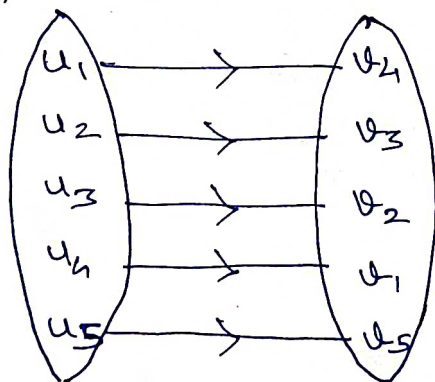
G_2



$$V(G_1) = \{u_1, u_2, u_3, u_4, u_5\}$$

$$V(G_2) = \{v_1, v_2, v_3, v_4, v_5\}$$

Let, $f: V(G_1) \rightarrow V(G_2)$ be defined by, $f(u_1) = v_4$, $f(u_2) = v_3$, $f(u_3) = v_2$, $f(u_4) = v_1$, $f(u_5) = v_5$.



Representation of $f: V(G_1) \rightarrow V(G_2)$, clearly f is one-to-one and onto.

Let A_1 be the adjacency matrices for the vertex ordering u_1, u_2, u_3, u_4, u_5 of the vertices of G_1 .

$$A_1 = \begin{matrix} & \begin{matrix} u_1 & u_2 & u_3 & u_4 & u_5 \end{matrix} \\ \begin{matrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} \end{matrix}$$

Let A_2 be the adjacency matrix for the vertex ordering (2)
 $u_1', u_2', u_3', u_4', u_5'$ of the vertices of G_2

$$A_2 = \begin{matrix} & u_4' & u_3' & u_2' & u_1' & u_5' \\ \begin{matrix} u_4' \\ u_3' \\ u_2' \\ u_1' \\ u_5' \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} \end{matrix}$$

$\therefore$ The adjacency matrices A_1 and A_2 are ~~not~~ identical.
 So, f is ~~not~~ an isomorphism. Hence G_1 and G_2 are ~~not~~
 isomorphic.

5. Theorem :-

A Connected graph G is Eulerian if and only if every
 vertex of G is of even degree.

Proof:-

Let G be an Eulerian graph. we have to prove all vertices
 are of even degree.

Since G is Eulerian, G contains an Euler circuit

$$v_0 e_1 v_1 e_2 \dots v_{n-1} e_n v_0$$

Both the edges e_1 and e_n contribute one to the degree
 of v_0 . So $\deg(v_0)$ is atleast two.

In tracing this circuit, we find an edge enters a vertex
 and another edge leaves the vertex contributing 2 to the
 degree of the vertex.

This is true for all vertices and so each vertex is of degree
 2, an even integer.

Conversely, let the graph G be such that all its vertices
 are of even degrees.

we have to prove G is an Euler graph.

We shall construct an Euler circuit and prove.
 Let V be an arbitrary vertex in G .

Beginning with V form a circuit $C: V, V_1, V_2, \dots, V_{n-1}, V$.

This is possible because every vertex of even degree. We can leave a vertex along an edge not used to enter it. This tracing clearly stops only at the vertex V , because V is also of even degree and it is started from V . Thus we get a cycle or circuit C .

If C includes all the edges of G , then C is an Euler circuit and so G is Eulerian.

If C does not contain all the edges of G , consider the subgraph H of G obtained by deleting all the edges of C from G and vertices not incident with the remaining edges. Note that, all the vertices of H have even degree. Since G is connected, H is C must have a common vertex u . Beginning with u , construct a circuit C_1 for H .

Now combine C and C_1 to form a larger circuit C_2 . If it is Eulerian, i.e., if it contains all the edges of G , then G is Eulerian.

Else continue this process until we get an Eulerian circuit.

Since G is finite this procedure must come to an end with an Eulerian circuit.

Hence G is Eulerian.

6. Check the given graph is Euler or not.

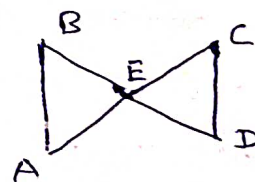
Solution:-

Consider the cycle,

$A \rightarrow E \rightarrow C \rightarrow D \rightarrow E \rightarrow B \rightarrow A$

Since it includes each of the edges exactly once, the above cycle is an Eulerian ~~graph~~ cycle.

∴ The graph contains Eulerian cycle, it is a Euler graph

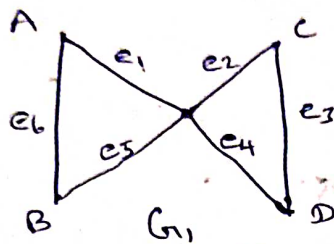


7. Give an example of a graph which is,
- (i) Eulerian but not Hamiltonian Graph.
 - (ii) Hamiltonian but not Eulerian.
 - (iii) Both Eulerian and Hamiltonian.
 - (iv) Non-Eulerian and Non-Hamiltonian.

(22)

Solution:

(i) Eulerian Graph but not a Hamiltonian Graph



G_1 contains Eulerian cycle, $A \xrightarrow{e_1} B \xrightarrow{e_2} C \xrightarrow{e_3} D \xrightarrow{e_4} B \xrightarrow{e_6} A$
 Since all the edges occupy exactly once.

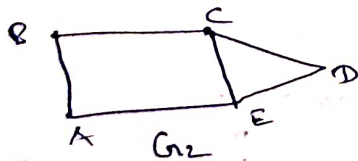
$\therefore G_1$ is an Eulerian Graph.

We cannot find Hamiltonian cycle as the vertex be repeated twice.

$\therefore G_1$ has not a Hamiltonian Graph.

$\therefore G_1$ is Eulerian but not Hamiltonian.

(ii) Hamiltonian Graph but not ~~Hamiltonian~~ Eulerian.



Since, G_2 contains the Hamilton cycle.

$A - B - C - D - E - A$.

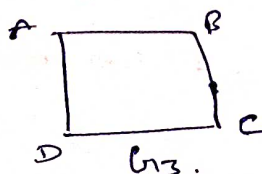
All the vertices occurs exactly once.

$\therefore G_2$ is a Hamilton Graph.

Since the degree of the vertices are not even

$\Rightarrow G_2$ is not an Eulerian Graph.

(iii) Both Eulerian and Hamiltonian?

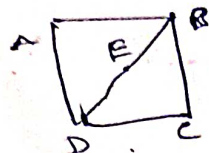


In G_3 , $A-B-C-D-A$

$\therefore$ the cycle contains all the edges, G_3 is Eulerian.

Since, the cycle contains all the vertices exactly once G_3 is Hamiltonian.

8. (iv) Neither Hamiltonian or Eulerian



In G_4 : $\deg(5) \Rightarrow \deg(5)^4$

Since degree of B and D are not even. G_4 is not a Hamiltonian

$\therefore G_4$ is not an Eulerian group.

No cycle with all the vertices at exactly once, G_4 is not a Hamilton Graph.

$\therefore G_4$ is neither Euler nor ~~Hamiltonian~~ Hamiltonian Graph.

8 Prove that the number of odd degree vertices is always even.

Solution:

Let $G = (V, E)$ be a graph with n vertices and e edges.

Let $v_1, v_2, \dots, v_k$ be the vertices of odd degree and $v'_1, v'_2, \dots, v'_m$ be the vertices of even degree.

Now to prove k is an even, we know that

$$\sum_{i=1}^n d(v_i) = 2|E| = 2e \Rightarrow \sum_{i=1}^k d(v_i) + \sum_{j=1}^m d(v'_j) = 2e$$

Each of $d(v'_j)$ is even $\Rightarrow \sum_{j=1}^m d(v'_j)$ and $2e$ are even numbers

$\therefore \sum_{i=1}^k d(v_i) + \text{an even number} = \text{an even number}$

$\sum_{i=1}^k d(v_i) = \text{an even number}$, $[\because \text{each term } d(v_i) \text{ is odd}]$

Hence, the number of terms in Left hand side must be even

$\therefore k$ is even.

9. Prove that there is a simple path between every pair of distinct vertices of a connected undirected graph.

Proof:

Let u and v be two distinct vertices of the connected undirected graph $G = (V, E)$.

Because G is connected, there is at least one path between u and v .

Let $x_0, x_1, x_2, \dots, x_n$ where $x_0 = u$ and $x_n = v$ be the vertex sequence of a path of least length.

This path of least is simple.

Suppose it is not simple, then $x_i = x_j$ for some i and j with $0 \leq i < j$.

This means that there is a path from u to v of shorter length with vertex sequence $x_0, x_1, x_2, \dots, x_{i-1}, x_j, x_{j+1}, \dots, x_n$ obtained by deleting the edges corresponding to the vertex sequence $x_i, \dots, x_{j-1}$.

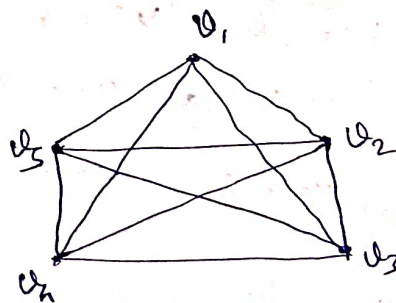
Hence the proof.

10. If the graph G has no loops or parallel edges, if $|V(G)| = n \geq 3$, and if $\deg(v) \geq n/2$ for each vertex v of G , then prove that G is Hamiltonian.

Solution:

Let us consider a graph.

This graph has no longer or parallel edges. Thus $\deg(v) = 4$ for each $v \in V$.



As, $\deg(v_1) = 4, \deg(v_2) = 4, \deg(v_3) = 4, \deg(v_4) = 4, \deg(v_5) = 4$.

As $|V(G)| = 5$.

Hence, it satisfies the statement.

As, $|V(G)| = 5$ and has a vertex of degree 2.

Hence, the given graph has a Hamiltonian circuit $v_1 v_2 v_3 v_4 v_5$.

11. An undirected graph has an even number of vertices of odd degree.

Proof:-

Let V_1 and V_2 be the set of vertices of even degree and the set of vertices of odd degree respectively, in an undirected graph $G=(V,E)$.

Then,

$$2e = \sum_{u \in V} \deg(u) = \sum_{u \in V_1} \deg(u) + \sum_{u \in V_2} \deg(u)$$

Because, $\deg(u)$ is even for $u \in V_1$, the first term on the right hand side is even. The sum of the two terms on the right hand side of the equality is even, because this sum is $2e$. Hence, the second term in the sum is also even.

Because all the terms in this sum are odd, there must be an even number of such terms.

Thus, there are an even number of vertices of odd degree.

12. The Handshaking Theorem:-

Let $G=(V,E)$ be an undirected graph with e edges then $\sum_{v \in V} \deg(v) = 2e$. The sum of degrees of the vertices of an undirected graph is twice the no. of edges of the graph and hence it is even.

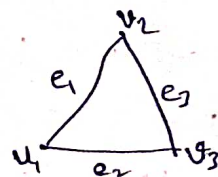
Proof:- Since each edge is incident with exactly two vertices, every edge contributes 2 to the sum of the degree of the vertices.

$\therefore$ All the e edges contribute $(2e)$ to the sum of the degree of vertices.

$$\therefore \sum \deg(v) = 2e.$$

For example:-

$$\Rightarrow \sum_{i=1}^3 \deg(v_i) = \deg(v_1) + \deg(v_2) + \deg(v_3) = 2 + 2 + 2 = 6 //$$



13. The maximum number of edges in a simple disconnected graph G with n vertices and k components is $\frac{(n-k)(n-k+1)}{2}$. [III+1]

Proof:-

Let the number of vertices in the i^{th} component of G be n_i ($n_i \geq 1$).

Then $n_1 + n_2 + \dots + n_k = n$ (or) $\sum_{i=1}^k n_i = n$

Hence, $\sum_{i=1}^k (n_i - 1) = n - k$

$$\therefore \left\{ \sum_{i=1}^k (n_i - 1) \right\}^2 = (n^2 - 2nk + k^2)$$

$$\Rightarrow \sum_{i=1}^k (n_i - 1)^2 + 2 \sum_{i=1}^k (n_i - 1) = n^2 - 2nk + k^2 \longrightarrow (1)$$

$$\Rightarrow \sum_{i=1}^k (n_i - 1)^2 \leq n^2 - 2nk + k^2 \quad [\because \text{equ. (1)} \geq 0, \text{ as each } n_i > 1]$$

$$\Rightarrow \sum_{i=1}^k (n_i^2 - 2n_i + 1) \leq n^2 - 2nk + k^2$$

$$\Rightarrow \sum_{i=1}^k n_i^2 \leq n^2 - 2nk + k^2 + 2n - k \longrightarrow (2)$$

Now the maximum number of edges in the i^{th} component of G is $\frac{1}{2} n_i (n_i - 1)$

$\therefore$ maximum number of edges of G .

$$= \frac{1}{2} \sum_{i=1}^k n_i (n_i - 1)$$

$$= \frac{1}{2} \sum_{i=1}^k n_i^2 - \frac{1}{2} n, \quad [\because$$

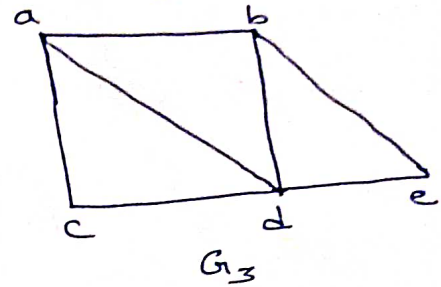
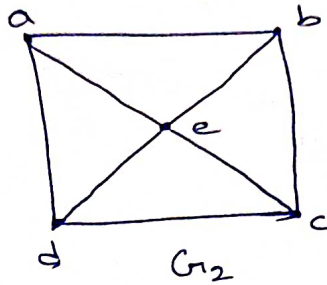
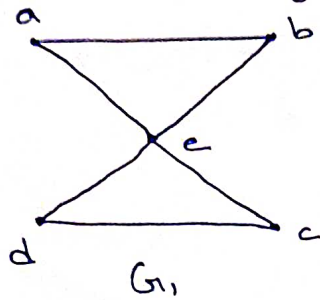
$$\leq \frac{1}{2} (n^2 - 2nk + k^2 + 2n - k) - \frac{1}{2} n, \quad [\because \text{by (2)}]$$

$$\leq \frac{1}{2} (n^2 - 2nk + k^2 - n - k)$$

$$\leq \frac{1}{2} [(n-k)^2 + (n-k)]$$

$$\leq \frac{1}{2} (n-k) (n-k+1).$$

14. Which of the graphs in Figures has an Euler circuit?



Solution:-

The Graph G_1 has an Euler circuit,
for example, a, e, c, d, e, b, a .

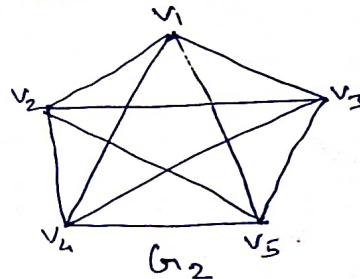
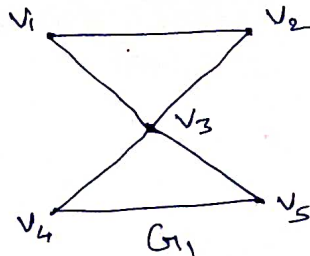
The Graph G_2 does not have an Euler circuit.

The Graph G_3 does not have an Euler circuit.

However, G_3 has an Euler path, namely a, c, d, e, b, d, a, b .

G_2 does not have an Euler path.

15. Let G be a graph, verify that G has an Eulerian circuit.



Solution:-

Here G_1 is connected and all the vertices are having even degree

$$\deg(v_1) = \deg(v_2) = \deg(v_3) = \deg(v_4) = \deg(v_5) = 2.$$

Thus G_1 has an Eulerian circuit, we find the Eulerian circuit.

$$v_1 - v_2 - v_3 - v_4 - v_5 - v_3 - v_1$$

The G_2 is also a connected graph and all the vertex are having even degree.

$$\deg(v_1) = \deg(v_2) = \deg(v_3) = \deg(v_4) = \deg(v_5) = 4.$$

Part-A:-

1. Define Binary and n-ary operations.

A mapping $f: X^n \rightarrow X$ is called n-ary operation and n is called the order of the operation.

→ For $n=1$, $f: X \rightarrow X$ is called unary operation.

→ For $n=2$, $f: X^2 \rightarrow X$ is called a binary operation on X .

→ Any element of X such as an identity element or a zero element with respect to a binary operation is considered as a 0-ary operation.

2. Algebraic System:-

A system consisting of a set and one or more n-ary operations on the set is called as algebraic system or simply an algebra.

→ The algebraic system is represented by $\langle S, f_1, f_2, f_3, \dots \rangle$ where S is a non-empty set and $f_1, f_2, f_3, \dots$ are operations on S .

3. Algebraic structure:-

The operations and relations on the set S define a structure on the element of S , an algebraic system is called an algebraic structure.

4. Operations of Addition:-

(i) For any $a, b, c \in I$, then $(a+b)+c = a+(b+c)$.
[Associative].

(ii) For any $a, b \in I$, then $a+b = b+a$.
[Commutative].

(iii) There exists an element $0 \in I$ such that for any $a \in I$, $a+0 = 0+a = a$. [Identity element].
Here $0 \in I$ is the identity element with respect to addition.

(iv) For each $a \in I$, there exists an element $-a \in I$, called the negative of " a " such that $a+(-a) = 0$.
[Inverse element].

⑤ Operations of Multiplication:

- (i) For any $a, b, c \in I$, $(a \times b) \times c = a \times (b \times c)$. [Associative]
- (ii) For any $a, b \in I$, then $a \times b = b \times a$ [Commutativity]
- (iii) There exists an element $1 \in I$, such that for any $a \in I$,
 $a \times 1 = 1 \times a = a$. [Identity element].
- (iv) For any $a, b, c \in I$, such that
 $a \times (b + c) = (a \times b) + (a \times c)$. [Distributivity]
- (v) The operations ' $\times$ ' distributes over '+' and '•'.
For any $a, b, c \in I$ and $a \neq 0$, then
 $a \times b = a \times c \Rightarrow b = c$ [Left cancellation]
- (vi) For any $a, b, c \in I$ and
 $b \times a = c \times a \Rightarrow b = c$ [Right cancellation].

The algebraic system $\langle I, +, \times \rangle$ expressed as $\langle I, +, \times, 0, 1 \rangle$.

6. Define Homomorphism :-

Let $\langle X, \circ \rangle$ and $\langle Y, * \rangle$ be two algebraic systems, where ' $\circ$ ' and ' $*$ ' are binary operations.

A mapping $g: X \rightarrow Y$ is called a homomorphism or morphism, from $\langle X, \circ \rangle$ to $\langle Y, * \rangle$ if for any $x_1, x_2 \in X$ then
 $g(x_1 \circ x_2) = g(x_1) * g(x_2)$.

7. Epimorphism, Monomorphism and Isomorphism :-

Let g be a homomorphism from $\langle X, \circ \rangle$ and $\langle Y, * \rangle$.

→ If $g: X \rightarrow Y$ is onto, then g is called ~~Epimorphism~~ Epimorphism.

→ If $g: X \rightarrow Y$ is one-to-one, then g is called Monomorphism.

→ If $g: X \rightarrow Y$ is one-to-one and onto then g is called an Isomorphism.

8. Isomorphic :-

Let $\langle X, \circ \rangle$ and $\langle Y, * \rangle$ be two algebraic systems. If there exists an isomorphic mapping $g: X \rightarrow Y$, then $\langle X, \circ \rangle$ and $\langle Y, * \rangle$ are said to be isomorphic.

9. Let $\langle X, \cdot \rangle$ and $\langle Y, * \rangle$ be two algebraic systems such that $Y \subseteq X$. (25)

→ A homomorphism $g: \langle X, \cdot \rangle \rightarrow \langle Y, * \rangle$ is called an endomorphism, if $Y \subseteq X$.

→ An isomorphism $g: \langle X, \cdot \rangle \rightarrow \langle Y, * \rangle$ is called an Automorphism, if $Y = X$.

10. ~~Equivalence~~ Equivalence relation:-

A relation R in a set X is called an equivalence relation, if it is reflexive, Symmetric and transitive.

11. Semigroup:-

Let S be a non-empty set and $*$ be a binary operation on S . The algebraic system $\langle S, * \rangle$ is called a Semigroup if the binary operation $*$ satisfies the following conditions.

(i) closure property: If $a, b \in S$, then $a * b \in S$, $\forall a, b \in S$.

(ii) Associative property: If $a, b, c \in S$, then $(a * b) * c = a * (b * c)$.
 $\forall a, b, c \in S$.

example:- The set of all integers under the operation multiplication is a semi-group. i.e., $\langle \mathbb{Z}, \cdot \rangle$ is a semi-group.

(2) Monoid:-

A non-empty set M , together with the binary operation $*$ is called a monoid if the binary operation $*$ satisfies the following conditions.

(i) closure Property:- If $a, b \in M$, $a * b \in M$ $\forall a, b \in M$

(ii) Associative property:- If $a, b, c \in M$, then

$$(a * b) * c = a * (b * c) \quad \forall a, b, c \in M.$$

(iii) Identity property:- If $a \in M$, there exists an element $e \in M$ such that $a * e = e * a = a$.

example:- The set of all integers under the operation addition, is a monoid. i.e., $\langle \mathbb{Z}, + \rangle$ is a monoid.

13. Commutative Monoid:-

A monoid $\langle M, *, e \rangle$ is said to be commutative or abelian if $a * b = b * a$ for all $a, b \in M$.

example:- A set of real numbers under addition and multiplication are abelian group monoids.

Note:- $\langle \mathbb{Z}, \cdot \rangle$ is a monoid and hence it is a semigroup.

14. Define Group:-

A non-empty set G , together with a binary operation $*$ is said to be a group, if it satisfies the following conditions.

(i) If $a, b \in G$ then $a * b \in G$ $\forall a, b \in G$. [closure property]

(ii) If $a, b, c \in G$ then $a * (b * c) = (a * b) * c$ $\forall a, b, c \in G$. [Associative Property]

(iii) If $a \in G$, then there exists $e \in G$ such that
 $a * e = e * a = a$ $\forall a \in G$ [Identity Property]

(iv) If $a \in G$, then there exists $a^{-1} \in G$ such that
 $a * a^{-1} = a^{-1} * a = e$ for $a \in G$ [Inverse property].

Example:-

$\rightarrow \langle \mathbb{Z}, + \rangle$ is a group

$\rightarrow \langle \mathbb{Z}, \cdot \rangle$ is not a group.

The set of all integers $\mathbb{Z}$ is not a group under the operation multiplication. Because there is no multiplicative inverse in $\mathbb{Z}$.

But $\langle \mathbb{Z}, + \rangle$ is a monoid and hence it is a semi-group.

15. Abelian Group:-

A group $\langle G, * \rangle$ is said to be an abelian group or commutative group if $a * b = b * a$ for all $a, b \in G$.

example:-

(i) $\langle \mathbb{Z}, + \rangle$ is an abelian group.

(ii) $\langle \mathbb{Q}, + \rangle$ is an abelian group.

(iii) $\langle \mathbb{R}, + \rangle$ is an abelian group. [Additive group of rational].

(iv) $\langle \mathbb{C}, + \rangle$ is an abelian group. [Additive group of complex numbers].

16. Subgroup :-

(26)

A non-empty subset H of a group G is said to be a Subgroup if the following conditions are satisfied.

(i) $e \in H$, where e is the identity element of G .

(ii) For any $a \in H$, $a^{-1} \in H$.

(iii) For $a, b \in H$, $a * b \in H$.

Example :- $\langle \mathbb{Z}, + \rangle$ is a subgroup of the group $\langle \mathbb{R}, + \rangle$.

17. $\rightarrow$ Any groups $\langle G, * \rangle$ and $\langle \{e\}, * \rangle$ are called Improper Subgroups or trivial Subgroups and all the other Subgroups are called proper Subgroups or non-trivial Subgroups.

$\rightarrow$ The number of elements in a group G is called the order of the group and it is denoted by $O(G)$ or $|G|$.

$\rightarrow$ If the number of elements in a group is finite, then the group is said to be a finite group.

$\rightarrow$ If the number of elements in a group is infinite, then the group is said to be an infinite group.

18. Group Homomorphism :-

Let $\langle G, * \rangle$ and $\langle H, \Delta \rangle$ be any two groups. A mapping $\chi: G \rightarrow H$ is called a group homomorphism from $\langle G, * \rangle$ to $\langle H, \Delta \rangle$ if for any $a, b \in G$ then $\chi(a * b) = \chi(a) \Delta \chi(b)$.

19. Cyclic Group :-

A group G is called a cyclic group, if there exists an element $a \in G$, such that each element of G , is written as power of " a ".
i.e., a^n for some integer n .

A cyclic group is said to be generated by " a " or " a " is a generator of the group G .

A cyclic group is abelian, because for any $p, q \in G$.

$p = a^r$ and $q = a^s$ for some $r, s \in \mathbb{I}$ and

$$p * q = a^r * a^s = a^{r+s} = a^{s+r} = a^s * a^r = q * p.$$

20. Subsemigroup and submonoid :-

Let $\langle S, * \rangle$ be a semigroup and $T \subseteq S$. If the set T is closed under the operation $*$ then $\langle T, * \rangle$ is said to be a subsemigroup of $\langle S, * \rangle$.

Let $\langle M, *, e \rangle$ be a monoid and $T \subseteq M$. If T is closed under the operation $*$ and $e \in T$, then $\langle T, *, e \rangle$ is said to be a submonoid of $\langle M, *, e \rangle$.

21. Permutation :-

Any one-to-one mapping of a set S on to S is called a permutation of S .

example :- Let $N = \{1, 2, 3\}$. Then all the permutations of N are.

$$I_N = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, P_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, P_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}.$$

$$P_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, P_4 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, P_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}.$$

→ The set of permutations which form a group under their operation are called permutation groups.

22. Symmetric Group S_n :-

Let S be a finite set containing n elements. The set of all permutations of S is a group under composition of functions and this group is called the symmetric group of degree n and is denoted by S_n or P_n .

23. Left cosets and Right cosets :-

Let $\langle H, * \rangle$ be a subgroup of $\langle G, * \rangle$. For any $a \in G$, the set aH defined by $aH = \{a * h / h \in H\}$ is called Left coset of H in G , $Ha = \{h * a / h \in H\}$ is called Right coset of H in G , determined by the element $a \in G$.

24. Index and Normal Subgroup :-

A subgroup $\langle H, * \rangle$ of $\langle G, * \rangle$ is called a normal subgroup if for any $a \in G$, $aH = Ha$.

Let $\langle H, * \rangle$ be a subgroup of $\langle G, * \rangle$. Then the number of different left cosets of H in G is called the index of H in G .

25. Define Ring :-

(27)

An algebraic system $\langle R, +, \cdot \rangle$ where R is a non-empty set and $+$ and $\cdot$ are two binary operations on R , is called a ring if the following conditions are satisfied.

(i) $\langle R, + \rangle$ is an abelian group.

(ii) $\langle R, \cdot \rangle$ is a semigroup.

(iii) The operation $\cdot$ is distributed over $+$, i.e., for any $a, b, c \in R$,

$$a \cdot (b + c) = a \cdot b + a \cdot c \text{ and}$$

$$(a + b) \cdot c = a \cdot c + b \cdot c$$

26. Define Field :-

If $\langle R, +, \cdot \rangle$ is a commutative ring with at least two elements and if $\langle R \setminus \{0\}, \cdot \rangle$ is a group, then the ring R is called a field.

27. Prove that every subgroup of an abelian group is normal.

Solution :-

Let G be an abelian group and let H be a subgroup of G .
Let x be any element of G and $h \in H$.

$$\begin{aligned} \text{Then } xhx^{-1} &= x(hx^{-1}) \\ &= x(x^{-1}h) \\ &= (xx^{-1})h \\ &= eh \\ &= h \in H \end{aligned}$$

[$\because G$ is abelian]

Thus, for $x \in G$ and $h \in H$, $xhx^{-1} \in H$.

$\therefore H$ is normal Subgroup of G .

28. Prove that every subgroup of a cyclic group is normal.

Solution :-

We know that every cyclic group is abelian and also we know that every subgroup of an abelian group is normal.

$\therefore$ Every subgroup of a cyclic group is normal.

29. Show that the group $G = \{1, -1, i, -i\}$ is cyclic and also find its generators of G .

Solution:-

We know that, $1 = (i)^4$, $-1 = (i)^2$, $i = (i)^1$ and $-i = (i)^3$.

∴, all the elements of G can be expressed as some integral powers of the element i .

∴ G is a cyclic group generated by i .

Since i is the generator of G , $(i)^{-1}$ is also generator of G .

Hence, G is a cyclic group and its generators are i and

$$i^{-1} = \frac{1}{i} = \frac{i}{i^2} = -i.$$

∴ G is cyclic and its generators are i and $-i$.

30. Prove that the identity element of a group G is unique.

Proof:-

Let $\langle G, * \rangle$ be a group. Let e_1 and e_2 be two identities of the group $\langle G, * \rangle$.

Since e_1 is the identity element, we can write

$$a * e_1 = e_1 * a = a \quad \forall a \in G.$$

In particular, $e_2 * e_1 = e_1 * e_2 = e_2 \quad \longrightarrow \textcircled{1}$

Since e_2 is the identity element, we have

$$a * e_2 = e_2 * a = a \quad \forall a \in G$$

In particular,

$$e_1 * e_2 = e_2 * e_1 = e_1 \quad \longrightarrow \textcircled{2}$$

∴ From equ. $\textcircled{1}$ and $\textcircled{2}$, $e_1 = e_2$

Hence the identity element is unique in a group.

Part B:-

1. State and prove Lagrange's Theorem.

Statement:-

The order of a subgroup of a finite group divides the order of the group.

(or)

If G is a finite group, then $\frac{o(H)}{o(G)}$ for all subgroups H of G .

Proof:-

Let $o(G) = n$ and $G = \{a_1 = e; a_2; a_3; \dots; a_n\}$

Also, let H be a subgroup of G , whose order is m .

i.e., $o(H) = m$

Consider the left cosets as follows,

$$e * H = \{e * h / h \in H\}$$

$$a_2 * H = \{a_2 * h / h \in H\}$$

$$\vdots$$

$$\therefore a_n * H = \{a_n * h / h \in H\}$$

We know that, any two left cosets are either identical or disjoint.

$$\text{Also, } o[e * H] = o[H]$$

$$\therefore o[a_2 * H] = o[H] \quad \forall a_i \in G$$

If $a_i * h_i = a_j * h_j$ for $i \neq j$; by cancellation laws,

we have $h_i = h_j$; which is a contradiction.

Let there be ' k ' distinct cosets of ' H ' in ' G ', clearly their union equals G .

$$\text{i.e., } G = (a_1 * H) \cup (a_2 * H) \cup \dots \cup (a_k * H)$$

$$\therefore o(G) = o(H) + o(H) + \dots + o(H)$$

$$o(G) = k \cdot o(H)$$

$\Rightarrow o(H)$ is a divisor of $o(G)$.

Hence proved.

2. State and prove Cayley's Theorem:-

Every finite group of order n is isomorphic to Permutation group of degree n .

Proof:-

We prove this theorem in 3-steps.

Step-1 $\rightarrow$ Find a set G' of Permutation.

Step-2 $\rightarrow$ Prove G' is a group.

Step-3 $\rightarrow$ Exhibit an isomorphism $\phi: G \rightarrow G'$.

Step-1:-

Let G be a finite group of order n and let $a \in G$.

Define: $f_a: G \rightarrow G$ by $f_a(x) = ax$

$$\text{Since, } f_a(x) = f_a(y)$$

$$\Rightarrow ax = ay$$

$$\Rightarrow x = y$$

$\therefore f_a$ is one-to-one.

Since, if $y \in G$, then $f_a(a^{-1}y) = a^{-1}y \cdot a = y$

$\therefore f_a$ is onto.

$\Rightarrow f_a$ is a bijection [$\because$ one-to-one and onto]

$\therefore G$ has n elements, f_a is just Permutation on n symbols.

Let $G' = \{f_a / a \in G\}$

Step-2:-

Let G' be a group and Let $f_a, f_b \in G'$

$$f_a \circ f_b(x) = f_a[f_b(x)] = f_a[bx] = abx = f_{ab}(x).$$

$$\therefore f_a \circ f_b = f_{ab}(x)$$

$$\Rightarrow f_a \circ f_b = f_{ab}.$$

Hence G is closed, $f_e \in G'$ is the identity element.

The inverse of f_a in G' is f_a^{-1}

$\therefore G$ is a group.

Step-3 :-

To prove G and G' are isomorphic.

Define $\phi : G \rightarrow G'$ by $\phi(a) = f_a$.

$$\begin{aligned}\phi(a) = \phi(b) &\Rightarrow f_a = f_b \\ &\Rightarrow f_a(x) = f_b(x) \\ &\Rightarrow ax = bx \\ &\Rightarrow a = b\end{aligned}$$

$\therefore \phi$ is one-one.

Since, f_a is onto, ϕ is onto.

$$\text{Also, } \phi(ab) = f_{ab}$$

$$\Rightarrow f_a \circ f_b = \phi(a) \circ \phi(b)$$

$\therefore \phi : G \rightarrow G'$ is an isomorphism

$$\therefore G \cong G'$$

Hence proved.

3. Show that $[Z, +]$ is an abelian group.

Solution :-

Given $[Z, +]$ is said to be an abelian group if the following Properties are satisfied.

- (i) Closure Property :- The sum of any two integers in Z is an integer in Z . Hence, the closure property is true with respect to addition in Z .
- (ii) Associative Property :- clearly, $a + (b + c) = (a + b) + c$ for any three integers $a, b, c \in Z$.
- (iii) Existence of identity :- There exists an element 0 in Z such that $a + 0 = 0 + a = a$, for all $a \in Z$. Therefore, 0 is the identity element with respect to addition in Z .
- (iv) Existence of inverse :- For each $a \in Z$, there exists an element $-a$ in Z such that $a + (-a) = (-a) + a = 0$. Therefore, every element a has an additive inverse $-a$ in Z .

(V) commutative property :- usual addition is commutative in $\mathbb{Z}$,
 $\Rightarrow a+b=b+a$ for all $a, b \in \mathbb{Z}$.

Hence, $[\mathbb{Z}, +]$ is an abelian group.

4. Prove that any two right (or left) cosets H in G are either identical or disjoint.

Proof:-

Let H be a subgroup of a group G . Let $x, y \in G$ and Hx and H_y be two right cosets of H in G .

We need to prove that either $Hx = Hy$ or $Hx \cap Hy = \emptyset$.

Suppose that $Hx \cap Hy = \emptyset$,

Then there exists an element $a \in Hx \cap Hy$.

$\Rightarrow a \in Hx$ and $a \in Hy$,

$a = h_1 x$ and $a = h_2 y$ for some $h_1, h_2 \in H$.

$$\Rightarrow h_1 x = h_2 y \longrightarrow \textcircled{1}$$

Multiplying h_1^{-1} on both sides in equ (1), we have

$$h_1^{-1} h_1 x = h_1^{-1} h_2 y$$

$$x = h_1^{-1} h_2 y \longrightarrow \textcircled{2}$$

||^{ly}, Multiplying h_2^{-1} on both sides in equ. (1), we have

$$h_2^{-1} h_1 x = h_2^{-1} h_2 y$$

$$h_2^{-1} h_1 x = y \longrightarrow \textcircled{3}$$

Let, $b \in Hx$,

$b = hx$ for some $h \in H$.

$$= h(h_1^{-1} h_2 y)$$

[$\because$ equ. (2)]

$$= (h h_1^{-1} h_2) y.$$

[$\because$ where, $h h_1^{-1} h_2 \in H$]

Thus, $b \in Hy$.

$b = hy$ for some $h \in H$

$$= h(h_2^{-1} h_1^{-1} x)$$

$$= (h h_2^{-1} h_1^{-1}) x.$$

$\therefore Hx \subseteq Hy.$

$\longrightarrow \textcircled{4}$

Let $c \in Hy$

$\Rightarrow c = hy$ for some $h \in H$

$$= h(h_2^{-1} h_1 x)$$

$$= (h h_2^{-1} h_1) x$$

[$\because$ where $h h_2^{-1} h_1 \in H$]

$\Rightarrow c \in Hx$

$\therefore Hy \subseteq Hx \longrightarrow \textcircled{5}$

Hence, from equ. $\textcircled{4}$ and $\textcircled{5}$, we have

$$Hx = Hy.$$

5. Prove that the intersection of two normal subgroups of a group G is also a normal subgroup in G .

Proof:-

Let H_1 and H_2 be two normal subgroups of a group G . Since H_1 and H_2 are subgroups of G , $H_1 \cap H_2$ is also a subgroup of G . we need to prove that, $H_1 \cap H_2$ is normal.

Let $x \in G$ and $h \in H_1 \cap H_2$

$\Rightarrow x \in G$ and $h \in H_1$ and $x \in G$ and $h \in H_2$.

$\Rightarrow xhx^{-1} \in H_1$ and $xhx^{-1} \in H_2$ [$\because H_1$ and H_2 are normal subgroups]

$\Rightarrow xhx^{-1} \in H_1 \cap H_2$

$\therefore H_1 \cap H_2$ is a normal subgroup of G .

6. Prove that subgroup H of a group G is a normal subgroup in G if and only if each left coset of H in G is equal to the right coset of H in G .

Proof:-

Let H be a normal subgroup of G .

$\therefore xHx^{-1} = H, \forall x \in G$.

$$\Rightarrow (xHx^{-1})x = Hx$$

$$\Rightarrow xH(x^{-1}x) = Hx$$

$$\Rightarrow xH(e) = Hx, \forall x \in G$$

$$\Rightarrow xH = Hx, \forall x \in G$$

$\Rightarrow$ Every left coset of H in G is equal to the right coset of H in G .

Conversely, we assume that each left coset of H in G is a right coset of H in G .

$$\Rightarrow xH = Hx, \forall x \in G$$

$$\Rightarrow (xH)x^{-1} = (Hx)x^{-1}$$

$$\Rightarrow xHx^{-1} = H(x x^{-1})$$

$$\Rightarrow xHx^{-1} = He$$

$$\Rightarrow xHx^{-1} = H$$

$\therefore H$ is a normal subgroup of G .

Hence proved.

6. Show that the group $G = \{1, -1, i, -i\}$ is cyclic and also find its generators of G .

Solution:-

$$\text{We know that } 1 = (i)^4, -1 = (i)^2, i = (i)^1 \text{ and } (-i) = (i)^3$$

That is, all the elements of G can be expressed as some integral powers of the element i .

$\therefore G$ is a cyclic group generated by i .

Since ' i ' is the generator of G , $(i)^{-1}$ is also a generator of G .

Hence, G is a cyclic group and its generators are ' i ' and

$$i^{-1} = \frac{1}{i} = \frac{i}{i^2} = -i$$

$\therefore G$ is cyclic and its generators are ' i ' and ' $-i$ '.

7. Show that $\langle \mathbb{Z}, * \rangle$ is a group, where $*$ is defined by $a * b = a + b + 1$.

Solution:-

(i) closure Property:-

$$\text{If } a, b \in \mathbb{Z}, \text{ then } a + b + 1 \in \mathbb{Z}$$

$$\therefore a, b \in \mathbb{Z} \Rightarrow a * b \in \mathbb{Z}.$$

Hence, $*$ is a binary operation on $\mathbb{Z}$.

(ii) Associative Property:-

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Let a, b and $c \in \mathbb{Z}$. we need to prove that $a * (b * c) = (a * b) * c$.

$$\begin{aligned}\text{Now, } a * (b * c) &= a * (b + c + 1) \\ &= a + (b + c + 1) + 1 \\ &= a + b + c + 2 \longrightarrow \textcircled{1}\end{aligned}$$

$$\begin{aligned}\text{and, } (a * b) * c &= (a + b + 1) * c \\ &= (a + b + 1) + c + 1 \\ &= a + b + c + 2. \longrightarrow \textcircled{2}\end{aligned}$$

$\therefore$ from equ. ① and ②, we have

$$a * (b * c) = (a * b) * c. \quad \forall a, b, c \in \mathbb{Z}.$$

Hence $*$ is associative.

(iii) Existence of an identity:-

Let e be the identity element in $\mathbb{Z}$.

Let $a \in \mathbb{Z}$, then $a * e = a$.

$$\Rightarrow a + e + 1 = a \Rightarrow e + 1 = 0 \text{ and } e = -1$$

$$\text{Also, } a * (-1) = a + (-1) + 1 = a$$

$$(-1) * a = (-1) + a + 1 = a$$

$$\therefore a * (-1) = a * 1 = a \quad [\forall a \in \mathbb{Z}].$$

(iv) Existence of inverse:-

Let b be the inverse of a in $\mathbb{Z}$

$$\therefore a * b = e = -1$$

$$\Rightarrow a + b + 1 = -1$$

$$\Rightarrow b = -2 - a.$$

$$\text{Also, } a * (-2 - a) = a + (-2 - a) + 1 = -1$$

$$\text{and } (-2 - a) * a = (-2 - a) + a + 1 = -1.$$

$$\therefore a * (-2 - a) = (-2 - a) + a + 1 = -1 \text{ for every } a \in \mathbb{Z}.$$

$$\therefore (-2 - a) \text{ is the inverse of } a \text{ in } \mathbb{Z}.$$

This ~~includes~~ ^{indicates} that inverse of exists

Hence, $\langle \mathbb{Z}, * \rangle$ is a group.

8. Prove that every subgroup of a cyclic group is cyclic. (14)

Proof:

Let G be a cyclic group with a generator a ,
i.e. $G = \{a^n / n \in \mathbb{Z}\}$.

Let H be a subgroup of G . We need to prove that H is cyclic.

If $H = \{e\}$, then the result is trivial.

If $H \neq \{e\}$, then ~~there~~ ^{there} exists $x \in H$ such that $x \neq e$.
The elements of H are integral powers of a .

$$\therefore x = a^s \in H$$

$\Rightarrow$ The inverse of $a^s = a^{-s} \in H$

$\therefore H$ contains elements that are positive as well as negative integral powers of a .

Let m be the least positive integer that $a^m \in H$. We need to prove that H is cyclic.

Ans: It is enough to prove that H is generated by a^m .

Let y be any element of $H \Rightarrow y = a^r$ for some positive integer r .

We know that division algorithm states that "If a is any integer and n is a positive integer, then there exists unique q and r such that $a = qn + r$, with $0 \leq r < n$ ".

Thus, by division algorithm, there exists unique integers q and r such that $t = mq + r$, with $0 \leq r < m$.

$$\text{Now, } a^t \in H \Rightarrow (a^m)^q \in H$$

[$\because$ by closure property]

$$\Rightarrow a^{mq} \in H$$

$$\Rightarrow a^{-mq} \in H$$

$$\text{Also, } a^t \in H, a^{-mq} \in H$$

$$\Rightarrow a^t \times a^{-mq} \in H$$

$$\Rightarrow a^{t-mq} \in H$$

$$\Rightarrow a^r \in H$$

That is, m is the least positive integer such that $a^m \in H$ and

$$0 \leq r < m.$$

Ans: The possible only when $r=0$.

Hence, $t=m$

$$\therefore a^t = a^{mq} = (a^m)^q$$

Thus, every element $a^t \in H$, is of the form $(a^m)^q$.

$\therefore H$ is cyclic and is generated by a^m .

Hence, $H = \langle a^m \rangle$.

9. Show that a non-empty subset H of the group $(G, *)$ is a subgroup of G and only if $a * b^{-1} \in H$ for all $a, b \in H$.

Solution:

Let us assume that H is a subgroup. If $a, b \in H$, then $b^{-1} \in H$ and

$$a * b^{-1} \in H.$$

To prove the converse,

Let us assume that $a, b \in H$, and $a * b^{-1} \in H$ for all $a, b \in H$.

Considering $b = a$, $a * a^{-1} = e \in H$. From $e, a, b \in H$.

we have $e * a^{-1} = a^{-1} \in H$.

||^{ly}, $b^{-1} \in H$. Finally, because a and b^{-1} are in H , we have $a * b^{-1} \in H$.

Hence $(H, *)$ is a subgroup of $(G, *)$.

10. Show that Kernel of a group homomorphism is a normal subgroup of the group.

Solution:

Let K be the Kernel of the homomorphism f .

$$K = \{x \in G / f(x) = e', e' \in H \text{ is the identity element in } H\}$$

To prove K is a subgroup, Let $x, y \in K$ then $f(x) = e'$ and $f(y) = e'$.

By the definition of homomorphism

$$\begin{aligned} f(x * y^{-1}) &= f(x) \Delta f(y^{-1}) \\ &= f(x) \Delta [f(y)]^{-1} \\ &= e' \Delta (e')^{-1} \\ &= e' \Delta e'. \end{aligned}$$

$$f(x * y^{-1}) = e'$$

Hence $x * g' \in K$ and this prove that K is a subgroup of G .
 To prove that K is normal subgroup.

Let $x \in K, g \in G$, then $f(x) = e'$

$$\Rightarrow g * x * g^{-1} \in K$$

$$\begin{aligned} \Rightarrow f(g * x * g^{-1}) &= f(g) * f(x) * f(g^{-1}) \\ &= f(g) * e' [f(g^{-1})]^{-1} \\ &= f(g) [f(g)]^{-1} \end{aligned}$$

$$f(g * x * g^{-1}) = e'$$

$$\therefore g * x * g^{-1} \in K.$$

Hence K is normal subgroup of G .

11. If G is a group, then $(a * b)^{-1} = b^{-1} * a^{-1}$ for all $a, b \in G$. (or) The inverse of the product of two elements of a group is the product of their inverse in reverse order.

Proof:-

Let $(G, *)$ be a group. Let $a, b \in G$ and let a^{-1} be the inverse of a and b^{-1} be the inverse of b .

$$\Rightarrow a^{-1}, b^{-1} \in G$$

$$\therefore a^{-1} * a = a * a^{-1} = e \text{ and } b * b^{-1} = b^{-1} * b = e.$$

$$\text{Also, } a * b \in G \text{ and } (a * b)^{-1} \in G$$

$$\text{Now, } (a * b) * (b^{-1} * a^{-1}) = a * [b * (b^{-1} * a^{-1})]$$

$$= a * [(b * b^{-1}) * a^{-1}]$$

[by associative property]

$$= a * [e * a^{-1}]$$

$$= a * a^{-1}$$

$$= e.$$

$$\text{Also, } (b^{-1} * a^{-1}) * (a * b) = (b^{-1} * a^{-1}) * (a * b) =$$

$$= b^{-1} * [a^{-1} * (a * b)]$$

$$= b^{-1} * [a^{-1} * a] * b$$

$$= b^{-1} * (e * b)$$

$$= b^{-1} * b$$

$$= e.$$

Thus, $(a * b) * (b^{-1} * a^{-1}) = (b^{-1} * a^{-1}) * (a * b) = e$

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$$\Rightarrow [a * b]^{-1} = b^{-1} * a^{-1}.$$

Hence proved.

12. Show that $\langle \mathbb{Q}, * \rangle$ is an abelian group, where $*$ is defined by $a * b = \frac{ab}{2}$, $\forall a, b \in \mathbb{Q}$.

Solution

From the definition of $\langle \mathbb{Q}, * \rangle$ is closed.

$$\text{Let } a, b, c \in \mathbb{Q}, \text{ then } a * (b * c) = a * \left(\frac{bc}{2}\right) = \frac{a \left(\frac{bc}{2}\right)}{2} = \frac{abc}{4} \rightarrow \textcircled{1}$$

$$\text{Now, } (a * b) * c = \left(\frac{ab}{2}\right) * c = \frac{\left(\frac{ab}{2}\right)c}{2} = \frac{abc}{4} \rightarrow \textcircled{2}$$

$\therefore$ From $\textcircled{1}$ and $\textcircled{2}$,

$$a * (b * c) = (a * b) * c. \text{ It is associative.}$$

$$\text{Let } a, b \in \mathbb{Q}, \text{ then } a * b = \frac{ab}{2} = \frac{ba}{2} = b * a.$$

$$\therefore a * b = b * a.$$

$\therefore \langle \mathbb{Q}, * \rangle$ is commutative.

Hence, the system $\langle \mathbb{Q}, * \rangle$ is a commutative (abelian) ^{semi} group.

13. If f is a homomorphism of a group G_1 into a group G_2 , then prove that,

$$\textcircled{i} [f(a)]^{-1} = f(a^{-1}) \quad \forall a \in G_1$$

$$\textcircled{ii} f(e) \text{ is an identity of } G_2 \text{ when } e \text{ is identity of } G_1$$

Solution

Let $f: \langle G_1, * \rangle \rightarrow \langle G_2, \Delta \rangle$ be a group homomorphism.

$$\therefore f(a * b) = f(a) \Delta f(b), \quad \forall a, b \in G_1.$$

$$\textcircled{i} \text{ Let } a \in G_1, \text{ then } a^{-1} \in G_1 \text{ and } a * a^{-1} = a^{-1} * a = e$$

$$\therefore e' = f(e) = f(a * a^{-1}) \\ = f(a) \Delta f(a^{-1}).$$

[$\because f$ is homomorphism]

$$\Rightarrow f(a) \Delta f(a^{-1}) = e'$$

Thus, $f(a^{-1})$ is the inverse of $f(a)$ in G_2 .

$$\Rightarrow [f(a^{-1})] = [f(a)]^{-1} \quad \forall a \in G_1.$$

(ii) Let e and e' be the identity elements of G_1 and G_2 , respectively,

Let $a \in G_1$, Then $f(a) \in G_2$.

$$\therefore f(a) \Delta e' = f(a) = f(a * e) = f(a) \Delta f(e)$$

$$\Rightarrow f(e) = e'.$$

[$\therefore$ by left cancellation law]

14. Show that $S = \left\{ \begin{pmatrix} x & x \\ x & x \end{pmatrix} / x \in \mathbb{R}^* \right\}$ is a group under multiplication of matrices. It is commutative?

Solution:-

Clearly, $S \neq \emptyset$. Since $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in S$.

(i) Closure property:-

Let $A, B \in S$, Let $A = \begin{pmatrix} x & x \\ x & x \end{pmatrix}$ and $B = \begin{pmatrix} y & y \\ y & y \end{pmatrix}$

$$AB = \begin{pmatrix} x & x \\ x & x \end{pmatrix} \begin{pmatrix} y & y \\ y & y \end{pmatrix} = \begin{pmatrix} 2xy & 2xy \\ 2xy & 2xy \end{pmatrix} \in S.$$

Hence, matrix multiplication is closed in S .

(ii) Associative property:

Let $A = \begin{pmatrix} x & x \\ x & x \end{pmatrix}$, $B = \begin{pmatrix} y & y \\ y & y \end{pmatrix}$, and $C = \begin{pmatrix} z & z \\ z & z \end{pmatrix}$ be any three elements in S .

We need to prove that, $A(BC) = (AB)C$.

$$\begin{aligned} \text{Now, } A(BC) &= \begin{pmatrix} x & x \\ x & x \end{pmatrix} \cdot \left[\begin{pmatrix} y & y \\ y & y \end{pmatrix} \begin{pmatrix} z & z \\ z & z \end{pmatrix} \right] \\ &= \begin{pmatrix} x & x \\ x & x \end{pmatrix} \begin{bmatrix} yz+yz & yz+yz \\ yz+yz & yz+yz \end{bmatrix} \end{aligned}$$

$$= \begin{pmatrix} x & x \\ x & x \end{pmatrix} \begin{pmatrix} 2yz & 2yz \\ 2yz & 2yz \end{pmatrix}$$

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$$= \begin{pmatrix} 2xyz + 2xyz & 2xyz + 2xyz \\ 2xyz + 2xyz & 2xyz + 2xyz \end{pmatrix}$$

$$A(BC) = \begin{pmatrix} 4xyz & 4xyz \\ 4xyz & 4xyz \end{pmatrix}$$

$$\text{iii) } (AB)C = \left[\begin{pmatrix} x & x \\ x & x \end{pmatrix} \begin{pmatrix} y & y \\ y & y \end{pmatrix} \right] \begin{bmatrix} z & z \\ z & z \end{bmatrix} = \begin{pmatrix} 2xy + 2xy & 2xy + 2xy \\ 2xy + 2xy & 2xy + 2xy \end{pmatrix} \begin{bmatrix} z & z \\ z & z \end{bmatrix}$$

$$= \begin{pmatrix} 4xy & 4xy \\ 4xy & 4xy \end{pmatrix} \begin{bmatrix} z & z \\ z & z \end{bmatrix} = \begin{pmatrix} 4xyz + 4xyz & 4xyz + 4xyz \\ 4xyz + 4xyz & 4xyz + 4xyz \end{pmatrix}$$

$$(AB)C = \begin{pmatrix} 4xyz & 4xyz \\ 4xyz & 4xyz \end{pmatrix}$$

$\therefore (AB)C = A(BC)$, Thus matrix multiplication is associative in S .

(ii) Existence of identity element:

Let $\begin{pmatrix} e & e \\ e & e \end{pmatrix}$ be the identity element of S .

$$\begin{pmatrix} x & x \\ x & x \end{pmatrix} \begin{pmatrix} e & e \\ e & e \end{pmatrix} = \begin{pmatrix} x & x \\ x & x \end{pmatrix}$$

$$\begin{pmatrix} xe + xe & xe + xe \\ xe + xe & xe + xe \end{pmatrix} = \begin{pmatrix} x & x \\ x & x \end{pmatrix}$$

$$\begin{pmatrix} 2xe & 2xe \\ 2xe & 2xe \end{pmatrix} = \begin{pmatrix} x & x \\ x & x \end{pmatrix}$$

$$2xe = x \Rightarrow 2e = 1 \Rightarrow e = 1/2$$

$$\therefore \begin{pmatrix} e & e \\ e & e \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

$$\text{now, } \begin{pmatrix} x & x \\ x & x \end{pmatrix} \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} = \begin{pmatrix} 1/2x + 1/2x & 1/2x + 1/2x \\ 1/2x + 1/2x & 1/2x + 1/2x \end{pmatrix} = \begin{pmatrix} x & x \\ x & x \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \text{ is an identity element of } S.$$

(iv) Existence of Inverse:-

Let $\begin{pmatrix} x & x \\ x & x \end{pmatrix} \in S$, Let $\begin{pmatrix} x_1 & x_1 \\ x_1 & x_1 \end{pmatrix}$ be the inverse of $\begin{pmatrix} x & x \\ x & x \end{pmatrix}$ in S .

$$\begin{pmatrix} x & x \\ x & x \end{pmatrix} \begin{pmatrix} x_1 & x_1 \\ x_1 & x_1 \end{pmatrix} = \begin{pmatrix} y_2 & y_2 \\ y_2 & y_2 \end{pmatrix}$$

$$\begin{pmatrix} 2xx_1 & 2xx_1 \\ 2xx_1 & 2xx_1 \end{pmatrix} = \begin{pmatrix} y_2 & y_2 \\ y_2 & y_2 \end{pmatrix} \Rightarrow 2xx_1 = y_2, \Rightarrow x_1 = \frac{1}{2}x$$

$$\begin{pmatrix} x_1 & x_1 \\ x_1 & x_1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}x & \frac{1}{2}x \\ \frac{1}{2}x & \frac{1}{2}x \end{pmatrix}$$

$$\text{Also, } \begin{pmatrix} x & x \\ x & x \end{pmatrix} \begin{pmatrix} \frac{1}{2}x & \frac{1}{2}x \\ \frac{1}{2}x & \frac{1}{2}x \end{pmatrix} = \begin{pmatrix} \frac{x}{2}x + \frac{x}{2}x & \frac{x}{2}x + \frac{x}{2}x \\ \frac{x}{2}x + \frac{x}{2}x & \frac{x}{2}x + \frac{x}{2}x \end{pmatrix} = \begin{pmatrix} \frac{2x}{2} & \frac{2x}{2} \\ \frac{2x}{2} & \frac{2x}{2} \end{pmatrix} = \begin{pmatrix} y_2 & y_2 \\ y_2 & y_2 \end{pmatrix}$$

Thus, $\begin{pmatrix} \frac{1}{2}x & \frac{1}{2}x \\ \frac{1}{2}x & \frac{1}{2}x \end{pmatrix}$ is the inverse of $\begin{pmatrix} x & x \\ x & x \end{pmatrix}$.

Hence $\langle S, \cdot \rangle$ is a group.

(v) Commutative property:-

Let $\begin{pmatrix} x & x \\ x & x \end{pmatrix}$ and $\begin{pmatrix} y & y \\ y & y \end{pmatrix}$ be any two elements in S .

$$\text{Now, } \begin{pmatrix} x & x \\ x & x \end{pmatrix} \begin{pmatrix} y & y \\ y & y \end{pmatrix} = \begin{pmatrix} 2xy & 2xy \\ 2xy & 2xy \end{pmatrix} = \begin{pmatrix} y & y \\ y & y \end{pmatrix} \begin{pmatrix} x & x \\ x & x \end{pmatrix}$$

$\therefore$ matrix multiplication is commutative in S .

Thus $\langle S, \cdot \rangle$ is an abelian group.

15. show that $(\mathbb{Z}, *)$ is a group, where $*$ is defined by $\boxed{a \cdot b + 1}$
 $a * b = a + b + 1$.

Solution:

(i) closure property:

If $a, b \in \mathbb{Z}$, then $a + b + 1 \in \mathbb{Z}$

$$\therefore a, b \in \mathbb{Z} \Rightarrow a * b \in \mathbb{Z}$$

Hence, $*$ is a binary operation on $\mathbb{Z}$.

(ii) Associative property:

Let a, b and $c \in \mathbb{Z}$, we need to prove that

$$a * (b * c) = (a * b) * c$$

$$\begin{aligned} \text{L.H.S.} &= a * (b * c) \\ &= a * (b + c + 1) \\ &= a + (b + c + 1) + 1 \\ &= a + b + c + 2 \end{aligned}$$

$$\begin{aligned} \text{R.H.S.} &= (a * b) * c \\ &= (a + b + 1) * c \\ &= (a + b + 1) + c + 1 \\ &= a + b + c + 2 \end{aligned}$$

$$\therefore \text{L.H.S.} = \text{R.H.S.}$$

(iii) Existence of an identity:

Let $e \in \mathbb{Z}$, Let $a \in \mathbb{Z}$, Then $a * e = a$.

$$\Rightarrow a + e + 1 = a$$

$$\Rightarrow e + 1 = a - a$$

$$\Rightarrow e + 1 = 0$$

$$\Rightarrow e = -1$$

$$\text{Also, } a * (-1) = a + (-1) + 1 = a$$

$$(-1) * a = (-1) + a + 1 = a$$

$$\therefore a * (-1) = (-1) * a = a \quad (\because \forall a \in \mathbb{Z})$$

Hence, -1 is an identity element to $\mathbb{Z}$.

(iv) Existence of Inverse!

Let b be the inverse of a in $\mathbb{Z}$

$$\therefore a * b = e = -1$$

$$\Rightarrow a + b + 1 = -1$$

$$\Rightarrow b = -2 - a$$

$$\text{Also, } a * (-2 - a) = a + (-2 - a) + 1 = -1$$

$$(-2 - a) * a = (-2 - a) + a + 1 = -1$$

$$\therefore a * (-2 - a) = (-2 - a) * a = -1 \quad \forall a \in \mathbb{Z}.$$

$\therefore (-2 - a)$ is the inverse of a in $\mathbb{Z}$.

$\Rightarrow$ Inverse exist

Hence $(\mathbb{Z}, *)$ is a group.

16. Prove that the intersection of two normal subgroup is a normal subgroup.

Proof:-

Let H and K be any two normal subgroups of a group G .

We have to prove that $H \cap K$ is normal in G .

Since H and K are subgroups of G , $e \in H$ and $e \in K$.

Hence $e \in H \cap K$. Thus $H \cap K$ is a non-empty set.

Let $a, b \in H \cap K$.

Claim:- $ab^{-1} \in H \cap K$

Since, $a, b \in H \cap K$, both a, b belong to H and K .

Since H and K are subgroups of G , $ab^{-1} \in H$ and $ab^{-1} \in K$.

So that $ab^{-1} \in H \cap K$

Hence $H \cap K$ is a subgroup of G , by a criterion for Subgroup.

To prove $\therefore H \cap K$ is normal,

Let $x \in H \cap K$, and let $g \in H$.

Since $x \in H \cap K$ and $x \in H$ and $x \in K$.

Since $x \in H$, $g \in G \Rightarrow gxg^{-1} \in H$ [$\because$ as H is normal] (IV) 12
 Likewise $x \in K$, $g \in G \Rightarrow gxg^{-1} \in K$ [$\because$ as K is normal]
 Hence $x \in H \cap K$ and $g \in G \Rightarrow gxg^{-1} \in H \cap K$
 This $H \cap K$ is a normal subgroup of G .

17 Fundamental Theorem of homomorphism of G onto G'
 with kernel K , then $G/K \cong G'$

Proof:

Let $f: G \rightarrow G'$ be a homomorphism from the group $(G, *)$ to the group (G', Δ) .

Then $K = \text{Ker}(f) = \{x \in G \mid f(x) = e'\}$ is a normal subgroup of $(G, *)$.

Also we know that the quotient set $(G/K, \otimes)$ is a group.

Define $\phi: G/K \rightarrow G'$ is a mapping from the group $(G/K, \otimes)$ to the group (G', Δ) given by

$$\phi(Ka) = f(a), \text{ for any } a \in G$$

Since, if $Ka = Kb \Rightarrow a * b^{-1} \in K$

$$\Rightarrow f(a * b^{-1}) = e'$$

$$\therefore f(a) \Delta f(b^{-1}) = e'$$

$$f(a) \Delta [f(b)]^{-1} \Delta [f(b)] = e' \Delta f(b)$$

$$\Rightarrow f(a) \Delta e' = f(b)$$

$$\Rightarrow f(a) = f(b)$$

$$\Rightarrow \phi(Ka) = \phi(Kb)$$

ϕ is well defined

claim: ϕ is a homomorphism.

Let $Ka, Kb \in G/K$.

$$\begin{aligned} \text{Now } \phi(Ka \otimes Kb) &= \phi[K(a * b)] \\ &= f[a * b] \\ &= f(a) \Delta f(b) \end{aligned}$$

$$= \phi(Ka) \Delta(Kb)$$

$\therefore \phi$ is a homomorphism.

claim ! ϕ is one-to-one.

If $\phi(Ka) = \phi(Kb)$, then $f(a) = f(b)$

$$f(a) \Delta f(b^{-1}) = f(b) \Delta f(b^{-1})$$

$$f(a * b^{-1}) = f(b * b^{-1})$$

$$= f(e)$$

$$f(a * b^{-1}) = e'$$

$$\therefore a * b^{-1} \in K \Rightarrow Ka = Kb$$

$\therefore \phi$ is one-to-one.

claim ! ϕ is onto.

Let y be any element of G' .

Since $f: G \rightarrow G'$ is a homomorphism from G onto G' , therefore there exists an element $a \in G$ such that $f(a) = y$.

$\therefore$ For every $a \in G$, $Ka \in G/K$.

We get $\phi(Ka) = f(a)$, for all $f(a) = y \in G'$.

$\therefore \phi$ is onto

$\therefore \phi: G/K \rightarrow G'$ is an isomorphism.

18. Prove that the set $Z_4 = \{[0], [1], [2], [3]\}$ is a commutative ring with respect to the binary operation addition modulo and multiplication modulo $+_4, *_4$

Solution:-

The composition tables for additive modulo 4

$+_4$	$[0]$	$[1]$	$[2]$	$[3]$
$[0]$	0	1	2	3
$[1]$	1	2	3	0
$[2]$	2	3	0	1
$[3]$	3	0	1	2

The composition table for ~~additive~~ ^{Multiplicative} modulo 4

$|1V+3|$

X_4	[0]	[1]	[2]	[3]
[0]	0	0	0	0
[1]	0	1	2	3
[2]	0	2	0	2
[3]	0	3	2	1

From the above tables we get

(i) All the entries in both the tables belong to Z_4 .

$\therefore Z_4$ is closed under the operations $+_4$ and $\times_4$.

(ii) In both the tables,

Entries in the first row = Entries in the first column.

Entries in the second row = Entries in the second column.

Entries in the third row = Entries in the third column.

Entries in the fourth row = Entries in the fourth column.

$\therefore$ The operations $+_4$ and $\times_4$ are commutative in Z_4 .

(iii) Also, for any $a, b, c \in Z_4$, we have

$$a +_4 (b +_4 c) = (a +_4 b) +_4 c$$

$$\text{and } a \times_4 (b \times_4 c) = (a \times_4 b) \times_4 c$$

$$\text{Since, } 0 +_4 (1 +_4 2) = 0 +_4 3 = 3$$

$$\text{and } (0 +_4 (1 +_4 2)) = 0 +_4 (1 +_4 1) = 3.$$

$$\text{Also, } 1 \times_4 (2 \times_4 3) = 1 \times_4 2 = 2$$

$$\text{and } (1 \times_4 2) \times_4 3 = 2 \times_4 3 = 2$$

$$\therefore 1 \times_4 (2 \times_4 3) = (1 \times_4 2) \times_4 3$$

Thus, the operations $+_4$ and $\times_4$ are associative in Z_4 .

(iv) '0' is the additive identity of Z_4 and 1 is the multiplicative identity of Z_4 .

(v) Additive inverse of 0, 1, 2, 3 are respectively 0, 3, 2, 1. Multiplicative inverses of the non-zero elements 1, 2 and 3 are 1, 2, and 3 respectively.

(vi) If $a, b, c \in Z_4$, then

$$a \times_4 (b +_4 c) = (a \times_4 b) +_4 (a \times_4 c)$$

$$\text{and } (a +_4 b) \times_4 c = (a \times_4 c) +_4 (b \times_4 c)$$

Thus, the operation of $\times_4$ and $+$ is distributed over $+$ in Z_4 .

Hence, $(Z_4, +_4, \times_4)$ is a commutative ring with unity.

Part-A:-1. Define Lattice:-

A Lattice is a partially ordered set $\langle L, \leq \rangle$ in which every pair of elements $a, b \in L$ has a greatest lower bound (GLB) and a least upper bound (LUB).

→ The greatest lower bound (GLB) of a subset $\{a, b\} \subseteq L$ is denoted by $a \times b$. The $\text{GLB}\{a, b\} = a \times b$ the meet or product of a and b .

→ The Least Upper Bound (LUB) of a subset $\{a, b\} \subseteq L$ is denoted by $a \oplus b$. The $\text{LUB}\{a, b\} = a \oplus b$ the join or sum of a and b .

2. State De Morgan's Law in any Boolean Algebra.

Let $a, b \in B$,

$$(a \oplus b)' = a' \times b'$$

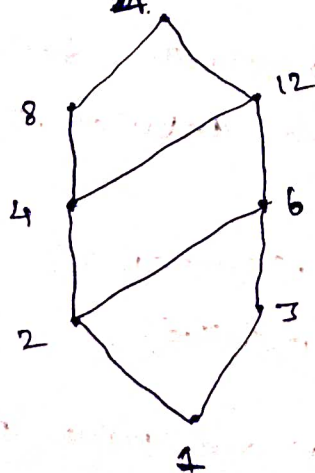
$$(a \times b)' = a' \oplus b' \quad , \quad \forall a, b \in B.$$

3. Draw the Hasse diagram representing for $(\mathbb{D}_{24}, |)$, where $\mathbb{D}_{24} = \{1, 2, 3, 4, 6, 8, 12, 24\}$.

Solution:-

The poset for $\mathbb{D}_{24}$ is the set of all divisors of 24.

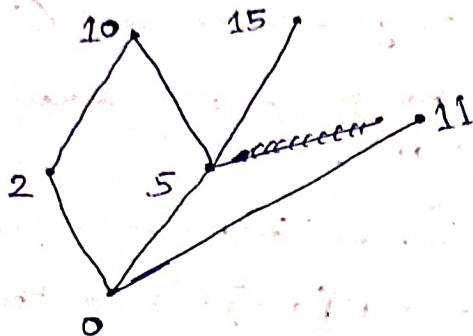
∴ $\mathbb{D}_{24} = \{1, 2, 3, 4, 6, 8, 12, 24\}$. Then the Hasse diagram is



Hasse diagram $(\mathbb{D}_{24})$.

4. Draw the Hasse diagram for ' $\leq$ ' relation on $\{0, 2, 5, 10, 11, 15\}$

Solution:-



5. Show that in a lattice if $a \leq b$ and $c \leq d$ then $a * c \leq b * d$.

Solution:-

Let $(L, \leq)$ be a lattice and $a, b, c, d \in L$.

Given $a \leq b$, $c \leq d$. To show that $a * c \leq b * d$.

By definition, we have

$$a * c = \text{GLB} \{a, c\}$$

$$\therefore a * c \leq a \text{ and } a * c \leq c.$$

$$\text{Now, } a * c \leq a \leq b \Rightarrow a * c \leq b$$

$$\text{and } a * c \leq c \leq d \Rightarrow a * c \leq d.$$

$$\Rightarrow a * c \text{ is a Lower bound of } \{b, d\}.$$

$$\text{But the GLB of } \{b, d\} = b * d.$$

$$\text{Thus, } a * c \leq b * d.$$

6. Define Partial Ordered Set.

Let X be any set, ' R ' be a relation defined on ' X '. The ' R ' is said to be Partial ordered relation. If it satisfies,

$$(i) x R x \Rightarrow x \text{ [Reflexive]}$$

$$(ii) x R y \text{ and } y R x \Rightarrow x = y \text{ [Antisymmetric]}$$

$$(iii) x R y \text{ and } y R z \Rightarrow x R z \text{ [Transitive]}.$$

7. Show that the Absorption Laws are valid in a Boolean algebra.

Solution:-

$$\text{Absorption Laws, (i) } a * (a + b) = a$$

$$(ii) a + (a * b) = a \quad \forall a, b \in B.$$

Let $a, b \in B$, then

(3b)

$$\begin{aligned} \text{(i)} \quad a * (a \oplus b) &= (a \oplus 0) * (a \oplus b) & [\because a \oplus 0 = a] \\ &= a \oplus (a * b) \\ &= a \oplus 0 \\ a * (a \oplus b) &= a \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad a \oplus (a * b) &= (a * 1) \oplus (a * b) & [\because a * 1 = a] \\ &= a * (1 \oplus b) \\ &= a * (b \oplus 1) \\ &= a * 1 \\ a * (a \oplus b) &= a. \end{aligned}$$

8. In any lattice $\langle L, \leq \rangle$, $\forall a, b \in L$, show that $a * (a \oplus b) = a$, where $a * b = \text{glb}(a, b)$ and $a \oplus b = \text{lub}(a, b)$.

Solution:-

Since $a * b = \text{glb of } \{a, b\}$, we have $a * b \leq a$ and $a * b \leq b$.

Also, $a \leq a$.

$$\therefore a \leq a \text{ and } a * b \leq a$$

$$\Rightarrow a \oplus (a * b) \leq a \quad \rightarrow \textcircled{1}$$

Also, by the definition of lub , we have

$$a \oplus (a * b) = \text{lub of } \{a, (a * b)\}$$

$$\therefore a \leq a \oplus (a * b) \quad \rightarrow \textcircled{2}$$

From equ. $\textcircled{1}$ and $\textcircled{2}$, we have

$$a \oplus (a * b) = a$$

$[\because \leq \text{ is antisymmetric}]$

Hence, the first part,

By the principle of duality, we have

$$a * (a \oplus b) = a.$$

9. State Modular Inequality of Lattices.

If $\langle L, \leq \rangle$ is a lattice, then for any $a, b, c \in L$,

$$a \leq c \Leftrightarrow a \oplus (b * c) \leq (a \oplus b) * c.$$

Proof:- Let us assume that $a \leq c$.

$$\therefore a \oplus c = c$$

we need to prove that $a \oplus (b * c) \leq (a \oplus b) * c \rightarrow \textcircled{1}$

$$\text{Now, } a \oplus (b * c) \leq (a \oplus b) * (a \oplus b)$$

$$\Rightarrow a \oplus (b * c) \leq (a \oplus b) * c$$

$$\text{Thus, } a \leq c \Rightarrow a \oplus (b * c) \leq (a \oplus b) * c \rightarrow \textcircled{2}$$

Conversely, we assume that,

$$a \oplus (b * c) \leq (a \oplus b) * c$$

we need to prove that $a \leq c$.

Now, by the definition of lub $a \leq a \oplus (b * c)$

$$\leq (a \oplus b) * c$$

$$\leq c$$

$$\Rightarrow a \leq c$$

$$\therefore a \oplus (b * c) \leq (a \oplus b) * c \Rightarrow a \leq c. \rightarrow \textcircled{3}$$

$\therefore$ from $\textcircled{2}$ and $\textcircled{3}$, we get

$$a \leq c \Leftrightarrow a \oplus (b * c) \leq (a \oplus b) * c.$$

10. Prove that in a lattice $\langle L, \leq \rangle$, $a * (a \oplus b) = a$, where $*$ and $\oplus$ denote the meet and join.

Proof:-

Let $a \leq b$, since $b \in L$, $b \leq b$.

$$\therefore a \oplus b \leq b \rightarrow \textcircled{1}$$

Since, $a \oplus b$ is the lub of $\{a, b\}$, we have

$$b \leq a \oplus b \rightarrow \textcircled{2}$$

$\therefore$ From $\textcircled{1}$ and $\textcircled{2}$, we get,

$$\Rightarrow a \leq b \Rightarrow a \oplus b = b \rightarrow \textcircled{3}$$

Conversely, let $a \oplus b = b$

Since, $a \oplus b$ is the lub of $\{a, b\}$, we have

$$a \leq a \oplus b$$

$$\Rightarrow a \leq b.$$

$$\therefore a \oplus b = b \Rightarrow a \leq b \rightarrow \textcircled{4}$$

$\therefore$ from $\textcircled{3}$ and $\textcircled{4}$, we get

$$a \leq b \Leftrightarrow a \oplus b = b, \rightarrow \textcircled{5}$$

similarly, we get, $a \leq b \Leftrightarrow a * b = a \rightarrow \textcircled{6}$

(37)

To prove that, $a * (a \oplus b) = a$,

$$a * b = a$$

[$\because$ eqn. (5)]

$$\boxed{a = a}$$

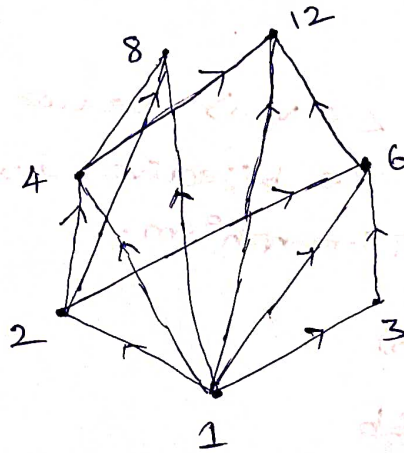
[$\because$ eqn. (6)]

11. Draw a Hasse diagram representing the partial ordering $\{(a,b) / a \text{ divides } b\}$ on the set $\{1, 2, 3, 4, 6, 8, 12\}$.

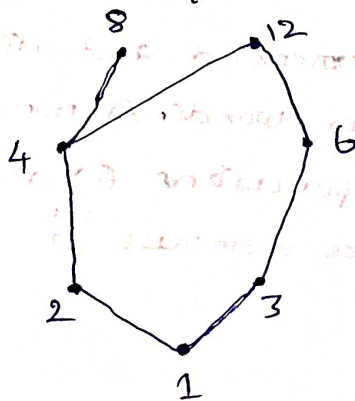
Solution:-

$$R = \{(a,b) / a|b\}$$

$$R = \{(1,2), (1,3), (1,4), (1,6), (1,8), (1,12), (2,4), (2,6), (2,8), (2,12), (3,6), (3,12), (4,8), (4,12), (6,12)\}.$$



Delete all the edges implied by the transitive property.
we have



12. Define Principle of Duality:-

The lattice $\langle L, \leq \rangle$ and $\langle L, \geq \rangle$ are called the duals of each other. Similarly, the operations $\vee$ and $\wedge$ are dual of each other and the relations $\leq$ and $\geq$ are duals of each other.

13. Define Algebraic System:

A lattice may be defined as an algebraic system $\langle L, \oplus, * \rangle$ with two binary operations $\oplus$ and $*$ on L , which satisfy the commutative, associative and absorption laws.

14. Definitions:

Lattice Homomorphism :-

Let $\langle L, \oplus, * \rangle$ and $\langle S, \vee, \wedge \rangle$ be two lattices. A mapping $f: L \rightarrow S$ is called a lattice homomorphism from L to S if for any $a, b \in L$.

$$f(a \oplus b) = f(a) \vee f(b)$$

$$\text{and } f(a * b) = f(a) \wedge f(b).$$

Lattice Isomorphism :-

Let $\langle L, \oplus, * \rangle$ and $\langle S, \vee, \wedge \rangle$ be two lattices. If a homomorphism $g: L \rightarrow S$ is bijective, i.e., one-to-one and onto, then g is called an isomorphism.

$$a \leq b \Rightarrow g(a) \leq g(b)$$

$$\text{Also, } g(a) \leq g(b) \Rightarrow a \leq b.$$

Boolean Algebra :-

A lattice that is complemented and distributive is called a Boolean algebra, or in other words, A non-empty set B , together with two binary operations $\oplus$ and $*$ one unary operation $'$ and two distinct elements $'0'$ and $'1'$ is called a Boolean algebra.

1 show that every chain is a distributive lattice.

Solution:-

Let $\langle L, \leq \rangle$ be a chain and let $a, b, c \in L$.

Then, $a \leq b \leq c$ and $a \geq b \geq c$.

when $a \leq b \leq c$, we have,

$$a * (b \oplus c) = a * c \\ = a$$

[$\because b \leq c$]

$$\text{Also, } (a * b) \oplus (a * c) = a \oplus a = a$$

$$\text{Thus, } a * (b \oplus c) = (a * b) \oplus (a * c)$$

$$\text{Again, } a \oplus (b * c) = a \oplus b \\ = b$$

[$\because b \leq c$]

[$\because a \leq b$]

$$\text{Also, } (a \oplus b) * (a \oplus c) = b * c \\ = b$$

$$\therefore a \oplus (b * c) = (a \oplus b) * (a \oplus c)$$

$\Rightarrow \langle L, \oplus, * \rangle$ is a distributive lattice.

when $a \geq b \geq c$, we have

$$a * b = b \text{ and } a \oplus b = a$$

$$a * c = c \text{ and } a \oplus c = a$$

$$\text{Now, } a * (b \oplus c) = a * b = b$$

$$(a * b) \oplus (a * c) = b \oplus c = b$$

$$\therefore a * (b \oplus c) = (a * b) \oplus (a * c)$$

$$\text{Also, } a \oplus (b * c) = (a \oplus c) = a$$

$$(a \oplus b) * (a \oplus c) = a * a = a$$

$$\therefore a \oplus (b * c) = (a \oplus b) * (a \oplus c)$$

Hence, $\langle L, \oplus, * \rangle$ is a distributive lattice.

This indicates that every chain is a distributive lattice.

2. State and prove Demorgan's Law in any Boolean algebra.

Statement:-

Demorgan's Law, $(a \oplus b)' = a' * b'$

and $(a * b)' = a' \oplus b' \quad \forall a, b \in B$

Proof:-

Let $a, b \in B$. Then we can write.

$$\begin{aligned}(a \oplus b) \oplus (a' * b') &= [(a \oplus b) \oplus a'] * [(a \oplus b) \oplus b'] \\&= [(b \oplus a) \oplus a'] * [(a \oplus b) \oplus b'] \\&= [b \oplus (a \oplus a')] * [a \oplus (b \oplus b')] \\&= [b \oplus 1] * [a \oplus 1] \\&= 1 * 1\end{aligned}$$

$$\therefore (a \oplus b) \oplus (a' * b') = 1 \quad \longrightarrow \textcircled{1}$$

Also,

$$\begin{aligned}(a \oplus b) * (a' * b') &= (a' * b') * (a \oplus b) \\&= [(a' * b') * a] \oplus [(a' * b') * b] \\&= [(b' * a') * a] \oplus [(a' * b') * b] \\&= [b' * (a' * a)] \oplus [a' * (b' * b)] \\&= [b' * 0] \oplus [a' * 0] \\&= 0 \oplus 0\end{aligned}$$

$$\therefore (a \oplus b) * (a' * b') = 0 \quad \longrightarrow \textcircled{2}$$

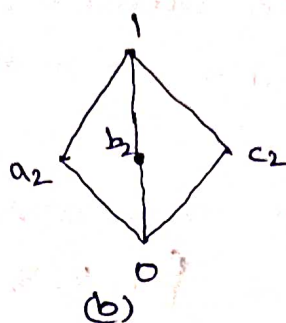
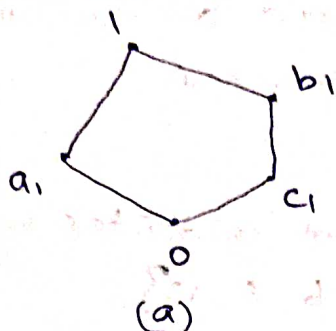
$\therefore$ from equ. $\textcircled{1}$ and $\textcircled{2}$, we have

$$(a \oplus b)' = a' * b'$$

The second part can be proved in the same manner.

Hence proved.

3. Verify whether the lattices given by the following Hasse diagrams are distributive or not.



Solution:-

(i) Let us consider the Hasse diagram (a).

$$c_1 * (a_1 \oplus b_1) = c_1 * 1 = c_1$$

$$\text{Also, } (c_1 * a_1) \oplus (c_1 * b_1) = 0 \oplus c_1 = c_1$$

$$\therefore c_1 * (a_1 \oplus b_1) = (c_1 * a_1) \oplus (c_1 * b_1) \longrightarrow \textcircled{1}$$

$$a_1 * (b_1 \oplus c_1) = a_1 * 1 = a_1$$

$$\text{Also, } (a_1 * b_1) \oplus (a_1 * c_1) = 0 \oplus 0 = 0$$

$$\therefore a_1 * (b_1 \oplus c_1) \neq (a_1 * b_1) \oplus (a_1 * c_1) \longrightarrow \textcircled{2}$$

Let us consider,

$$b_1 * (c_1 \oplus a_1) = b_1 * 1 = b_1$$

$$\text{and } (b_1 * c_1) \oplus (b_1 * a_1) = c_1 \oplus 0 = c_1$$

$$\therefore b_1 * (c_1 \oplus a_1) \neq (b_1 * c_1) \oplus (b_1 * a_1)$$

Hence the lattice given in (a) is not distributive.

(ii) Let us consider the Hasse diagram (b)

$$a_2 * (b_2 * c_2) = a_2 * 1 = a_2$$

$$\text{But, } (a_2 * b_2) \oplus (a_2 * c_2) = 0 \oplus 0 = 0$$

$$\therefore a_2 * (b_2 \oplus c_2) \neq (a_2 * b_2) \oplus (a_2 * c_2)$$

Thus, the lattice given is not ~~distributive~~ distributive.

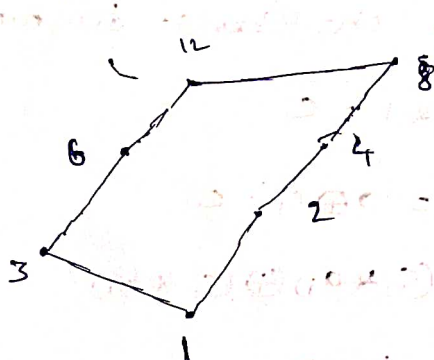
4. If S_n is the set of all divisors of the positive integer n and D is the ~~sublattice~~ relation of division, then prove that $\langle S_{24}, D \rangle$ is a lattice. Also, find all the sublattices of D_{24} that contain five or more elements.

Solution:-

Let $a, b \in S_n$. Then $a D b$ means a divides b .

now, $\langle S_{24}, D \rangle = \{1, 2, 3, 4, 6, 8, 12\}$.

The Hasse Diagram for $\langle S_{24}, D \rangle$.



clearly, $\langle S_{24}, D \rangle$ is a lattice.

The sublattices containing five elements:

$\{1, 2, 3, 4, 6, 12\}$, $\{1, 2, 3, 6, 12\}$, $\{1, 2, 3, 4, 12, 24\}$ and $\{1, 2, 4, 8, 24\}$.

$\therefore$ The sublattices containing six elements $\{1, 2, 3, 4, 6, 12, 24\}$.

5. Let $\langle L, \leq \rangle$ be a lattice in which $\oplus$ and $\times$ denote the operations of join and meet, respectively. Thus for any $a, b \in L$, $a \leq b \Leftrightarrow a \oplus b = b \Leftrightarrow a \times b = a$.

Proof:-

(i) Let $a \leq b$, since $b \in L$, $b \leq b$.

$\therefore a \oplus b \leq b \rightarrow \textcircled{1}$

Since $a \oplus b$ is the lub of $\{a, b\}$, we have

$b \leq a \oplus b \rightarrow \textcircled{2}$

$\therefore$ From equ. $\textcircled{1}$ and $\textcircled{2}$, we have

$\Rightarrow a \leq b \Rightarrow a \oplus b = b \rightarrow \textcircled{3}$

Conversely, let $a \oplus b = b$.

Since, $a \oplus b$ is the lub of $\{a, b\}$, we have

$$a \leq a \oplus b \Rightarrow a \leq b$$

$$\therefore a \oplus b = b \Rightarrow a \leq b \quad \longrightarrow \textcircled{1} \quad [\because \text{by our assumption}]$$

$\therefore$ from $\textcircled{1}$ and $\textcircled{2}$ we get

$$a \leq b \Leftrightarrow a \oplus b = b$$

(ii) Let $a \leq b$, since $a \in L$, $a \leq a$.

$$\therefore a \leq a \times b \quad \longrightarrow \textcircled{5}$$

Since $a \times b$ is the glb of $\{a, b\}$, we have

$$a \times b \leq a \quad \longrightarrow \textcircled{6}$$

$\therefore$ from equ. $\textcircled{5}$ and $\textcircled{6}$, we have.

$$a \times b = a.$$

$$\Rightarrow a \leq b \Rightarrow a \times b = a. \quad \longrightarrow \textcircled{7}$$

Conversely, let $a \times b = a$

Since, $a \times b$ is the glb of $\{a, b\}$, we have

$$a \times b \leq b$$

$$\Rightarrow a \leq b$$

$$\therefore a \times b = a \Rightarrow a \leq b \quad \longrightarrow \textcircled{8}$$

$\therefore$ from $\textcircled{7}$ and $\textcircled{8}$, we have

$$a \leq b \Leftrightarrow a \times b = a.$$

(iii) Let $a \oplus b = b$

$$\Leftrightarrow a \leq b \quad [\because \text{from (i)}]$$

$$\Leftrightarrow a \oplus b = a \quad [\because \text{from (ii)}]$$

Hence proved.

6. If $\langle L, \leq \rangle$ is a lattice, then for any $a, b, c \in L$, the following properties hold good: (i), $b \leq c$.

(i) $a \oplus b \leq a \oplus c$ and (ii) $a * b \leq a * c$.

Proof:-

Let us assume that $b \leq c$.

(i) We need to prove that $a \oplus b \leq a \oplus c$.

By definition, we have

$$a \oplus c = \text{lub of } \{a, c\}.$$

$$\therefore a \leq a \oplus c \text{ and } c \leq a \oplus c. \quad \longrightarrow \textcircled{1}$$

But $b \leq c$ by our assumption.

$$\therefore b \leq c, c \leq a \oplus c \Rightarrow b \leq a \oplus c$$

from $\textcircled{1}$, $a \leq a \oplus c$

$$\therefore \text{lub of } \{a, b\} \leq a \oplus c$$

$$\Rightarrow a \oplus b \leq a \oplus c$$

Hence, $b \leq c \Rightarrow a \oplus b \leq a \oplus c$.

(ii) We need to prove that $a * b \leq a * c$.

By definition,

$$a * b = \text{glb of } \{a, b\}$$

$$\therefore a * b \leq a \text{ and } a * b \leq b.$$

By our assumption, $b \leq c$ $\longrightarrow \textcircled{2}$

$\therefore$ from $\textcircled{1}$ and $\textcircled{2}$, we have

$$a * b \leq b \text{ and } b \leq c, \text{ we get}$$

$$a * b \text{ is the glb of } \{a, c\}$$

$$\Rightarrow (a * b) \leq a * c$$

Hence, $a \leq b \Rightarrow a * b \leq a * c$.

7. If L is a distributive Lattice, prove that:

$$(a * b) \oplus (b * c) \oplus (c * a) = (a \oplus b) * (b \oplus c) * (c \oplus a)$$

Solution:-

Let us consider the L.H.S.

$$(a * b) \oplus (b * c) \oplus (c * a) = (a * b) \oplus [(b * c) \oplus c] * [(b * c) \oplus a]$$

$$= (a \times b) \oplus [c \times (a \oplus (b \times c))]$$

$$= [(a \times b) \oplus c] \times [(a \times b) \oplus a] \oplus [(a \times b) \oplus (b \times c)]$$

$$= (a \oplus c) \times (b \oplus c) \times [(a \oplus ((a \times b) \oplus (b \times c)))]$$

$$= (a \oplus c) \times (b \oplus c) \times (a \oplus (b \times c))$$

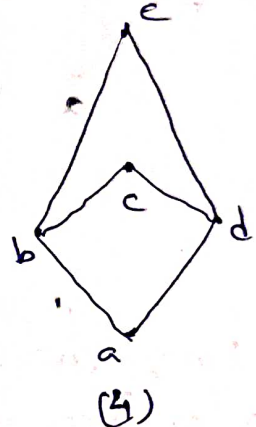
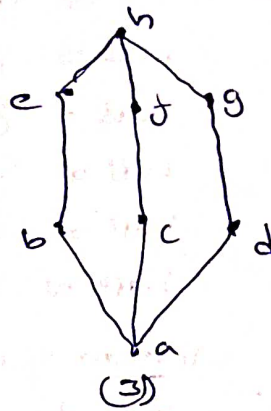
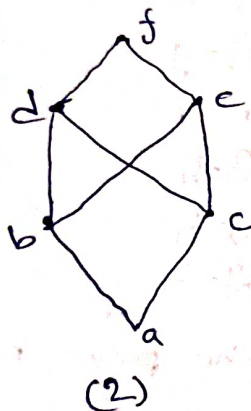
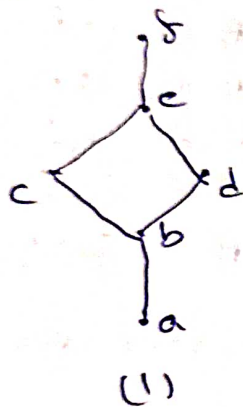
$$= (a \oplus c) \times (b \oplus c) \times (a \oplus b) \times (a \oplus c)$$

$$(a \times b) \oplus (b \times c) \oplus (c \times a) = (a \oplus b) \times (b \oplus c) \times (c \oplus a)$$

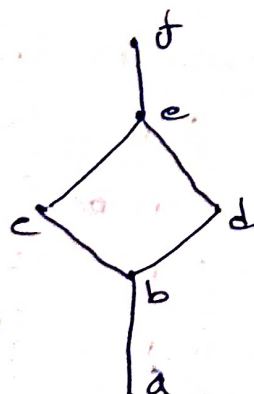
$$\therefore L.H.S = R.H.S.$$

Hence proved.

8. Determine whether the following (Poset) Partial ordering set with the Hasse diagrams are Lattices.



Proof:
(1)



$$\text{LUB of } \{a, b\} = b$$

$$\text{LUB of } \{b, c\} = c$$

$$\text{LUB of } \{b, d\} = d$$

$$\text{LUB of } \{c, e\} = c$$

$$\text{LUB of } \{d, e\} = e$$

$$\text{LUB of } \{b, e\} = e$$

$$\text{LUB of } \{c, d\} = c$$

$$\text{LUB of } \{e, f\} = f$$

$$\therefore \text{GLB of } \{a, b\} = a$$

$$\therefore \text{GLB of } \{b, c\} = b$$

$$\therefore \text{GLB of } \{b, d\} = b$$

$$\therefore \text{GLB of } \{c, e\} = c$$

$$\therefore \text{GLB of } \{d, e\} = d$$

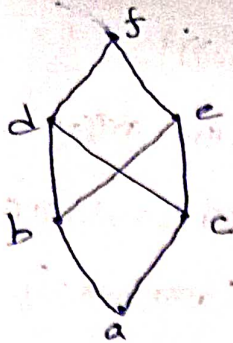
$$\therefore \text{GLB of } \{b, e\} = b$$

$$\therefore \text{GLB of } \{c, d\} = b$$

$$\therefore \text{GLB of } \{e, f\} = e$$

$\therefore$ Every pair of elements of the poset has atleast upper bound and a greatest lower bound. Thus this Hasse diagram is a lattice.

(2)



From the Hasse diagram, we know that,

$$\text{LUB of } \{a, b\} = b$$

$$\text{LUB of } \{a, c\} = c$$

$$\text{LUB of } \{b, d\} = d$$

$$\text{LUB of } \{c, e\} = e$$

$$\text{LUB of } \{b, c\} = e$$

$$\text{LUB of } \{d, e\} = f$$

$$\text{LUB of } \{d, f\} = f$$

$$\text{LUB of } \{c, f\} = f$$

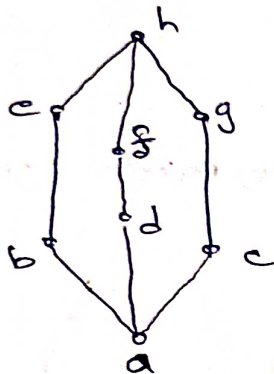
$$\text{LUB of } \{b, f\} = d, e$$

$$\text{LUB of } \{d, e\} = f$$

$$\text{LUB of } \{a, f\} = f$$

Here the pair of elements $\{b, c\}$ does not have a least upper bound. Hence the poset in the given Hasse diagram is not a lattice.

(3)



From the Hasse diagram, we know that

$$\text{LUB of } \{a, b\} = b$$

$$\text{LUB of } \{a, c\} = c$$

$$\text{LUB of } \{a, d\} = d$$

$$\text{LUB of } \{b, e\} = e$$

$$\text{LUB of } \{c, g\} = g$$

$$\text{LUB of } \{d, f\} = f$$

$$\text{LUB of } \{e, b\} = h$$

$$\text{LUB of } \{g, h\} = h$$

$$\text{LUB of } \{f, h\} = h$$

$$\text{GLB of } \{a, b\} = a$$

$$\text{GLB of } \{a, c\} = a$$

$$\text{GLB of } \{a, d\} = a$$

$$\text{GLB of } \{b, e\} = b$$

$$\text{GLB of } \{c, g\} = c$$

$$\text{GLB of } \{d, f\} = d$$

$$\text{GLB of } \{e, h\} = e$$

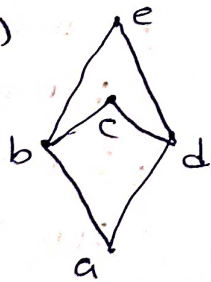
$$\text{GLB of } \{f, h\} = f$$

$$\text{GLB of } \{g, h\} = g$$

Every pair of elements of the poset has a least upper bound and a greatest lower bound.

Hence the poset given in the Hasse diagram is a lattice.

(4)



From the Hasse diagram, we know that

$$\text{LUB of } \{a, b\} = b$$

$$\text{LUB of } \{a, d\} = d$$

$$\text{LUB of } \{b, c\} = c$$

$$\text{LUB of } \{d, c\} = c$$

$$\text{LUB of } \{d, e\} = e$$

$$\text{LUB of } \{b, d\} = c$$

$$\text{GLB of } \{a, b\} = a$$

$$\text{GLB of } \{a, d\} = a$$

$$\text{GLB of } \{b, c\} = b$$

$$\text{GLB of } \{d, c\} = d$$

$$\text{GLB of } \{d, e\} = d$$

$$\text{GLB of } \{b, d\} = a$$

Every pair of elements of the poset has a least upper bound and a greatest lower bound.

Hence the poset given in the Hasse diagram is a lattice.

9. In a Boolean algebra, Prove that (i) $b \leq c \Rightarrow a \cdot b \leq a \cdot c$
 (ii) $b \leq c \Rightarrow a + b \leq a + c$

Solution:-

(i) $b \leq c \Rightarrow a \cdot b \leq a \cdot c$

$$b \leq c \Rightarrow b \cdot c = b \longrightarrow \text{①}$$

Now, $a \cdot b = a \cdot (b \cdot c)$

$$= (a \cdot a) \cdot (b \cdot c)$$

$$= a \cdot (a \cdot b) \cdot c$$

$$= a \cdot (b \cdot a) \cdot c$$

$$= (a \cdot b) \cdot (a \cdot c)$$

$$\therefore (a \cdot b) \leq (a \cdot c)$$

(ii) $b \leq c \Rightarrow b + c = c$

Now, $(a + b) + (a + c) = a + (b + a) + c$

$$= a + (a + b) + c$$

$$= (a + a) + (b + c)$$

$$= a + c$$

$$\Rightarrow (a + b) \leq a + c$$

10. Show that Every distributive lattice is modular. whether the converse is true? Justify your claim.

Solution:-

Let $(L, \leq)$ be a distributive lattice.

For all $a, b, c \in L$, we have

$$a \oplus (b * c) = (a \oplus b) * (a \oplus c)$$

Thus if $a \leq c$, then $a \oplus c = c$ and

$$a \oplus (b * c) = (a \oplus b) * c$$

So if $a \leq c$, the modular equation is satisfied and L is modular.

However, the converse is not true.

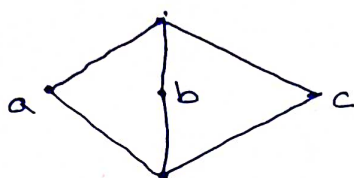
Example of lattice which is modular but not a distributive.

$M_5 \rightarrow$ diamond lattice

In M_5 , $a \vee (b \wedge c) = a \vee 0 = a$ while $(a \vee b) \wedge (a \vee c) = 1 \wedge 1 = 1$

So M_5 is not distributive

As N_5 is not a sublattice of M_5 , M_5 is modular



11. Let $D_{100} = \{1, 2, 4, 5, 10, 20, 25, 50, 100\}$ be the divisors of 100. Draw the Hasse diagram of $(D_{100}, /)$ where $/$ is the relation "division". Find.

(i) GLB $\{10, 20\}$,

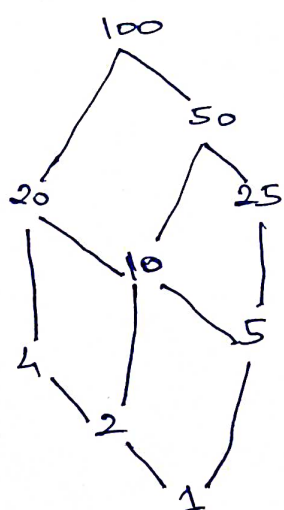
(ii) LUB $\{10, 20\}$

(iii) GLB $\{5, 10, 20, 25\}$

(iv) LUB $\{5, 10, 20, 25\}$

Solution:-

$$D_{100} = \{1, 2, 5, 4, 10, 20, 25, 50, 100\}$$



$$(i) \text{GLB} = \{10, 20\} = 10$$

$$(ii) \text{LUB} = \{10, 20\} = 20$$

$$(iii) \text{GLB} = \{5, 10, 20, 25\} = 5$$

$$(iv) \text{LUB} = \{5, 10, 20, 25\} = 100$$

12. Prove that any chain L is modular lattice.

$\boxed{\sqrt{V}+2}$

Proof:-

(i) To prove every chain is a distributive lattice.

Let $(L, \leq)$ be a chain, $a, b, c \in L$

consider the following possible cases

(1) $a \leq b$ or $a \leq c$ and

(2) $a \geq b$ and $a \geq c$

We shall now show that the distributive lattice

$$a * (b \oplus c) = (a * b) \oplus (a * c)$$

In case (1), $a \leq b$ or $a \leq c$ then we have

$$a * b = a$$

$$a \oplus b = a, \quad a * c = a$$

$$\Rightarrow a \leq b \oplus c$$

$$\text{So } a * (b \oplus c) = a \quad \longrightarrow \textcircled{1}$$

$$\text{and } (a * b) \oplus (a * c) = a \oplus a = a \quad \longrightarrow \textcircled{2}$$

Equ. $\textcircled{1} + \textcircled{2}$, we get

$$a * (b \oplus c) = (a * b) \oplus (a * c)$$

In case (2), If $a \geq b$ and $a \geq c$ then we have

$$a * b = b, \quad a * c = c,$$

$$\text{and } b \oplus c \leq a$$

$$\text{So that, } a * (b \oplus c) = b \oplus c \quad \longrightarrow \textcircled{3}$$

$$(a * b) \oplus (a * c) = b \oplus c \quad \longrightarrow \textcircled{4}$$

Equ. $\textcircled{3}$ and $\textcircled{4}$, we get

$$a * (b \oplus c) = (a * b) \oplus (a * c)$$

(ii) To prove every distributive lattice is modular.

Let $(L, \leq)$ be a distributive lattice

for all $a, b, c \in L$, we have

$$a \oplus (b * c) = (a \oplus b) * (a \oplus c)$$

Thus if $a \leq c$, then $a \oplus c = c$ and

$$a \oplus (b * c) = (a \oplus b) * c$$

So if $a \leq c$, the modular equation is satisfied and L is modular.

13. State and prove the distributive Inequalities of a lattice.

Statement:-

(1) Distributive Inequalities,

$$(i) a \oplus (b * c) \leq (a \oplus b) * (a \oplus c)$$

$$(ii) a * (b \oplus c) \geq (a * b) \oplus (a * c)$$

(2) Modular Inequalities,

$$(i) a \leq c \Leftrightarrow a \oplus (b * c) \leq (a \oplus b) * c$$

$$(ii) a \geq c \Leftrightarrow a * (b \oplus c) \geq (a * b) \oplus c$$

Proof:-

As (i) in 1, (ii) in 2 are duals of (i) in 1 and (i) in 2 respectively it is enough to prove (i) and (i) in 2 only.

consider (i) in 1,

Let $a, b, c \in L$. As $a \leq a \oplus b$ and $a \leq a \oplus c$

we have $a \leq [(a \oplus b) * (a \oplus c)]$

As $b * c \leq b \leq a \oplus b$ and $b * c \leq c \leq a \oplus c$,

we have $(b * c) \leq (a \oplus b) * (a \oplus c)$

$(a \oplus b) * (a \oplus c)$ is an upper bound for a and $b * c$ and hence

$$a \oplus (b * c) \leq (a \oplus b) * (a \oplus c)$$

Thus (i) in 1 is proved.

The inequality (i) in 2 is special case of (i) in 1.

If $a \leq c$, then $a \oplus c = c$ and from (i) in 1, we have

$$a \oplus (b * c) \leq (a \oplus b) * (a \oplus c) = (a \oplus b) * c$$

which is inequality (i) in 2.

Hence proved.

14. If (A, R) is a partially ordered set then show that the set (A, R^1) is also a partially ordered set, where $R^1 = \{(b, a) \mid (a, b) \in R\}$

Solution:-

Let $R = \{(b, a)\}$ is a partial order relation on A .

Claim:-

$R^1 = \{(a, b)\}$ is a partial order relation. $[\because (a, a) \in R]$

$\Rightarrow R^{-1} = \{(a,a)\}$ is a reflexive.

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Given R is Antisymmetric:

To prove R^{-1} is Antisymmetric

$(a,b) \in R$ and $(b,a) \in R \Rightarrow a=b \longrightarrow \textcircled{1}$

Since $(a,b) \in R \Rightarrow (b,a) \in R^{-1}$

$(b,a) \in R \Rightarrow (a,b) \in R^{-1}$

$(a,b) \in R^{-1} \text{ \& } (b,a) \in R^{-1}$

$\Rightarrow a=b$ [$\because$ by equ. $\textcircled{1}$]

$\therefore R^{-1}$ is Antisymmetric.

Given R is transitive, To prove R^{-1} is transitive

$(a,b) \in R$ and $(b,c) \in R \Rightarrow (a,c) \in R \longrightarrow \textcircled{2}$

Since $(a,b) \in R \Rightarrow (b,a) \in R^{-1}$

and $(b,c) \in R \Rightarrow (c,b) \in R^{-1}$

Since $(c,b) \in R \Rightarrow (b,c) \in R^{-1}$

$\Rightarrow (c,b) \in R^{-1} \text{ \& } (b,a) \in R^{-1} \Rightarrow (c,a) \in R^{-1}$ [$\because$ by $\textcircled{2}$]

$\therefore R^{-1}$ is transitive.

Since R^{-1} is reflexive, Antisymmetric and transitive, R^{-1} is a Partial order relation.

15. Let $(L, *, \oplus)$ and $(M, \wedge, \vee)$ be two lattices. Then prove that,

$(L \times M, \Delta, \nabla)$ is a lattice, where

$(x,y) \Delta (a,b) = (x * a, y \wedge b)$ and

$(x,y) \nabla (a,b) = (x \oplus a, y \vee b)$, $\forall (x,y), (a,b) \in L \times M$

Solution:-

We prove ^{that} the binary operations Δ and ∇ satisfies

commutative, associative and absorption laws in $L \times M$.

Let $\langle x_1, y_1 \rangle, \langle x_2, y_2 \rangle, \langle x_3, y_3 \rangle \in L \times M$

i) $\langle x_1, y_1 \rangle + \langle x_2, y_2 \rangle = \langle x_1 \vee x_2, y_1 \oplus y_2 \rangle$

$= \langle x_2 \vee x_1, y_2 \oplus y_1 \rangle$

$= \langle x_2, y_2 \rangle + \langle x_1, y_1 \rangle$

ii) $\langle x_1, y_1 \rangle \cdot \langle x_2, y_2 \rangle = \langle x_2, y_2 \rangle \cdot \langle x_1, y_1 \rangle$

$\therefore +$ and $\cdot$ are commutative

$$\begin{aligned} \text{(i)} \quad \langle x_1, y_1 \rangle + [\langle x_2, y_2 \rangle + \langle x_3, y_3 \rangle] &= \langle x_1, y_1 \rangle + \langle x_2 \vee x_3, y_2 \vee y_3 \rangle \\ &= \langle x_1 \vee (x_2 \vee x_3), y_1 \vee (y_2 \vee y_3) \rangle \\ &= \langle (x_1 \vee x_2) \vee x_3, (y_1 \vee y_2) \vee y_3 \rangle \\ &= [\langle x_1, y_1 \rangle + \langle x_2, y_2 \rangle] + \langle x_3, y_3 \rangle \end{aligned}$$

$$\text{ii)} \quad \langle x_1, y_1 \rangle \cdot [\langle x_2, y_2 \rangle \cdot \langle x_3, y_3 \rangle] = [\langle x_1, y_1 \rangle \cdot \langle x_2, y_2 \rangle] \cdot \langle x_3, y_3 \rangle$$

$\therefore +$ and $\cdot$ are associative

$$\begin{aligned} \text{(ii)} \quad \langle x_1, y_1 \rangle + [\langle x_1, y_1 \rangle \cdot \langle x_2, y_2 \rangle] &= \langle x_1, y_1 \rangle + \langle x_1 \wedge x_2, y_1 \wedge y_2 \rangle \\ &= \langle x_1 \vee (x_1 \wedge x_2), y_1 \vee (y_1 \wedge y_2) \rangle \\ &= \langle x_1, y_1 \rangle \quad [\text{Absorption in } L \times M] \end{aligned}$$

iii)

$$\langle x_1, y_1 \rangle \cdot [\langle x_1, y_1 \rangle + \langle x_2, y_2 \rangle] = \langle x_1, y_1 \rangle$$

$\therefore +$ and $\cdot$ are absorption law.

Hence, $(L \times M, \cdot, +)$ is a Lattice.

16. Let $(L, \leq)$ be a lattice. If we define $a * b = \text{glb}(a, b)$ and $a \oplus b = \text{lub}(a, b)$ for $a, b \in L$, then show that both $*$ and $\oplus$ satisfy commutative, associative, absorption and idempotent laws.

Solution:-

Property - 1:-

Let $(L, \leq)$ be a lattice, for any $a, b, c \in L$
we have $a * a = a$ and $a \oplus a = a$ [Idempotent laws]

Proof:-

Let $a, b, c \in L$ by one definition of GLB of a and b we have

$$a * b \leq a \longrightarrow \textcircled{1}$$

and if, $a \leq a$ and $a \leq b$, then

$$a \leq a * b \longrightarrow \textcircled{2}$$

As $a \leq a$ from ① and ② we have

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$$a * a \leq a \text{ and } a \leq a * a$$

By the antisymmetric property it follows that $a = a * a$
i.e., we can prove that $a \oplus a = a$.

Property 2 :-

Show that the operation of meet and join on a lattice are associative.

Solution :-

$$\text{To prove } (a * b) * c = a * (b * c)$$

Let $a, b, c \in L$, by the definition, we have

$$(a * b) * c \leq a * b$$

$$\text{and } (a * b) * c \leq c$$

By the definition of GLB of a and b , we have

$$a * b \leq a$$

$$\text{and } a * b \leq b,$$

by the transitive property of $\leq$ we have

$$(a * b) * c \leq a$$

$$\text{and } (a * b) * c \leq b$$

$$\text{As } (a * b) * c \leq b \text{ and } (a * b) * c \leq c$$

we see that, $(a * b) * c$ is lower bound for b and c . From the definition of $b * c$ it follows that $(a * b) * c \leq b * c$

$$\text{As } (a * b) * c \leq a \text{ and } (a * b) * c \leq b * c$$

from the definition of $a * (b * c)$, we have

$$(a * b) * c \leq a * (b * c) \longrightarrow \textcircled{3}$$

$$\text{Now, } a * (b * c) \leq a \text{ and } a * (b * c) \leq b * c$$

$$\text{As } b * c \leq b, \text{ by transitivity } a * (b * c) \leq b$$

$$\text{Since } a * (b * c) \leq a \text{ and } a * (b * c) \leq b$$

$$\text{we have } a * (b * c) \leq (a * b) * c$$

$$\text{As } a * (b * c) \leq (b * c) \leq c$$

$$a * (b * c) \leq (a * b) * c. \longrightarrow \textcircled{4}$$

■

from equ. (3) and (4), by antisymmetric ^{Property} ~~Property~~, it follows that

$$a * (b * c) = (a * b) * c.$$

iii^{ly}, we can prove that $a \oplus (b \oplus c) = (a \oplus b) * c$.

Property-3:-

Show that the operation of meet and join on a lattice are commutative law. i.e., $a * b = b * a$ and $a \oplus b = b \oplus a$.

Solution:-

Given $a, b \in L$ both $a * b$ and $b * a$ are GLB of a and b . By the uniqueness of GLB of a and b , we have $a * b = b * a$.

iii^{ly}, $a \oplus b = b \oplus a$ holds good.

Property-4:-

Absorption law $a * (a \oplus b) = a$ and $a \oplus (a * b) = a$

Solution:-

Let $a, b \in L$. Then $a \leq a$ and $a \leq (a \oplus b)$ so $a \leq a * (a \oplus b)$. So $a \leq a * (a \oplus b)$. On the other hand $a * (a \oplus b) \leq a$. By antisymmetric property of $\leq$ we have $a = a * (a \oplus b)$.

iii^{ly}, we have $a = a \oplus (a * b) \forall a, b \in L$.